COMPARISON OF TWO FORMULAS FOR METRIC CONNECTIONS IN THE BUNDLE OF DIRAC SPINORS.

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Abstract. Two explicit formulas for metric connections in the bundle of Dirac spinors are studied. Their equivalence is proved. The explicit formula relating the spinor curvature tensor with the Riemann curvature tensor is rederived.

1. Basic notations and definitions.

Dirac spinors play crucial role in modern particle physics. However, this crucial application of Dirac spinors is based mostly on the special relativity, where the base manifold M is the flat Minkowski space. Passing to the general relativity, we get a little bit more complicated theory of spinors.

Let M be a *space-time* manifold of the general relativity. It is a four-dimensional orientable manifold equipped with a pseudo-Euclidean Minkowski-type metric $\mathbf{g}$ and with a *polarization*. The polarization of M is responsible for distinguishing the *Future light cone* from the *Past light cone* at each point $p \in M$ (see [1] for more details). Let's denote by DM the bundle of Dirac spinors over M (see [2] and [3] for detailed description). In addition to the metric tensor $\mathbf{g}$ inherited from M, the Dirac bundle DM is equipped with four other basic spin-tensorial fields:

Symbol	Name	Spin-tensorial type
$\mathbf{g}$	Metric tensor	$(0,0\|0,0\|0,2)$
$\mathbf{d}$	Skew-symmetric metric tensor	$(0,2\|0,0\|0,0)$
$\mathbf{H}$	Chirality operator	$(1,1\|0,0\|0,0)$
$\mathbf{D}$	Dirac form	$(0,1\|0,1\|0,0)$
$\boldsymbol{\gamma}$	Dirac γ-field	$(1,1\|0,0\|1,0)$

(1.1)

The spin-tensorial type in the above table (1.1) reflects the number of indices in coordinate representation of fields. The first two numbers are the numbers of upper and lower spinor indices, the second two numbers are the numbers of upper and lower conjugate spinor indices, and the last two numbers are the numbers of upper and lower tensorial indices (they are also called spacial indices). The metric tensor

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$\mathbf{g}$ is interpreted as a spin-tensorial field of the type $(0,0|0,0|0,2)$, i. e. it has no spinor indices and no conjugate spinor indices, but has two lower spacial indices.

The Dirac bundle is a complex bundle over a real manifold. For this reason spin-tensorial bundles produced from DM are equipped with the involution of complex conjugation τ which exchanges spinor and conjugate spinor indices:

$$D^{\alpha}_{\beta}\bar{D}^{\nu}_{\gamma}T^{m}_{n}M \;\underset{\tau}{\overset{\tau}{\rightleftarrows}}\; D^{\nu}_{\gamma}\bar{D}^{\alpha}_{\beta}T^{m}_{n}M. \tag{1.2}$$

Two fields $\mathbf{g}$ and $\mathbf{D}$ in (1.1) are real fields, i. e. they are invariant with respect to the involution of complex conjugation (1.2):

$$\tau(\mathbf{g})=\mathbf{g}, \qquad\qquad \tau(\mathbf{D})=\mathbf{D}.$$

Other fields $\mathbf{d}$, $\mathbf{H}$, and $\boldsymbol{\gamma}$ in (1.1) are not real fields. For them we denote

$$\bar{\boldsymbol{\gamma}}=\tau(\boldsymbol{\gamma}), \qquad\qquad \bar{\mathbf{d}}=\tau(\mathbf{d}), \qquad\qquad \bar{\mathbf{H}}=\tau(\mathbf{H}).$$

Definition 1.1. A metric connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ in DM is a spinor connection which is real in the sense of the involution (1.2) and concordant with $\mathbf{d}$ and $\boldsymbol{\gamma}$, i. e.

$$\tau\circ\nabla=\nabla\circ\tau, \qquad\qquad \nabla\mathbf{d}=0, \qquad\qquad \nabla\boldsymbol{\gamma}=0. \tag{1.3}$$

Note that we use three symbols $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ for denoting a spinor connection. It is because we use three different types of connection components for three groups of indices when writing covariant derivatives in coordinates:

$$\left.\begin{aligned}
&\nabla_i X^k = L_{\boldsymbol{\Upsilon}_i}X^k+\sum_{j=0}^{3}\Gamma^k_{ij}\,X^j,\\
&\nabla_i X_k = L_{\boldsymbol{\Upsilon}_i}X_k-\sum_{j=0}^{3}\Gamma^j_{i\,k}\,X_j,
\end{aligned}\right\}\ \text{for spacial indices,} \tag{1.4}$$

$$\left.\begin{aligned}
&\nabla_i \psi^a = L_{\boldsymbol{\Upsilon}_i}\psi^a+\sum_{b=1}^{4}\mathrm{A}^a_{ib}\ \psi^b,\\
&\nabla_i \psi_a = L_{\boldsymbol{\Upsilon}_i}\psi_a-\sum_{b=1}^{4}\mathrm{A}^b_{i\,a}\ \psi_b,
\end{aligned}\right\}\ \text{for spinor indices,} \tag{1.5}$$

$$\left.\begin{aligned}
&\nabla_i \psi^{\bar{a}} = L_{\boldsymbol{\Upsilon}_i}\psi^{\bar{a}}+\sum_{\bar{b}=1}^{4}\bar{\mathrm{A}}^{\bar{a}}_{i\bar{b}}\ \psi^{\bar{b}},\\
&\nabla_i \psi_{\bar{a}} = L_{\boldsymbol{\Upsilon}_i}\psi_{\bar{a}}-\sum_{\bar{b}=1}^{4}\bar{\mathrm{A}}^{\bar{b}}_{i\,\bar{a}}\ \psi_{\bar{b}},
\end{aligned}\right\}\ \begin{array}{l}\text{for conjugate}\\ \text{spinor indices.}\end{array} \tag{1.6}$$

In the case of a field of some mixed spin-tensorial type the formulas (1.4), (1.5), and (1.6) are combined (see formula (7.10) in [3]).

Note that in (1.4), (1.5), and (1.6) we have no partial derivatives. They are replaced by the derivatives $L_{\boldsymbol{\Upsilon}_i}$ along four vector fields $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ forming some local frame of the tangent bundle TM. The spacial indices i, j, and k in (1.4), (1.5), and (1.6) are also relative to this frame. The spinor and conjugate spinor indices a, b, $\bar{a}$, and $\bar{b}$ in these formulas are relative to some spinor frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$, $\mathbf{\Psi}_3$, $\mathbf{\Psi}_4$ of the Dirac bundle DM. Local frames of the tangent bundle TM are generalizations of local coordinates (they are also called non-holonomic coordinates). Indeed, once some local coordinates x^0, x^1, x^2, x^3 in M are given, we have their associated frame of coordinate vector fields:

$$\mathbf{E}_0 = \frac{\partial}{\partial x^0}, \qquad \mathbf{E}_1 = \frac{\partial}{\partial x^1}, \qquad \mathbf{E}_2 = \frac{\partial}{\partial x^2}, \qquad \mathbf{E}_3 = \frac{\partial}{\partial x^3}. \tag{1.7}$$

The local coordinates x^0, x^1, x^2, x^3 are called holonomic, since their associated vector fields (1.7) commute with each other:

$$[\mathbf{E}_i, \mathbf{E}_j] = 0. \tag{1.8}$$

Unlike (1.8), the vector fields $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$ of a general (non-holonomic) frame do not commute. In this case we have

$$[\mathbf{\Upsilon}_i, \mathbf{\Upsilon}_j] = \sum_{k=0}^{3} c^k_{ij}\, \mathbf{\Upsilon}_k. \tag{1.9}$$

As for spinor frames, they are always non-holonomic since they are composed by spinor fields, while the commutator of spinor fields is not defined at all.

Theorem 1.1. *Any metric connection* $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ *is concordant with all of the basic spin tensorial fields* $\mathbf{g}$*,* $\mathbf{d}$*,* $\mathbf{H}$*,* $\mathbf{D}$*, and* $\boldsymbol{\gamma}$ *listed in the table* (1.1)*.*

The theorem 1.1 means that from (1.3) it follows that

$$\nabla\mathbf{g} = 0, \qquad \nabla\mathbf{H} = 0, \qquad \nabla\mathbf{D} = 0. \tag{1.10}$$

Applying the first identity (1.3) to (1.10) and to other identities (1.3), we derive

$$\nabla\bar{\boldsymbol{\gamma}} = 0, \qquad \nabla\bar{\mathbf{d}} = 0, \qquad \nabla\bar{\mathbf{H}} = 0.$$

The general relativity (the Einstein's theory of gravity) is a theory with zero torsion. Exactly for this case we have the following theorem.

Theorem 1.2. *There is a unique metric connection* $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ *of the bundle of Dirac spinors* DM *whose torsion is zero.*

The metric connection with zero torsion is called the *Levi-Civita* connection. The proof of both theorems 1.1 and 1.2 can be found in [3]. The first identity (1.10) means that Γ^k_{ij} in (1.4) are the components of the standard Levi-Civita connection. In a holonomic frame (1.7) they are given by the standard formula

$$\Gamma^k_{ij} = \sum_{r=0}^{3} \frac{g^{kr}}{2} \left(\frac{\partial g_{rj}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r} \right). \tag{1.11}$$

The quantities (1.11) are symmetric, i. e. $\Gamma^k_{ij} = \Gamma^k_{j\,i}$. In the case of a non-holonomic frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ the components of the Levi-Civita connection are not symmetric: $\Gamma^k_{ij} - \Gamma^k_{j\,i} = -c^k_{ij}$ with c^k_{ij} taken from (1.9). They are given by the formula

$$\begin{aligned}\Gamma^k_{ij} = \sum^{3}_{r=0} \frac{g^{kr}}{2} \left(L_{\boldsymbol{\Upsilon}_i}(g_{rj}) + L_{\boldsymbol{\Upsilon}_j}(g_{ir}) - L_{\boldsymbol{\Upsilon}_r}(g_{ij})\right) -\\ - \frac{c^k_{ij}}{2} + \sum^{3}_{r=0}\sum^{3}_{s=0} \frac{c^s_{ir}}{2}\, g^{kr}\, g_{sj} + \sum^{3}_{r=0}\sum^{3}_{s=0} \frac{c^s_{j\,r}}{2}\, g^{kr}\, g_{si}.\end{aligned} \tag{1.12}$$

Now let's proceed to the quantities $\mathrm{A}^a_{i\,b}$ and $\bar{\mathrm{A}}^{\bar{a}}_{i\,\bar{b}}$ in the formulas (1.5) and (1.6). Due to the theorem 1.2 for a torsion-free connection they are uniquely determined by the equalities (1.3). From the first equality (1.3) one easily derives that $\mathrm{A}^a_{i\,b}$ and $\bar{\mathrm{A}}^{\bar{a}}_{i\,\bar{b}}$ are related to each other by virtue of the complex conjugation:

$$\bar{\mathrm{A}}^{\bar{a}}_{i\,\bar{b}} = \overline{\mathrm{A}^{\bar{a}}_{i\,\bar{b}}}. \tag{1.13}$$

In one of my previous papers I have derived the following formula for the spinor components $\mathrm{A}^a_{i\,b}$ of the metric connection (see formula (8.34) in [3]):

$$\begin{aligned}\mathrm{A}^a_{ib} = \sum^{4}_{\alpha=1}\sum^{4}_{\beta=1} \frac{L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{d}_{\alpha\beta})\, \overset{\bullet}{d}{}^{\beta\alpha}\, \overset{\circ}{H}{}^a_b + L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{d}_{\alpha\beta})\, \overset{\circ}{d}{}^{\beta\alpha}\, \overset{\bullet}{H}{}^a_b}{4} +\\ + \sum^{3}_{m=0}\sum^{3}_{n=0}\sum^{4}_{\alpha=1} \frac{L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{bm})\, g^{mn}\, \overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n} + L_{\boldsymbol{\Upsilon}_i}(\overset{\circ\bullet}{\gamma}{}^{\alpha}_{bm})\, g^{mn}\, \overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}}{4} -\\ - \sum^{3}_{m=0}\sum^{3}_{n=0}\sum^{3}_{r=0}\sum^{4}_{\alpha=1} \frac{\Gamma^r_{im}\, \overset{\bullet\circ}{\gamma}{}^{\alpha}_{br}\, g^{mn}\, \overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n} + \Gamma^r_{im}\, \overset{\circ\bullet}{\gamma}{}^{\alpha}_{br}\, g^{mn}\, \overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}}{4}.\end{aligned} \tag{1.14}$$

However, in some other papers there are much more simple formulas for spinor connections. I choose the formula (5) from [4] for comparing it with (1.14). Being transformed to our notations, this formula looks like

$$A^a_{ib} = \frac{1}{4} \sum^{3}_{m=0}\sum^{3}_{n=0}\sum^{3}_{r=0}\sum^{4}_{\alpha=1} \Gamma^r_{im}\, \gamma^{\alpha}_{bn}\, g^{mn}\, \gamma^a_{\alpha r}. \tag{1.15}$$

Our main goal is to compare the formulas (1.14) and (1.15). Then we calculate the curvature tensors associated with these formulas.

2. Comparison of the formulas (1.14) and (1.15).

The formula (1.15) has no differentiations at all. This means that it is written for special frames where the components of the basic fields $\mathbf{d}$ and $\boldsymbol{\gamma}$ are constants. For this reason we could omit the terms with derivatives $L_{\boldsymbol{\Upsilon}_i}$ in (1.14) for the comparison purposes. However, we shall not do it. We shall transform the formula (1.14) to a form similar to (1.15) preserving all of its terms. As a result in each step we shall have a formula applicable for arbitrary frames.

Another feature of the formula (1.14), in contrast to (1.15), is that it uses special notations with circles and bullets. These notations were introduced in [3]. Now we need to reproduce them. Remember that the chirality operator $\mathbf{H}$ is a diagonalizable operator with two eigenvalues $\lambda_1 = \lambda_2 = 1$ and two other eigenvalues $\lambda_3 = \lambda_4 = -1$. Therefore, $\mathbf{H}^2 = 1$. Due to this equality one can define two projection operators

$$\overset{\bullet}{\mathbf{H}} = \frac{1+\mathbf{H}}{2}, \qquad\qquad \overset{\circ}{\mathbf{H}} = \frac{1-\mathbf{H}}{2} \tag{2.1}$$

(see (8.1) in [3]). The components $\overset{\circ}{H}{}^a_b$ and $\overset{\bullet}{H}{}^a_b$ of these two projection operators (2.1) are used in the formula (1.14). Other special symbols in (1.14) are defined as

$$\begin{aligned}
&\overset{\bullet}{d}{}^{\beta\alpha} = \sum_{a=1}^{4}\sum_{b=1}^{4} \overset{\bullet}{H}{}^{\beta}_{b}\, d^{ba}\, \overset{\bullet}{H}{}^{\alpha}_{a}, &&\overset{\circ}{d}{}^{\beta\alpha} = \sum_{a=1}^{4}\sum_{b=1}^{4} \overset{\circ}{H}{}^{\beta}_{b}\, d^{ba}\, \overset{\circ}{H}{}^{\alpha}_{a},\\
&\overset{\bullet}{d}{}_{\alpha\beta} = \sum_{a=1}^{4}\sum_{b=1}^{4} \overset{\bullet}{H}{}^{a}_{\alpha}\, d_{ab}\, \overset{\bullet}{H}{}^{b}_{\beta}, &&\overset{\circ}{d}{}_{\alpha\beta} = \sum_{a=1}^{4}\sum_{b=1}^{4} \overset{\circ}{H}{}^{a}_{\alpha}\, d_{ab}\, \overset{\circ}{H}{}^{b}_{\beta}
\end{aligned} \tag{2.2}$$

(see (8.19) in [3]). Here d_{ab} and d^{ba} are the components of two mutually inverse skew-symmetric matrices. The first of them represents the spinor metric tensor $\mathbf{d}$ (see table (1.1)) and the second one corresponds to its dual metric tensor which is denoted by the same symbol $\mathbf{d}$. Similarly, we have

$$\overset{\circ\bullet}{\gamma}{}^{\alpha}_{\beta m} = \sum_{a=1}^{4}\sum_{b=1}^{4} \overset{\circ}{H}{}^{\alpha}_{a}\, \overset{\bullet}{H}{}^{b}_{\beta}\, \gamma^{a}_{bm}, \qquad\qquad \overset{\bullet\circ}{\gamma}{}^{\alpha}_{\beta m} = \sum_{a=1}^{4}\sum_{b=1}^{4} \overset{\bullet}{H}{}^{\alpha}_{a}\, \overset{\circ}{H}{}^{b}_{\beta}\, \gamma^{a}_{bm} \tag{2.3}$$

(see (8.5) in [3]). The projection operators (2.1) obey some commutation and anticommutation relationships with $\mathbf{d}$ and $\boldsymbol{\gamma}$:

$$\begin{aligned}
&\sum_{b=1}^{4} d_{\alpha b}\, \overset{\bullet}{H}{}^{b}_{\beta} = \sum_{a=1}^{4} \overset{\bullet}{H}{}^{a}_{\alpha}\, d_{a\beta}, &&\sum_{a=1}^{4} d^{\beta a}\, \overset{\bullet}{H}{}^{\alpha}_{a} = \sum_{b=1}^{4} \overset{\bullet}{H}{}^{\beta}_{b}\, d^{b\alpha},\\
&\sum_{b=1}^{4} d_{\alpha b}\, \overset{\circ}{H}{}^{b}_{\beta} = \sum_{a=1}^{4} \overset{\circ}{H}{}^{a}_{\alpha}\, d_{a\beta}, &&\sum_{a=1}^{4} d^{\beta a}\, \overset{\circ}{H}{}^{\alpha}_{a} = \sum_{b=1}^{4} \overset{\circ}{H}{}^{\beta}_{b}\, d^{b\alpha},
\end{aligned} \tag{2.4}$$

$$\sum_{b=1}^{4} \gamma^{\alpha}_{bm}\, \overset{\bullet}{H}{}^{b}_{\beta} = \sum_{a=1}^{4} \overset{\circ}{H}{}^{\alpha}_{a}\, \gamma^{a}_{\beta m}, \qquad\qquad \sum_{b=1}^{4} \gamma^{\alpha}_{bm}\, \overset{\circ}{H}{}^{b}_{\beta} = \sum_{a=1}^{4} \overset{\bullet}{H}{}^{\alpha}_{a}\, \gamma^{a}_{\beta m} \tag{2.5}$$

(see (6.16) and (6.17) in [3]). Due to the relationships (2.4) and (2.5) the formulas (2.2) and (2.3) are written as follows:

$$\begin{aligned}
&\overset{\bullet}{d}{}_{\alpha\beta} = \sum_{b=1}^{4} d_{\alpha b}\, \overset{\bullet}{H}{}^{b}_{\beta} = \sum_{a=1}^{4} \overset{\bullet}{H}{}^{a}_{\alpha}\, d_{a\beta},\\
&\qquad\qquad \overset{\bullet}{d}{}^{\beta\alpha} = \sum_{a=1}^{4} d^{\beta a}\, \overset{\bullet}{H}{}^{\alpha}_{a} = \sum_{b=1}^{4} \overset{\bullet}{H}{}^{\beta}_{b}\, d^{b\alpha},
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
&\overset{\circ}{d}{}_{\alpha\beta} = \sum_{b=1}^{4} d_{\alpha b}\, \overset{\circ}{H}{}^{b}_{\beta} = \sum_{a=1}^{4} \overset{\circ}{H}{}^{a}_{\alpha}\, d_{a\beta},\\
&\qquad\qquad \overset{\circ}{d}{}^{\beta\alpha} = \sum_{a=1}^{4} d^{\beta a}\, \overset{\circ}{H}{}^{\alpha}_{a} = \sum_{b=1}^{4} \overset{\circ}{H}{}^{\beta}_{b}\, d^{b\alpha},
\end{aligned} \tag{2.7}$$

$$
\begin{gathered}
\overset{\circ\bullet}{\gamma}{}^{\alpha}_{\beta m}=\sum_{b=1}^{4}\gamma^{\alpha}_{bm}\,\overset{\bullet}{H}{}^{b}_{\beta}=\sum_{a=1}^{4}\overset{\circ}{H}{}^{\alpha}_{a}\,\gamma^{a}_{\beta m},\\
\overset{\bullet\circ}{\gamma}{}^{\alpha}_{\beta m}=\sum_{b=1}^{4}\gamma^{\alpha}_{bm}\,\overset{\circ}{H}{}^{b}_{\beta}=\sum_{a=1}^{4}\overset{\bullet}{H}{}^{\alpha}_{a}\,\gamma^{a}_{\beta m}.
\end{gathered}
\tag{2.8}
$$

Now we apply (2.6), (2.7), and (2.8) for to transform the formula (1.14). Using the formulas (2.8), we derive the following relationship:

$$
\begin{gathered}
\sum_{\alpha=1}^{4}\Bigl(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{br}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+\overset{\circ\bullet}{\gamma}{}^{\alpha}_{br}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\Bigr)=\sum_{\alpha=1}^{4}\Bigl(\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}\,\overset{\bullet\circ}{\gamma}{}^{\alpha}_{br}+\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\,\overset{\circ\bullet}{\gamma}{}^{\alpha}_{br}\Bigr)=\\
=\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\Bigl(\gamma^{a}_{cn}\,\overset{\bullet}{H}{}^{c}_{\alpha}\,\overset{\bullet}{H}{}^{\alpha}_{d}\,\gamma^{d}_{br}+\gamma^{a}_{cn}\,\overset{\circ}{H}{}^{c}_{\alpha}\,\overset{\circ}{H}{}^{\alpha}_{d}\,\gamma^{d}_{br}\Bigr).
\end{gathered}
\tag{2.9}
$$

Remember that $\overset{\bullet}{\mathbf{H}}$ and $\overset{\circ}{\mathbf{H}}$ are projection operators complementary to each other, i. e. $\overset{\bullet}{\mathbf{H}}{}^2=\overset{\bullet}{\mathbf{H}}$, $\overset{\circ}{\mathbf{H}}{}^2=\overset{\circ}{\mathbf{H}}$, and $\overset{\bullet}{\mathbf{H}}+\overset{\circ}{\mathbf{H}}=1$. Therefore, we have

$$
\sum_{\alpha=1}^{4}\Bigl(\overset{\bullet}{H}{}^{c}_{\alpha}\,\overset{\bullet}{H}{}^{\alpha}_{d}+\overset{\circ}{H}{}^{c}_{\alpha}\,\overset{\circ}{H}{}^{\alpha}_{d}\Bigr)=\overset{\bullet}{H}{}^{c}_{d}+\overset{\circ}{H}{}^{c}_{d}=\delta^{c}_{d}.
\tag{2.10}
$$

Applying this formula (2.10) to (2.9) we derive

$$
\sum_{\alpha=1}^{4}\Bigl(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{br}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+\overset{\circ\bullet}{\gamma}{}^{\alpha}_{br}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\Bigr)=\sum_{c=1}^{4}\gamma^{a}_{cn}\,\gamma^{c}_{br}.
\tag{2.11}
$$

Using the formula (2.11), we can transform the last term of (1.14) as follows:

$$
\begin{gathered}
\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\Gamma^{r}_{im}\,\overset{\bullet\circ}{\gamma}{}^{\alpha}_{br}\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+\Gamma^{r}_{im}\,\overset{\circ\bullet}{\gamma}{}^{\alpha}_{br}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}}{4}=\\
=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{c=1}^{4}\frac{\Gamma^{r}_{im}\,\gamma^{c}_{br}\,g^{mn}\,\gamma^{a}_{cn}}{4}.
\end{gathered}
\tag{2.12}
$$

Now let's proceed to the second term in the right hand side of the formula (1.14). Applying the formulas (2.8), we derive

$$
\begin{gathered}
\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\Bigl(L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ\bullet}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\Bigr)=\\
=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\gamma^{a}_{cn}\,\Bigl(\overset{\bullet}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d}\,\gamma^{d}_{bm})+\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d}\,\gamma^{d}_{bm})\Bigr)\,g^{mn}=\\
=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\gamma^{a}_{cn}\,\Bigl(\overset{\bullet}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})+\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\Bigr)\,g^{mn}\,\gamma^{d}_{bm}+\\
+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\gamma^{a}_{cn}\,\Bigl(\overset{\bullet}{H}{}^{c}_{\alpha}\,\overset{\bullet}{H}{}^{\alpha}_{d}+\overset{\circ}{H}{}^{c}_{\alpha}\,\overset{\circ}{H}{}^{\alpha}_{d}\Bigr)\,L_{\boldsymbol{\Upsilon}_i}(\gamma^{d}_{bm})\,g^{mn}.
\end{gathered}
$$

In order to continue our calculations we need the identity (6.21) from [3]:

$$
\begin{aligned}
\sum_{m=0}^{3}\sum_{n=0}^{3}\gamma^{d}_{bm}\,g^{mn}\,\gamma^{a}_{cn} &= \delta^{d}_{c}\,\delta^{a}_{b}-H^{d}_{c}\,H^{a}_{b}+\\
&+d^{da}\,d_{bc}-\sum_{r=1}^{4}\sum_{s=1}^{4}H^{d}_{r}\,d^{ra}\,d_{bs}\,H^{s}_{c}.
\end{aligned}
\tag{2.13}
$$

Applying (2.10) and (2.13) to the above formula, we derive

$$
\begin{aligned}
&\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\Bigl(L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ\bullet}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\Bigr)=\\
&\qquad=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm})\,g^{mn}\,\gamma^{a}_{\alpha n}+\\
&=\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\overset{\bullet}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\,\Biggl(\delta^{d}_{c}\,\delta^{a}_{b}-H^{d}_{c}\,H^{a}_{b}\,+\\
&\qquad\qquad+d^{da}\,d_{bc}-\sum_{r=1}^{4}\sum_{s=1}^{4}H^{d}_{r}\,d^{ra}\,d_{bs}\,H^{s}_{c}\Biggr)+\\
&+\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\,\Biggl(\delta^{d}_{c}\,\delta^{a}_{b}-H^{d}_{c}\,H^{a}_{b}\,+\\
&\qquad\qquad+d^{da}\,d_{bc}-\sum_{r=1}^{4}\sum_{s=1}^{4}H^{d}_{r}\,d^{ra}\,d_{bs}\,H^{s}_{c}\Biggr).
\end{aligned}
$$

Now remember that $\overset{\bullet}{\mathbf{H}}\circ\mathbf{H}=\mathbf{H}\circ\overset{\bullet}{\mathbf{H}}=\overset{\bullet}{\mathbf{H}}$ and $\overset{\circ}{\mathbf{H}}\circ\mathbf{H}=\mathbf{H}\circ\overset{\circ}{\mathbf{H}}=-\overset{\circ}{\mathbf{H}}$. These formulas are easily derived from (2.1). Applying them, we find

$$
\begin{aligned}
&\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\Bigl(L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ\bullet}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\Bigr)=\\
&=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm})\,g^{mn}\,\gamma^{a}_{\alpha n}+\sum_{\alpha=1}^{4}\sum_{d=1}^{4}\overset{\bullet}{H}{}^{d}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\,(\delta^{a}_{b}-H^{a}_{b})\,+\\
&+\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}d_{bc}\,\overset{\bullet}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\sum_{r=1}^{4}\bigl(\delta^{d}_{r}-H^{d}_{r}\bigr)\,d^{ra}+\sum_{\alpha=1}^{4}\sum_{d=1}^{4}\overset{\circ}{H}{}^{d}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\,\times\\
&\times\,(\delta^{a}_{b}+H^{a}_{b})+\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}d_{bc}\,\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\sum_{r=1}^{4}\bigl(\delta^{d}_{r}+H^{d}_{r}\bigr)\,d^{ra}.
\end{aligned}
$$

Note that $\delta^{a}_{b}-H^{a}_{b}=2\,\overset{\circ}{H}{}^{a}_{b}$ and $\delta^{a}_{b}+H^{a}_{b}=2\,\overset{\bullet}{H}{}^{a}_{b}$. Similarly, $\delta^{d}_{r}-H^{d}_{r}=2\,\overset{\circ}{H}{}^{d}_{r}$ and $\delta^{d}_{r}+H^{d}_{r}=2\,\overset{\bullet}{H}{}^{d}_{r}$. Moreover, we apply the following obvious formula

$$
L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm})\,g^{mn}=L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm}\,g^{mn})-\gamma^{\alpha}_{bm}\,L_{\boldsymbol{\Upsilon}_i}(g^{mn}).
$$

Therefore, the above formula is transformed to the following one:

$$
\begin{aligned}
&\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\Big(L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ\bullet}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\Big)=\\
&=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm}\,g^{mn})\,\gamma^{a}_{\alpha n}-\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\gamma^{\alpha}_{bm}\,L_{\boldsymbol{\Upsilon}_i}(g^{mn})\,\gamma^{a}_{\alpha n}+\\
&+2\sum_{\alpha=1}^{4}\sum_{d=1}^{4}\overset{\bullet}{H}{}^{d}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\,\overset{\circ}{H}{}^{a}_{b}+2\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}d_{bc}\,\overset{\bullet}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\,\overset{\circ}{H}{}^{d}_{r}\,d^{ra}+\\
&+2\sum_{\alpha=1}^{4}\sum_{d=1}^{4}\overset{\circ}{H}{}^{d}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\,\overset{\bullet}{H}{}^{a}_{b}+2\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}d_{bc}\,\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\,\overset{\bullet}{H}{}^{d}_{r}\,d^{ra}.
\end{aligned}
$$

For the derivative $L_{\boldsymbol{\Upsilon}_i}(g^{mn})$ in the above formula we write

$$
L_{\boldsymbol{\Upsilon}_i}(g^{mn})=-\sum_{s=0}^{3}\Gamma^{m}_{is}\,g^{sn}-\sum_{s=0}^{3}\Gamma^{n}_{is}\,g^{ms}. \tag{2.14}
$$

This formula is easily derived from $\nabla_i g^{mn}=0$. Applying (2.14), we get

$$
\begin{aligned}
&\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\Big(L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ\bullet}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\Big)=\\
&=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm}\,g^{mn})\,\gamma^{a}_{\alpha n}+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\gamma^{\alpha}_{bm}\,\Gamma^{m}_{is}\,g^{sn}\,\gamma^{a}_{\alpha n}+\\
&+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\gamma^{\alpha}_{bm}\,\Gamma^{n}_{is}\,g^{ms}\,\gamma^{a}_{\alpha n}+2\sum_{\alpha=1}^{4}\sum_{d=1}^{4}\overset{\bullet}{H}{}^{d}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\,\overset{\circ}{H}{}^{a}_{b}+\\
&+2\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}d_{bc}\,\overset{\bullet}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\,\overset{\circ}{H}{}^{d}_{r}\,d^{ra}+2\sum_{\alpha=1}^{4}\sum_{d=1}^{4}\overset{\circ}{H}{}^{d}_{\alpha}\times\\
&\times L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\,\overset{\bullet}{H}{}^{a}_{b}+2\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}d_{bc}\,\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\,\overset{\bullet}{H}{}^{d}_{r}\,d^{ra}.
\end{aligned} \tag{2.15}
$$

The last step in our calculations is to transform the first term in the right hand side of (1.14). Applying (2.6) and (2.7) to it, we get

$$
\begin{aligned}
&\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\Big(L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{d}_{\alpha\beta})\,\overset{\bullet}{d}{}^{\beta\alpha}\,\overset{\circ}{H}{}^{a}_{b}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{d}_{\alpha\beta})\,\overset{\circ}{d}{}^{\beta\alpha}\,\overset{\bullet}{H}{}^{a}_{b}\Big)=\\
&=\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{c}_{\alpha}\,d_{c\beta})\,d^{\beta d}\,\overset{\bullet}{H}{}^{\alpha}_{d}\,\overset{\circ}{H}{}^{a}_{b}+\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{c}_{\alpha}\,d_{c\beta})\times\\
&\times d^{\beta d}\,\overset{\circ}{H}{}^{\alpha}_{d}\,\overset{\bullet}{H}{}^{a}_{b}=\sum_{\alpha=1}^{4}\sum_{d=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{d}_{\alpha})\,\overset{\bullet}{H}{}^{\alpha}_{d}\,\overset{\circ}{H}{}^{a}_{b}+\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{d=1}^{4}L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta d}\,\overset{\bullet}{H}{}^{\alpha}_{d}\times\\
&\times\overset{\circ}{H}{}^{a}_{b}+\sum_{\alpha=1}^{4}\sum_{d=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{d}_{\alpha})\,\overset{\circ}{H}{}^{\alpha}_{d}\,\overset{\bullet}{H}{}^{a}_{b}+\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{d=1}^{4}L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta d}\,\overset{\circ}{H}{}^{\alpha}_{d}\,\overset{\bullet}{H}{}^{a}_{b}.
\end{aligned}
$$

Some terms in the above sum are zero. Indeed, we have

$$\sum_{\alpha=1}^{4}\sum_{d=1}^{4} L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{d}_{\alpha})\,\overset{\bullet}{H}{}^{\alpha}_{d}=\frac{1}{4}\sum_{\alpha=1}^{4}\sum_{d=1}^{4} L_{\boldsymbol{\Upsilon}_i}(H^{d}_{\alpha})\,(\delta^{\alpha}_{d}+H^{\alpha}_{d})=$$
$$=\frac{1}{4}\operatorname{tr}(L_{\boldsymbol{\Upsilon}_i}(\mathbf{H}))+\frac{1}{4}\operatorname{tr}(L_{\boldsymbol{\Upsilon}_i}(\mathbf{H})\circ\mathbf{H})=\frac{1}{4}L_{\boldsymbol{\Upsilon}_i}(\operatorname{tr}\mathbf{H})+\frac{1}{8}L_{\boldsymbol{\Upsilon}_i}(\operatorname{tr}\mathbf{H}^2).$$

But we know that $\operatorname{tr}\mathbf{H}=0$ and $\mathbf{H}^2=1$, which means $\operatorname{tr}\mathbf{H}^2=4$. Therefore, we have the following two relationships:

$$\begin{gathered}\sum_{\alpha=1}^{4}\sum_{d=1}^{4} L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{d}_{\alpha})\,\overset{\bullet}{H}{}^{\alpha}_{d}=0,\\ \sum_{\alpha=1}^{4}\sum_{d=1}^{4} L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{d}_{\alpha})\,\overset{\circ}{H}{}^{\alpha}_{d}=0.\end{gathered}\tag{2.16}$$

The second relationship (2.16) is derived in a quite similar way as the first one. Applying these relationships, we continue our previous calculations and obtain

$$\begin{gathered}\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4} L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{d}_{\alpha\beta})\,\overset{\bullet}{d}{}^{\beta\alpha}\,\overset{\circ}{H}{}^{a}_{b}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{d}_{\alpha\beta})\,\overset{\circ}{d}{}^{\beta\alpha}\,\overset{\bullet}{H}{}^{a}_{b}=\\ =\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{d=1}^{4}\Bigl(L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta d}\,\overset{\bullet}{H}{}^{\alpha}_{d}\,\overset{\circ}{H}{}^{a}_{b}+L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta d}\,\overset{\circ}{H}{}^{\alpha}_{d}\,\overset{\bullet}{H}{}^{a}_{b}\Bigr).\end{gathered}\tag{2.17}$$

Applying (2.16) to (2.15) we can simplify it either:

$$\begin{gathered}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\Bigl(L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet\circ}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{a}_{\alpha n}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ\bullet}{\gamma}{}^{\alpha}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{a}_{\alpha n}\Bigr)=\\ =\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4} L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm}\,g^{mn})\,\gamma^{a}_{\alpha n}+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\gamma^{\alpha}_{bm}\,\Gamma^{m}_{is}\,g^{sn}\,\gamma^{a}_{\alpha n}+\\ +\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\gamma^{\alpha}_{bm}\,\Gamma^{n}_{is}\,g^{ms}\,\gamma^{a}_{\alpha n}+2\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}d_{bc}\,\overset{\bullet}{H}{}^{c}_{\alpha}\,\times\\ \times\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\,\overset{\circ}{H}{}^{d}_{r}\,d^{ra}+2\sum_{\alpha=1}^{4}\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}d_{bc}\,\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\,\overset{\bullet}{H}{}^{d}_{r}\,d^{ra}.\end{gathered}\tag{2.18}$$

Moreover, note that $\overset{\bullet}{\mathbf{H}}$ and $\overset{\circ}{\mathbf{H}}$ are projectors, i. e. $\overset{\bullet}{\mathbf{H}}\circ\overset{\bullet}{\mathbf{H}}=\overset{\bullet}{\mathbf{H}}$ and $\overset{\circ}{\mathbf{H}}\circ\overset{\circ}{\mathbf{H}}=\overset{\circ}{\mathbf{H}}$. Differentiating these formulas, we derive the following identities:

$$\begin{gathered}\sum_{\alpha=1}^{4}\sum_{d=1}^{4}\overset{\bullet}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{d})\,\overset{\circ}{H}{}^{d}_{r}=\sum_{\alpha=1}^{4}\overset{\bullet}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{\alpha}_{r})=\sum_{d=1}^{4}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{c}_{d})\,\overset{\circ}{H}{}^{d}_{r},\\ \sum_{\alpha=1}^{4}\sum_{d=1}^{4}\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{d})\,\overset{\bullet}{H}{}^{d}_{r}=\sum_{\alpha=1}^{4}\overset{\circ}{H}{}^{c}_{\alpha}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{\alpha}_{r})=\sum_{d=1}^{4}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{c}_{d})\,\overset{\bullet}{H}{}^{d}_{r}.\end{gathered}\tag{2.19}$$

These formulas will be used below. Now we substitute (2.17), (2.18), and (2.12) into the formula (1.14). Meanwhile we apply (2.19) to (2.18). As a result we get

$$
\begin{gathered}
\mathrm{A}^a_{ib}=\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{d=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta d}\,\overset{\bullet}{H}{}^{\alpha}_{d}\,\overset{\circ}{H}{}^{a}_{b}+L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta d}\,\overset{\circ}{H}{}^{\alpha}_{d}\,\overset{\bullet}{H}{}^{a}_{b}}{4}\,+\\
+\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}\frac{d_{bc}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{c}_{d})\,\overset{\circ}{H}{}^{d}_{r}\,d^{ra}}{2}+\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}\frac{d_{bc}\,L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{c}_{d})\,\overset{\bullet}{H}{}^{d}_{r}\,d^{ra}}{2}\,+\\
+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm}\,g^{mn})\,\gamma^{a}_{\alpha n}}{4}+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\frac{\gamma^{\alpha}_{bm}\,\Gamma^{n}_{is}\,g^{ms}\,\gamma^{a}_{\alpha n}}{4}.
\end{gathered}\tag{2.20}
$$

Note that the last term in (2.20) coincides with (1.15), other terms contain derivatives $L_{\boldsymbol{\Upsilon}_i}$. Due to this observation we can formulate the following result.

Theorem 2.1. *The formulas* (1.14) *and* (1.15) *represent the same metric connection for Dirac spinors. The formula* (1.15) *applies to special frame where the components of the basic fields listed in the table* (1.1) *are constants. The formula* (1.14) *is a general formula applicable to all frames. It can be written as* (2.20).

3. Further transformations of the formula (1.14).

The third line in the formula (2.20) has no symbols with circle and bullet. However, previous two lines still have such symbols. We use the formulas (2.1) to remove bullets and circles from the formula (2.20) at all:

$$
\begin{gathered}
\overset{\bullet}{H}{}^{\alpha}_{d}\,\overset{\circ}{H}{}^{a}_{b}+\overset{\circ}{H}{}^{\alpha}_{d}\,\overset{\bullet}{H}{}^{a}_{b}=\frac{1}{2}\,(\delta^{\alpha}_{d}+H^{\alpha}_{d})\,\overset{\circ}{H}{}^{a}_{b}+\frac{1}{2}\,(\delta^{\alpha}_{d}-H^{\alpha}_{d})\,\overset{\bullet}{H}{}^{a}_{b}=\\
=\frac{1}{2}\,\delta^{\alpha}_{d}\,(\overset{\circ}{H}{}^{a}_{b}+\overset{\bullet}{H}{}^{a}_{b})+\frac{1}{2}\,H^{\alpha}_{d}\,(\overset{\circ}{H}{}^{a}_{b}-\overset{\bullet}{H}{}^{a}_{b})=\frac{1}{2}\,\delta^{\alpha}_{d}\,\delta^{a}_{b}-\frac{1}{2}\,H^{\alpha}_{d}\,H^{a}_{b},
\end{gathered}\tag{3.1}
$$

$$
\begin{gathered}
L_{\boldsymbol{\Upsilon}_i}(\overset{\bullet}{H}{}^{c}_{d})\,\overset{\circ}{H}{}^{d}_{r}+L_{\boldsymbol{\Upsilon}_i}(\overset{\circ}{H}{}^{c}_{d})\,\overset{\bullet}{H}{}^{d}_{r}=\frac{1}{2}\,L_{\boldsymbol{\Upsilon}_i}(\delta^{c}_{d}+H^{c}_{d})\,\overset{\circ}{H}{}^{d}_{r}\,+\\
+\frac{1}{2}\,L_{\boldsymbol{\Upsilon}_i}(\delta^{c}_{d}-H^{c}_{d})\,\overset{\bullet}{H}{}^{d}_{r}=\frac{1}{2}\,L_{\boldsymbol{\Upsilon}_i}(H^{c}_{d})\,(\overset{\circ}{H}{}^{d}_{r}-\overset{\bullet}{H}{}^{d}_{r})=-\frac{1}{2}\,L_{\boldsymbol{\Upsilon}_i}(H^{c}_{d})\,H^{d}_{r}.
\end{gathered}\tag{3.2}
$$

Applying (3.1) and (3.2) to (2.20), we obtain

$$
\begin{gathered}
\mathrm{A}^a_{ib}=\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta\alpha}}{8}\,\delta^{a}_{b}-\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{d=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta d}\,H^{\alpha}_{d}}{8}\,H^{a}_{b}\,-\\
-\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}\frac{d_{bc}\,L_{\boldsymbol{\Upsilon}_i}(H^{c}_{d})\,H^{d}_{r}\,d^{ra}}{4}+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm}\,g^{mn})}{4}\,\times\\
\times\,\gamma^{a}_{\alpha n}+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\frac{\gamma^{\alpha}_{bm}\,\Gamma^{n}_{is}\,g^{ms}\,\gamma^{a}_{\alpha n}}{4}.
\end{gathered}\tag{3.3}
$$

The theorem 1.1 says that the condition $\nabla\mathbf{H}=0$ follows from the conditions (1.3). Therefore, one can expect that the derivatives $L_{\boldsymbol{\Upsilon}_i}(H^{c}_{d})$ are expressed through the derivatives of the components of $\mathbf{d}$ and $\boldsymbol{\gamma}$. It is really so and the calculations of $L_{\boldsymbol{\Upsilon}_i}(H^c_d)$ are quite similar to those in proving the theorem 7.3 in [3]. These calculations are based on the formula (2.13). For the sake of brevity we omit them and give the ultimate result only. Here is the formula for $L_{\boldsymbol{\Upsilon}_i}(H^c_d)$:

$$
\begin{gathered}
L_{\boldsymbol{\Upsilon}_i}(H^c_d)=\sum_{\alpha=1}^{4}\sum_{r=1}^{4}\frac{H^c_r\,d^{r\alpha}\,L_{\boldsymbol{\Upsilon}_i}(d_{\alpha d})}{6}-\sum_{\beta=1}^{4}\sum_{s=1}^{4}\frac{d^{c\beta}\,L_{\boldsymbol{\Upsilon}_i}(d_{\beta s})\,H^s_d}{6}+\\
+\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(d^{c\alpha}\,d_{\beta d})\,H^\beta_\alpha}{6}-\sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\frac{L_{\boldsymbol{\Upsilon}_i}(\gamma^c_{\beta n}\,g^{mn}\,\gamma^\alpha_{dm})\,H^\beta_\alpha}{6}.
\end{gathered}
\tag{3.4}
$$

It is clear that substituting (3.4) into (3.3) will make this formula more huge and complicated. Therefore, we stop our transformations of the formula (1.14) at this point assuming that (3.3) is the most simple formula for the spinor components of the torsion-free metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$.

4. Special frames and curvature spin-tensors.

There are four types of special frames in the bundle of Dirac spinors DM. They are considered in [3]. A frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ of any one of these four types in DM is canonically associated with some definite frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ in the tangent bundle TM. The frame types association is given by the following diagram:

$$
\begin{array}{ccc}
\boxed{\begin{matrix}\text{Canonically orthonormal}\\ \text{chiral frames}\end{matrix}} & \longrightarrow & \boxed{\begin{matrix}\text{Positively polarized}\\ \text{right orthonormal frames}\end{matrix}}\\
\boxed{\begin{matrix}P\text{-reverse}\\ \text{anti-chiral frames}\end{matrix}} & \longrightarrow & \boxed{\begin{matrix}\text{Positively polarized}\\ \text{left orthonormal frames}\end{matrix}}\\
\boxed{\begin{matrix}T\text{-reverse}\\ \text{anti-chiral frames}\end{matrix}} & \longrightarrow & \boxed{\begin{matrix}\text{Negatively polarized}\\ \text{right orthonormal frames}\end{matrix}}\\
\boxed{\begin{matrix}PT\text{-reverse}\\ \text{chiral frames}\end{matrix}} & \longrightarrow & \boxed{\begin{matrix}\text{Negatively polarized}\\ \text{left orthonormal frames}\end{matrix}}
\end{array}
\tag{4.1}
$$

For the sake of certainty we choose some *canonically orthonormal chiral frame* $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ in DM. According to the diagram (4.1), it is associated with some *positively polarized right orthonormal frame* $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ in TM. Then $g_{ij}=g(\boldsymbol{\Upsilon}_i,\boldsymbol{\Upsilon}_j)$ and for the components of both metric tensors we have

$$
g_{ij}=g^{ij}=\begin{Vmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{Vmatrix}.
\tag{4.2}
$$

The formula (4.2) means that $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ is an orthonormal frame in TM. Moreover, it is a right frame regarding to the orientation in M. It is positively polarized, i. e. $\boldsymbol{\Upsilon}_0$ is a time-like unit vector directed to the future.

Canonically orthonormal chiral frames are simultaneously orthonormal, chiral, and self-adjoint frames in DM. The orthonormality of our frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$, $\mathbf{\Psi}_3$, $\mathbf{\Psi}_4$ means that the components of the metric tensor $\mathbf{d}$ are given by the following matrix:

$$d_{ij} = d(\mathbf{\Psi}_i, \mathbf{\Psi}_j) = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}. \tag{4.3}$$

Chiral frames in the bundle of Dirac spinors DM are those for which the chirality operator $\mathbf{H}$ is given by the matrix

$$H^i_j = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{4.4}$$

And finally, self-adjoint frames in the bundle of Dirac spinors DM are those for which the Dirac form $\mathbf{D}$ is given by the matrix

$$D_{i\bar{j}} = D(\mathbf{\Psi}_{\bar{j}}, \mathbf{\Psi}_i) = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}. \tag{4.5}$$

Our choice is a canonically orthonormal chiral frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$, $\mathbf{\Psi}_3$, $\mathbf{\Psi}_4$ in DM and its associated positively polarized right orthonormal frame $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$ in TM. Therefore, in our case the conditions (4.2), (4.3), (4.4), and (4.5) are fulfilled simultaneously. The components of Dirac's γ-field are uniquely fixed by our choice of frames. They are usually collected into four matrices:

$$\begin{aligned} \gamma_0 &= \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}, & \gamma_1 &= \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{Vmatrix}, \\ \gamma_2 &= \begin{Vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{Vmatrix}, & \gamma_3 &= \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}. \end{aligned} \tag{4.6}$$

The matrices (4.6) are enumerated by the spacial index k of γ^a_{bk}. Other two indices a and b represent the position of an element within the matrix γ_k, the index a being the row number and the index b being the column number.

Looking at the formulas (4.2), (4.3), (4.4), (4.5), and (4.6), we see that the components of all basic fields $\mathbf{g}$, $\mathbf{d}$, $\mathbf{H}$, $\mathbf{D}$, and $\boldsymbol{\gamma}$ are constants. It means that our choice of frames is that very case where the formula (1.15) is applicable and where the formula (1.14) reduces to (1.15). This choice of frames is convenient for calculating the curvature tensors. The first of them is the Riemann curvature tensor $\mathbf{R}$. Its components are given by the formula:

$$R^p_{qij} = L_{\mathbf{\Upsilon}_i}(\Gamma^p_{j\,q}) - L_{\mathbf{\Upsilon}_j}(\Gamma^p_{i\,q}) + \sum_{h=0}^{3} \left(\Gamma^p_{i\,h}\,\Gamma^h_{j\,q} - \Gamma^p_{j\,h}\,\Gamma^h_{i\,q}\right) - \sum_{k=0}^{3} c^k_{ij}\,\Gamma^p_{kq} \tag{4.7}$$

(see (6.27) in [5]). Here Γ^k_{ij} are the spacial components of the metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$. They are given by the formula (1.12). Apart from (4.7), there are two other curvature tensors $\mathfrak{R}$ and $\bar{\mathfrak{R}}$. Their components are given by the formulas

$$\mathfrak{R}^p_{qij} = L_{\boldsymbol{\Upsilon}_i}(\mathrm{A}^p_{j\,q}) - L_{\boldsymbol{\Upsilon}_j}(\mathrm{A}^p_{i\,q}) + \sum_{h=1}^{4}\left(\mathrm{A}^p_{i\,h}\ \mathrm{A}^h_{j\,q} - \mathrm{A}^p_{j\,h}\ \mathrm{A}^h_{i\,q}\right) - \sum_{k=0}^{3} c^k_{ij}\ \mathrm{A}^p_{kq}, \tag{4.8}$$

$$\bar{\mathfrak{R}}^p_{qij} = L_{\boldsymbol{\Upsilon}_i}(\bar{\mathrm{A}}^p_{j\,q}) - L_{\boldsymbol{\Upsilon}_j}(\bar{\mathrm{A}}^p_{i\,q}) + \sum_{h=1}^{4}\left(\bar{\mathrm{A}}^p_{i\,h}\ \bar{\mathrm{A}}^h_{j\,q} - \bar{\mathrm{A}}^p_{j\,h}\ \bar{\mathrm{A}}^h_{i\,q}\right) - \sum_{k=0}^{3} c^k_{ij}\ \bar{\mathrm{A}}^p_{kq} \tag{4.9}$$

(compare with (6.25) and (6.26) in [5]). Applying (1.13) to (4.8) and (4.9) and taking into account that Γ^k_{ij} and c^k_{ij} are purely real functions, we get

$$\bar{\mathfrak{R}}^p_{qij} = \overline{\mathfrak{R}^p_{qij}}. \tag{4.10}$$

In a coordinate-free form the relationship (4.10) is written as

$$\bar{\mathfrak{R}} = \tau(\mathfrak{R}), \tag{4.11}$$

while the Riemann curvature tensor $\mathbf{R}$ is a purely real field, i. e.

$$\tau(\mathbf{R}) = \mathbf{R}. \tag{4.12}$$

Keeping in mind that we deal with the special frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ and $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$, let's apply the formula (1.15) to (4.8). As a result we get

$$\begin{aligned}
\mathfrak{R}^p_{qij} = \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(\Gamma^r_{jm}) - L_{\boldsymbol{\Upsilon}_j}(\Gamma^r_{im})}{4}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{\alpha r}+\\
+\sum_{h=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\mathrm{A}^p_{i\,h}\ \Gamma^r_{jm}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^h_{\alpha r}}{4}\,-\\
-\sum_{h=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\mathrm{A}^h_{i\,q}\ \Gamma^r_{jm}\,\gamma^\alpha_{hn}\,g^{mn}\,\gamma^p_{\alpha r}}{4}\,-\\
-\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{c^k_{ij}}{4}\,\Gamma^r_{km}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{\alpha r}.
\end{aligned} \tag{4.13}$$

Since $\gamma^p_{\alpha r} = \text{const}$, the concordance condition $\nabla\boldsymbol{\gamma} = 0$ from (1.3) is written as

$$\nabla_i\,\gamma^p_{\alpha r} = \sum_{h=1}^{4}\mathrm{A}^p_{i\,h}\ \gamma^h_{\alpha r} - \sum_{h=1}^{4}\mathrm{A}^h_{i\,\alpha}\ \gamma^p_{hr} - \sum_{s=0}^{3}\Gamma^s_{ir}\,\gamma^p_{\alpha s} = 0.$$

From this identity we derive

$$\sum_{h=1}^{4}\mathrm{A}^p_{i\,h}\ \gamma^h_{\alpha r} = \sum_{h=1}^{4}\mathrm{A}^h_{i\,\alpha}\ \gamma^p_{hr} + \sum_{s=0}^{3}\Gamma^s_{ir}\,\gamma^p_{\alpha s}. \tag{4.14}$$

Applying (4.14) to the second term in the right hand side of (4.13), we write it as

$$\begin{aligned}
\mathfrak{R}^p_{qij} &= \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(\Gamma^r_{jm})-L_{\boldsymbol{\Upsilon}_j}(\Gamma^r_{im})}{4}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{\alpha r}+\\
&+\sum_{h=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\mathrm{A}^h_{i\,\alpha}\,\Gamma^r_{jm}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{hr}}{4}+\\
&+\sum_{s=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\Gamma^s_{ir}\,\Gamma^r_{jm}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{\alpha s}}{4}+\\
&-\sum_{h=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\mathrm{A}^h_{i\,q}\,\Gamma^r_{jm}\,\gamma^\alpha_{hn}\,g^{mn}\,\gamma^p_{\alpha r}}{4}-\\
&-\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{c^k_{ij}}{4}\,\Gamma^r_{km}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{\alpha r}.
\end{aligned}\tag{4.15}$$

Now we use the following formula equivalent to (4.14):

$$\sum_{\alpha=1}^{4}\mathrm{A}^h_{i\,\alpha}\ \gamma^\alpha_{qn}=\sum_{\alpha=1}^{4}\mathrm{A}^\alpha_{i\,q}\ \gamma^h_{\alpha n}+\sum_{s=0}^{3}\Gamma^s_{in}\,\gamma^h_{qs}.\tag{4.16}$$

Applying (4.16) to the second term in the right hand side of (4.15), we write it as

$$\begin{aligned}
\mathfrak{R}^p_{qij} &= \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(\Gamma^r_{jm})-L_{\boldsymbol{\Upsilon}_j}(\Gamma^r_{im})}{4}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{\alpha r}+\\
&+\sum_{h=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\mathrm{A}^\alpha_{i\,q}\,\Gamma^r_{jm}\,\gamma^h_{\alpha n}\,g^{mn}\,\gamma^p_{hr}}{4}+\\
&+\sum_{h=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\Gamma^s_{in}\,\Gamma^r_{jm}\,\gamma^h_{qs}\,g^{mn}\,\gamma^p_{hr}}{4}+\\
&+\sum_{s=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\Gamma^s_{ir}\,\Gamma^r_{jm}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{\alpha s}}{4}+\\
&-\sum_{h=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{\mathrm{A}^h_{i\,q}\,\Gamma^r_{jm}\,\gamma^\alpha_{hn}\,g^{mn}\,\gamma^p_{\alpha r}}{4}-\\
&-\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\frac{c^k_{ij}}{4}\,\Gamma^r_{km}\,\gamma^\alpha_{qn}\,g^{mn}\,\gamma^p_{\alpha r}.
\end{aligned}\tag{4.17}$$

By means of the formal exchanges of summation indices $h \leftrightarrow \alpha$ we find that the second and the fifth terms in the right hand side of (4.17) do cancel each other. In order to transform the third term in the right hand side of (4.17) we use the concordance condition $\nabla\mathbf{g}=0$ from (1.10). Since $g^{ms}=\text{const}$, this condition yields

$$\nabla_i g^{ms}=\sum_{n=0}^{3}\Gamma^m_{in}\,g^{ns}+\sum_{n=0}^{3}\Gamma^s_{in}\,g^{mn}=0.$$

This formula can be rewritten in the following way:

$$\sum_{n=0}^{3} \Gamma^{s}_{in}\, g^{mn} = -\sum_{n=0}^{3} \Gamma^{m}_{in}\, g^{ns}. \tag{4.18}$$

Applying the formula (4.18) to the third term in the right hand side of (4.17) and canceling the second and the fifth terms there, we get

$$\begin{aligned}\mathfrak{R}^{p}_{qij} = &\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} \frac{L_{\boldsymbol{\Upsilon}_i}(\Gamma^{r}_{jm}) - L_{\boldsymbol{\Upsilon}_j}(\Gamma^{r}_{im})}{4}\,\gamma^{\alpha}_{qn}\,g^{mn}\,\gamma^{p}_{\alpha r} -\\ &-\sum_{h=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} \frac{\Gamma^{m}_{in}\,\Gamma^{r}_{jm}\,\gamma^{h}_{qs}\,g^{ns}\,\gamma^{p}_{hr}}{4} +\\ &+\sum_{s=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} \frac{\Gamma^{s}_{ir}\,\Gamma^{r}_{jm}\,\gamma^{\alpha}_{qn}\,g^{mn}\,\gamma^{p}_{\alpha s}}{4} -\\ &-\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} \frac{c^{k}_{ij}}{4}\,\Gamma^{r}_{km}\,\gamma^{\alpha}_{qn}\,g^{mn}\,\gamma^{p}_{\alpha r}.\end{aligned} \tag{4.19}$$

Upon the formal change of summation indices $s \to n \to m \to h \to \alpha$ in the second term and $s \to r \to h$ in the third term respectively we write (4.19) as

$$\begin{aligned}\mathfrak{R}^{p}_{qij} = &\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} \frac{L_{\boldsymbol{\Upsilon}_i}(\Gamma^{r}_{jm}) - L_{\boldsymbol{\Upsilon}_j}(\Gamma^{r}_{im})}{4}\,\gamma^{\alpha}_{qn}\,g^{mn}\,\gamma^{p}_{\alpha r} -\\ &-\sum_{h=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} \frac{\Gamma^{r}_{jh}\,\Gamma^{h}_{im}\,\gamma^{\alpha}_{qn}\,g^{mn}\,\gamma^{p}_{\alpha r}}{4} +\\ &+\sum_{h=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} \frac{\Gamma^{r}_{ih}\,\Gamma^{h}_{jm}\,\gamma^{\alpha}_{qn}\,g^{mn}\,\gamma^{p}_{\alpha r}}{4} -\\ &-\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} \frac{c^{k}_{ij}}{4}\,\Gamma^{r}_{km}\,\gamma^{\alpha}_{qn}\,g^{mn}\,\gamma^{p}_{\alpha r}.\end{aligned} \tag{4.20}$$

And finally, here is the last transformation that brings the formula (4.20) to

$$\begin{aligned}\mathfrak{R}^{p}_{qij} = \frac{1}{4}&\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4}\Bigg(L_{\boldsymbol{\Upsilon}_i}(\Gamma^{r}_{jm}) - L_{\boldsymbol{\Upsilon}_j}(\Gamma^{r}_{im})\ +\\ &+\sum_{h=0}^{3}\left(\Gamma^{r}_{ih}\,\Gamma^{h}_{jm} - \Gamma^{r}_{jh}\,\Gamma^{h}_{im}\right) - \sum_{k=0}^{3}\frac{c^{k}_{ij}}{4}\,\Gamma^{r}_{km}\Bigg)\,\gamma^{\alpha}_{qn}\ g^{mn}\,\gamma^{p}_{\alpha r}.\end{aligned} \tag{4.21}$$

Comparing (4.21) with the formula (4.7), we can write the following ultimate result:

$$\mathfrak{R}^{p}_{qij} = \frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{\alpha=1}^{4} R^{r}_{mij}\,\gamma^{\alpha}_{qn}\ g^{mn}\,\gamma^{p}_{\alpha r}. \tag{4.22}$$

Note that the formula (4.22) is quite similar to (1.15). However, like (4.11) and (4.12) and unlike (1.15), it is a tensorial formula. For this reason, being proved for some special pair of frames, it remains valid for an arbitrary pair of frames.

Theorem 4.1. *The spinor curvature tensor $\mathfrak{R}$ of the torsion-free metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ in the bundle of Dirac spinors DM is related to the corresponding Riemann curvature tensor $\mathbf{R}$ by means of the formula* (4.22)*. This formula is valid for any two frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ and $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ no matter holonomic or non-holonomic, special or not special, and, if special, no matter being in frame association* (4.1) *or not.*

5. Conclusions.

The main result of this paper is that the formulas (1.14) and (1.15) represent the spinor components of the same metric connection in the bundle of Dirac spinors. The formula (1.14) is a general formula, while (1.15) is its specialization. The formula (1.15) is an important specialization since, for instance, it is convenient for proving the formula (4.22).

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