ON THE SPINOR STRUCTURE OF THE HOMOGENEOUS
AND ISOTROPIC UNIVERSE IN CLOSED MODEL.

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ABSTRACT. The closed homogeneous and isotropic universe is considered. The bundles of Weyl and Dirac spinors for this universe are explicitly described. Some explicit formulas for the basic fields and for the connection components in stereographic and in spherical coordinates are presented.

1. STEREOGRAPHIC PROJECTIONS OF THE SPHERE S^3.

It is known that the closed homogeneous and isotropic universe is described by a manifold diffeomorphic to the Cartesian product of the three-dimensional sphere S^3 by a straight line: $M = \mathbb{R} \times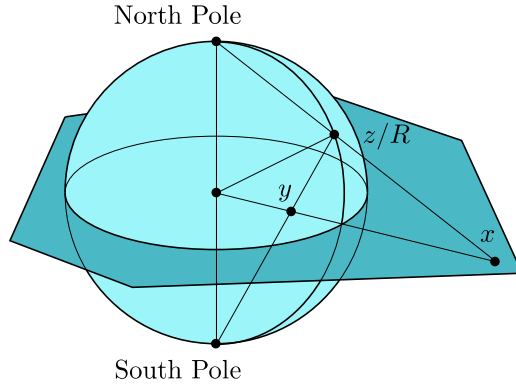 S^3$ (see §111 and §112 in [1]). The sphere S^3 is a manifold that can be covered with two local charts. We choose stereographic projections from two diametrically opposite points (we call them North and South poles) to their equatorial hyperplane. The sphere S^3 is naturally presented as a three-dimensional hypersurface in the four-dimensional space $\mathbb{R}^4$ with the standard Euclidean metric. Let z be a point on such a sphere with the radius R. Then z/R is a point on the unit sphere. Let x and y be the stereographic projections of z/R. Then the points x and y lie on some ray in the equatorial hyperplane coming out from the center of the sphere. It is rather easy to derive the following formulas relating their coordinates:

North Pole

South Pole

Fig. 1.1

$$\left\|\begin{matrix} y^1 \\ y^2 \\ y^3 \end{matrix}\right\| = \frac{1}{|x|^2} \left\|\begin{matrix} x^1 \\ x^2 \\ x^3 \end{matrix}\right\|, \qquad\qquad \left\|\begin{matrix} x^1 \\ x^2 \\ x^3 \end{matrix}\right\| = \frac{1}{|y|^2} \left\|\begin{matrix} y^1 \\ y^2 \\ y^3 \end{matrix}\right\|. \qquad (1.1)$$

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Here $|x|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and $|y|^2 = (y^1)^2 + (y^2)^2 + (y^3)^2$. As for the point z on the sphere, its coordinates can also be expressed through the coordinates of x and y. Here are the formulas for the coordinates of the point z:

$$
\begin{Vmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \end{Vmatrix} = \frac{2\,R}{|x|^2 + 1} \begin{Vmatrix} x^1 \\ x^2 \\ x^3 \\ 0 \end{Vmatrix} + \frac{|x|^2 - 1}{|x|^2 + 1} \begin{Vmatrix} 0 \\ 0 \\ 0 \\ R \end{Vmatrix}, \tag{1.2}
$$

$$
\begin{Vmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \end{Vmatrix} = \frac{2\,R}{|y|^2 + 1} \begin{Vmatrix} y^1 \\ y^2 \\ y^3 \\ 0 \end{Vmatrix} - \frac{|y|^2 - 1}{|y|^2 + 1} \begin{Vmatrix} 0 \\ 0 \\ 0 \\ R \end{Vmatrix}. \tag{1.3}
$$

The standard Euclidean metric in $\mathbb{R}^4$ is given by the formula

$$
ds^2 = (dz^1)^2 + (dz^2)^2 + (dz^3)^2 + (dz^4)^2. \tag{1.4}
$$

Differentiating (1.2) and (1.3) and substituting them into (1.4), we derive the formulas for the induced metric in the local coordinates x^1, x^2, x^3 and y^1, y^2, y^3:

$$
ds^2 = \frac{4\,R^2\,(dx^1)^2 + 4\,R^2\,(dx^2)^2 + 4\,R^2\,(dx^3)^2}{\left(|x|^2 + 1\right)^2}, \tag{1.5}
$$

$$
ds^2 = \frac{4\,R^2\,(dy^1)^2 + 4\,R^2\,(dy^2)^2 + 4\,R^2\,(dy^3)^2}{\left(|y|^2 + 1\right)^2}. \tag{1.6}
$$

Passing from $\mathbb{R}^4$ to the space-time manifold $M = \mathbb{R} \times S^3$, we add new coordinates x^0 and y^0 to the initial coordinates x^1, x^2, x^3 and y^1, y^2, y^3. The transition functions relating x^1, x^2, x^3 and y^1, y^2, y^3 are determined by the formulas (1.1). For the newly introduced coordinates we set

$$
x^0 = y^0. \tag{1.7}
$$

Using the local coordinates x^0, x^1, x^2, x^3 and y^0, y^1, y^2, y^3 and relying upon (1.5) and (1.6), we introduce the Minkowski type metric $\mathbf{g}$ to our model M of the universe:

$$
ds^2 = R^2\,(dx^0)^2 - \frac{4\,R^2\,(dx^1)^2 + 4\,R^2\,(dx^2)^2 + 4\,R^2\,(dx^3)^2}{\left(|x|^2 + 1\right)^2}, \tag{1.8}
$$

$$
ds^2 = R^2\,(dy^0)^2 - \frac{4\,R^2\,(dy^1)^2 + 4\,R^2\,(dy^2)^2 + 4\,R^2\,(dy^3)^2}{\left(|y|^2 + 1\right)^2}. \tag{1.9}
$$

From now on the parameter R in formulas is not a constant. We shall assume it to be a function of the newly introduced coordinates (1.7):

$$
R = R(x^0) = R(y^0). \tag{1.10}
$$

The parameter R in (1.10) is interpreted as the radius of the sphere S^3 in its Euclidean realization as a hypersurface in $\mathbb{R}^4$. This parameter is the only parameter

describing the evolution of the homogeneous and isotropic universe in closed model. One can introduce the time variable t through the following formula:

$$R\,dx^0 = R\,dy^0 = c\,dt \qquad (c \text{ is the light velocity}).$$

Then we can write (1.10) as $R = R(t)$. Depending on the function $R(t)$ we say: the universe is stable, the universe is expanding, or the universe is contracting. Oscillatory regimes are also admissible. Unlike the Newtonian mechanics, t is not an absolute time in the universe, but the most preferable time variable due to the symmetry of the model.

The signature of the metric $\mathbf{g}$ is $(+, -, -, -)$. Looking at (1.8) and (1.9), we see that the metric tensor $\mathbf{g}$ is diagonal in the stereographic projection charts. This means that the coordinate frames of these two charts

$$\frac{\partial}{\partial x^0}, \qquad\qquad \frac{\partial}{\partial x^1}, \qquad\qquad \frac{\partial}{\partial x^2}, \qquad\qquad \frac{\partial}{\partial x^3}, \qquad (1.11)$$

$$\frac{\partial}{\partial y^0}, \qquad\qquad \frac{\partial}{\partial y^1}, \qquad\qquad \frac{\partial}{\partial y^2}, \qquad\qquad \frac{\partial}{\partial y^3} \qquad (1.12)$$

are orthogonal frames. However, they are not orthonormal frames. We normalize them introducing the following two orthonormal frames:

$$
\begin{aligned}
\mathbf{X}_0 &= \frac{1}{R}\frac{\partial}{\partial x^0}, & \mathbf{X}_1 &= \frac{1+|x|^2}{2\,R}\frac{\partial}{\partial x^1}, \\
\mathbf{X}_2 &= \frac{1+|x|^2}{2\,R}\frac{\partial}{\partial x^2}, & \mathbf{X}_3 &= \frac{1+|x|^2}{2\,R}\frac{\partial}{\partial x^3},
\end{aligned}
\qquad (1.13)
$$

$$
\begin{aligned}
\mathbf{Y}_0 &= \frac{1}{R}\frac{\partial}{\partial y^0}, & \mathbf{Y}_1 &= \frac{1+|y|^2}{2\,R}\frac{\partial}{\partial y^1}, \\
\mathbf{Y}_2 &= \frac{1+|y|^2}{2\,R}\frac{\partial}{\partial y^2}, & \mathbf{Y}_3 &= \frac{1+|y|^2}{2\,R}\frac{\partial}{\partial y^3}.
\end{aligned}
\qquad (1.14)
$$

The frames (1.13) and (1.14) are orthonormal, i.e. the metric tensor $\mathbf{g}$ and its dual metric tensor are given by the standard Minkowski matrix

$$g_{ij} = g^{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix} \qquad (1.15)$$

in both of these two frames. But unlike the frames (1.11) and (1.12), these two frames are non-holonomic. The vector fields $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{Y}_0$, $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ composing these frames do not commute with each other:

$$[\mathbf{X}_i, \mathbf{X}_j] = \sum_{k=0}^{3} c_{ij}^{k}\,\mathbf{X}_k. \qquad (1.16)$$

Using (1.13), one can easily find the explicit formulas for the commutation coefficients c_{ij}^k in (1.16). Most of these coefficients are zero. Below is the list of those coefficients c_{ij}^k which are nonzero:

$$c_{01}^1 = -c_{10}^1 = c_{02}^2 = -c_{20}^2 = c_{03}^3 = -c_{30}^3 = -\frac{R'}{R^2},$$

$$c_{12}^1 = -c_{21}^1 = -\frac{(x^2)}{R^2}, \qquad\qquad c_{12}^2 = -c_{21}^2 = \frac{(x^1)}{R^2},$$

$$c_{13}^1 = -c_{31}^1 = -\frac{(x^3)}{R^2}, \qquad\qquad c_{13}^3 = -c_{31}^3 = \frac{(x^1)}{R^2}, \qquad (1.17)$$

$$c_{23}^2 = -c_{32}^2 = -\frac{(x^3)}{R^2}, \qquad\qquad c_{23}^3 = -c_{32}^3 = \frac{(x^2)}{R^2}.$$

Here R' is the derivative of the function (1.10). The vector fields of the second frame $\mathbf{Y}_0$, $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ obey the commutation relationships similar to (1.16):

$$[\mathbf{Y}_i, \mathbf{Y}_j] = \sum_{k=0}^{3} c_{ij}^k \, \mathbf{Y}_k. \qquad (1.18)$$

Below is the list of all nonzero commutation coefficients c_{ij}^k for (1.18):

$$c_{01}^1 = -c_{10}^1 = c_{02}^2 = -c_{20}^2 = c_{03}^3 = -c_{30}^3 = -\frac{R'}{R^2},$$

$$c_{12}^1 = -c_{21}^1 = -\frac{(y^2)}{R^2}, \qquad\qquad c_{12}^2 = -c_{21}^2 = \frac{(y^1)}{R^2},$$

$$c_{13}^1 = -c_{31}^1 = -\frac{(y^3)}{R^2}, \qquad\qquad c_{13}^3 = -c_{31}^3 = \frac{(y^1)}{R^2}, \qquad (1.19)$$

$$c_{23}^2 = -c_{32}^2 = -\frac{(y^3)}{R^2}, \qquad\qquad c_{23}^3 = -c_{32}^3 = \frac{(y^2)}{R^2}.$$

The formula (1.1) complemented with the formula (1.7) determines the transition functions for two overlapping local charts x^0, x^1, x^2, x^3 and y^0, y^1, y^2, y^3. Differentiating these transition functions, we get the transition matrices relating the holonomic frames (1.11) and (1.12):

$$\frac{\partial}{\partial x^i} = \sum_{j=0}^{3} \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}, \qquad\qquad \frac{\partial}{\partial y^i} = \sum_{j=0}^{3} \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}. \qquad (1.20)$$

Then, applying (1.13) and (1.14) to (1.20), we derive the formulas relating the non-holonomic frames $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{Y}_0$, $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$:

$$\mathbf{X}_i = \sum_{j=0}^{3} T_i^j \, \mathbf{Y}_j, \qquad\qquad \mathbf{Y}_i = \sum_{j=0}^{3} S_i^j \, \mathbf{X}_j. \qquad (1.21)$$

Here is the explicit formula for the matrix S in (1.21):

$$
S = \left\|
\begin{array}{cccc}
1 & 0 & 0 & 0 \\[2mm]
0 & \dfrac{|y|^2 - 2\,(y^1)^2}{|y|^2} & \dfrac{-2\,(y^1)\,(y^2)}{|y|^2} & \dfrac{-2\,(y^1)\,(y^3)}{|y|^2} \\[4mm]
0 & \dfrac{-2\,(y^1)\,(y^2)}{|y|^2} & \dfrac{|y|^2 - 2\,(y^2)^2}{|y|^2} & \dfrac{-2\,(y^2)\,(y^3)}{|y|^2} \\[4mm]
0 & \dfrac{-2\,(y^1)\,(y^3)}{|y|^2} & \dfrac{-2\,(y^2)\,(y^3)}{|y|^2} & \dfrac{|y|^2 - 2\,(y^3)^2}{|y|^2}
\end{array}
\right\| .
\tag{1.22}
$$

The matrix T is the inverse matrix for S, i.e. $T = S^{-1}$. Its components can be expressed in terms of the coordinates x^0, x^1, x^2, x^3:

$$
T = \left\|
\begin{array}{cccc}
1 & 0 & 0 & 0 \\[2mm]
0 & \dfrac{|x|^2 - 2\,(x^1)^2}{|x|^2} & \dfrac{-2\,(x^1)\,(x^2)}{|x|^2} & \dfrac{-2\,(x^1)\,(x^3)}{|x|^2} \\[4mm]
0 & \dfrac{-2\,(x^1)\,(x^2)}{|x|^2} & \dfrac{|x|^2 - 2\,(x^2)^2}{|x|^2} & \dfrac{-2\,(x^2)\,(x^3)}{|x|^2} \\[4mm]
0 & \dfrac{-2\,(x^1)\,(x^3)}{|x|^2} & \dfrac{-2\,(x^2)\,(x^3)}{|x|^2} & \dfrac{|x|^2 - 2\,(x^3)^2}{|x|^2}
\end{array}
\right\| .
\tag{1.23}
$$

In general case the matrices S and T are different. However, it is the feature of our particular charts that the matrices (1.22) and (1.23) do coincide:

$$
S = T, \qquad\qquad\qquad S^2 = T^2 = 1.
\tag{1.24}
$$

The frames (1.13) and (1.14) are two orthonormal frames of the spherical universe $M = \mathbb{R} \times S^3$. The vector $\mathbf{X}_0 = \mathbf{Y}_0$ is a time-like vector directed to the future. Therefore the frames (1.13) and (1.14) are called *positively polarized* (see definition in §5 of [2]). However, their orientations are different. Indeed, by means of direct calculations for S and T we find that

$$
\det S = \det T = -1.
\tag{1.25}
$$

The formula (1.25) is concordant with (1.24). It means that if we take (1.13) for a right oriented frame in M, then (1.14) is a left oriented frame. In the theory of Weyl spinors only positively polarized right orthonormal frames are admissible (see definition 5.2 in [2]). For this reason we introduce the following auxiliary frame:

$$
\tilde{\mathbf{Y}}_0 = \mathbf{Y}_0, \qquad \tilde{\mathbf{Y}}_1 = -\mathbf{Y}_1, \qquad \tilde{\mathbf{Y}}_2 = -\mathbf{Y}_2, \qquad \tilde{\mathbf{Y}}_3 = -\mathbf{Y}_3.
\tag{1.26}
$$

Like $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, the frame (1.26) is a positively polarized right orthonormal frame in M. It is related to $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ as follows:

$$\mathbf{X}_i = \sum_{j=0}^{3} \tilde{T}_i^j \, \tilde{\mathbf{Y}}_j, \qquad\qquad \tilde{\mathbf{Y}}_i = \sum_{j=0}^{3} \tilde{S}_i^j \, \mathbf{X}_j. \tag{1.27}$$

Here are the explicit formulas for the matrices $\tilde{S}$ and $\tilde{T}$ in (1.27):

$$\tilde{S} = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\[6pt] 0 & \dfrac{2\,(y^1)^2 - |y|^2}{|y|^2} & \dfrac{2\,(y^1)\,(y^2)}{|y|^2} & \dfrac{2\,(y^1)\,(y^3)}{|y|^2} \\[10pt] 0 & \dfrac{2\,(y^1)\,(y^2)}{|y|^2} & \dfrac{2\,(y^2)^2 - |y|^2}{|y|^2} & \dfrac{2\,(y^2)\,(y^3)}{|y|^2} \\[10pt] 0 & \dfrac{2\,(y^1)\,(y^3)}{|y|^2} & \dfrac{2\,(y^2)\,(y^3)}{|y|^2} & \dfrac{2\,(y^3)^2 - |y|^2}{|y|^2} \end{array} \right\|, \tag{1.28}$$

$$\tilde{T} = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\[6pt] 0 & \dfrac{|x|^2 - 2\,(x^1)^2}{|x|^2} & \dfrac{-2\,(x^1)\,(x^2)}{|x|^2} & \dfrac{-2\,(x^1)\,(x^3)}{|x|^2} \\[10pt] 0 & \dfrac{-2\,(x^1)\,(x^2)}{|x|^2} & \dfrac{|x|^2 - 2\,(x^2)^2}{|x|^2} & \dfrac{-2\,(x^2)\,(x^3)}{|x|^2} \\[10pt] 0 & \dfrac{-2\,(x^1)\,(x^3)}{|x|^2} & \dfrac{-2\,(x^2)\,(x^3)}{|x|^2} & \dfrac{|x|^2 - 2\,(x^3)^2}{|x|^2} \end{array} \right\|. \tag{1.29}$$

Like in (1.24), for the matrices (1.28) and (1.29) we have the relationships

$$\tilde{S} = \tilde{T}, \qquad\qquad \tilde{S}^2 = \tilde{T}^2 = 1. \tag{1.30}$$

However, instead of (1.25), their determinants now are equal to the unity:

$$\det \tilde{S} = \det \tilde{T} = 1. \tag{1.31}$$

Being transition matrices that relate two positively polarized right orthonormal frames, the matrices $\tilde{S}$ and $\det \tilde{T}$ belong to the special orthochronous Lorentz group $\text{SO}^+(1, 3, \mathbb{R})$. Our next step is to construct the bundle SM of Weyl spinors for the universe $M = \mathbb{R} \times S^3$. We use the frames $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\tilde{\mathbf{Y}}_0$, $\tilde{\mathbf{Y}}_1$, $\tilde{\mathbf{Y}}_2$, $\tilde{\mathbf{Y}}_3$ and their transition matrices $\tilde{S}$ and $\tilde{T}$ for this purpose.

2. Constructing the bundle of Weyl spinors.

The bundle of Weyl spinors SM is a two-dimensional complex vector-bundle related in some special way to the tangent bundle TM (see definition 5.2 in [2]).

According to this definition, each positively polarized right orthonormal frame of the tangent bundle TM should be associated with some frame of the spinor bundle in such a way that the transition matrices of the associated spinor frames would belong to the group $\mathrm{SL}(2,\mathbb{C})$ and would be linked to the transition matrices of the tangent frames by means of the group homomorphism

$$\phi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3,\mathbb{R}). \tag{2.1}$$

In our particular case we have two positively polarized right orthonormal frames (1.13) and (1.26) with the transition matrix (1.28) relating them. In order to prove the existence of the spinor bundle SM in the case of the spherical universe $M = \mathbb{R} \times S^3$ we need to find a matrix $\tilde{\mathfrak{S}} \in \mathrm{SL}(2,\mathbb{C})$ such that $\phi(\tilde{\mathfrak{S}}) = \tilde{S}$. The components of the matrix $\tilde{\mathfrak{S}}$ should be smooth functions in the intersection of the domains of two frames (1.13) and (1.26), i.e. they should be smooth functions on the whole sphere S^3 except for the poles. As for the homomorphism (2.1), this homomorphism is given by the explicit formulas (1.2), (1.3), (1.4), and (1.5) in paper [3]. Here are these explicit formulas expressing $\tilde{S} = \phi(\tilde{\mathfrak{S}})$ through $\tilde{\mathfrak{S}}$:

$$
\begin{aligned}
\tilde{S}^0_0 &= \frac{\overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^2_1 + \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^2_2}{2}, \\[4pt]
\tilde{S}^0_1 &= \frac{\overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^2_2 + \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^2_1}{2}, \\[4pt]
\tilde{S}^0_2 &= \frac{\overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^1_1 - \overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^2_2}{2\,i}, \\[4pt]
\tilde{S}^0_3 &= \frac{\overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^1_1 - \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^2_2}{2},
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
\tilde{S}^1_0 &= \frac{\overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^2_1 + \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^2_2}{2}, \\[4pt]
\tilde{S}^1_1 &= \frac{\overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^2_1 + \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^2_2}{2}, \\[4pt]
\tilde{S}^1_2 &= \frac{\overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^1_1 - \overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^2_2}{2\,i}, \\[4pt]
\tilde{S}^1_3 &= \frac{\overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^1_2 - \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^2_2}{2},
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
\tilde{S}^2_0 &= \frac{\overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^2_2 - \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^1_2}{2\,i}, \\[4pt]
\tilde{S}^2_1 &= \frac{\overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^2_2 - \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^1_1}{2\,i}, \\[4pt]
\tilde{S}^2_2 &= \frac{\overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^2_2 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^1_2 - \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^2_1}{2}, \\[4pt]
\tilde{S}^2_3 &= \frac{\overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^1_2 - \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^2_2}{2\,i},
\end{aligned}
\tag{2.4}
$$

$$\tilde{S}^3_0 = \frac{\overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^1_2 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^2_2}{2},$$

$$\tilde{S}^3_1 = \frac{\overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^1_1 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^2_2 - \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^2_1}{2},$$

$$\tilde{S}^3_2 = \frac{\overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^1_1 - \overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^1_2 + \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^2_2 - \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^2_1}{2\,i},$$

$$\tilde{S}^3_3 = \frac{\overline{\tilde{\mathfrak{S}}^1_1}\,\tilde{\mathfrak{S}}^1_1 + \overline{\tilde{\mathfrak{S}}^2_2}\,\tilde{\mathfrak{S}}^2_2 - \overline{\tilde{\mathfrak{S}}^2_1}\,\tilde{\mathfrak{S}}^2_1 - \overline{\tilde{\mathfrak{S}}^1_2}\,\tilde{\mathfrak{S}}^1_2}{2}.$$

(2.5)

The components of the matrix $\tilde{S}$ are known. Therefore, the formulas (2.2), (2.3), (2.4), and (2.5), are understood as the equations for the components of a complex 2×2 matrix $\tilde{\mathfrak{S}}$. As appears, these equations can be solved explicitly:

$$\tilde{\mathfrak{S}} = \frac{1}{|y|} \left\| \begin{matrix} i\,y^3 & i\,y^1 + y^2 \\ i\,y^1 - y^2 & -i\,y^3 \end{matrix} \right\|.$$

(2.6)

It is easy to see that $\det \tilde{\mathfrak{S}} = 1$, which means that $\tilde{\mathfrak{S}} \in \mathrm{SL}(2, \mathbb{C})$. The matrix (2.6) satisfying the equations (2.2), (2.3), (2.4), (2.5) and belonging to $\mathrm{SL}(2, \mathbb{C})$ is unique up to the change of sign: $\tilde{\mathfrak{S}} \to -\tilde{\mathfrak{S}}$.

Let's denote by $\tilde{\mathfrak{T}}$ the inverse matrix for the matrix (2.6), i. e. let $\tilde{\mathfrak{T}} = \tilde{\mathfrak{S}}^{-1}$. By means of the direct calculations, applying (1.1), we find

$$\tilde{\mathfrak{T}} = \frac{-1}{|x|} \left\| \begin{matrix} i\,x^3 & i\,x^1 + x^2 \\ i\,x^1 - x^2 & -i\,x^3 \end{matrix} \right\|.$$

(2.7)

When expressed back through y^0, y^1, y^2, y^3, the matrix (2.7) coincides with $-\tilde{\mathfrak{S}}$. This means that we have the relationships

$$\tilde{\mathfrak{S}} = -\tilde{\mathfrak{T}}, \qquad\qquad \tilde{\mathfrak{S}}^2 = \tilde{\mathfrak{T}}^2 = -1.$$

(2.8)

The determinants of the matrices (2.6) and (2.7) are equal to the unity:

$$\det \tilde{S} = \det \tilde{T} = 1.$$

(2.9)

The relationships (2.8) and (2.9) are concordant with the relationships (1.30) and (1.31) since $\phi(-\mathfrak{S}) = \phi(\mathfrak{S})$ for the homomorphism (2.1).

The mutually inverse matrices (2.6) and (2.7) are postulated to be the transition matrices for the spinor frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ and $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$:

$$\boldsymbol{\Psi}_i = \sum_{j=1}^{2} \tilde{\mathfrak{T}}^j_i\,\tilde{\boldsymbol{\Phi}}_j, \qquad\qquad \tilde{\boldsymbol{\Phi}}_i = \sum_{j=1}^{2} \tilde{\mathfrak{S}}^j_i\,\boldsymbol{\Psi}_j.$$

(2.10)

Thus, having found the matrices (2.6) and (2.7) and having equipped the local charts x^0, x^1, x^2, x^3 and y^0, y^1, y^2, y^3 with the spinor frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ and $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$

related to each other by means of the formulas (2.10), we have constructed the spinor bundle SM over the space-time manifold $M = \mathbb{R} \times S^3$.

3. Basic fields of the bundle of Weyl spinors.

The spinor bundle of Weyl spinors SM over any four-dimensional space-time manifold M is equipped with two special spin-tensorial fields. These basic spin-tensorial fields are presented in the following table:

| Symbol	Name	Spin-tensorial type
$\mathbf{d}$	Skew-symmetric metric tensor	$(0,2\|0,0\|0,0)$
$\mathbf{G}$	Infeld-van der Waerden field	$(1,0\|1,0\|0,1)$

(3.1)

The spin-tensorial type in the table (3.1) specifies the number of indices in coordinate representation of fields. The first two numbers are the numbers of upper and lower spinor indices, the second two numbers are the numbers of upper and lower conjugate spinor indices, and the last two numbers are the numbers of upper and lower tensorial indices (they are also called spacial indices).

Now let's return to our special case $M = \mathbb{R} \times S^3$. The spinor frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ and $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$ considered in section 2 are canonically associated with positively polarized right orthonormal frames in TM. For this reason they are orthonormal frames by definition, i.e. the skew symmetric metric tensor $\mathbf{d}$ is given by the matrix

$$d_{ij} = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\| \tag{3.2}$$

in both of these frames. The indices i and j in (3.2) are spinor indices. Therefore, the canonical presentation (3.2) of the metric tensor $\mathbf{d}$ depends on the choice of a spinor frame, but it is not sensitive to the choice of a tangent frame.

Unlike $\mathbf{d}$, the Infeld-van der Waerden field $\mathbf{G}$ has one lower spacial index in its coordinate presentation. Therefore, its coordinate presentation depends on the choice of two frames in TM and in SM. According to the definition of the spinor bundle SM (see definition 5.2 in [2]), each positively polarized right orthonormal frame of TM has its associated orthonormal frame in SM. We visualize this frame association through the following diagram:

$$\boxed{\text{Orthonormal frames}} \rightarrow \boxed{\begin{matrix} \text{Positively polarized} \\ \text{right orthonormal frames} \end{matrix}} \tag{3.3}$$
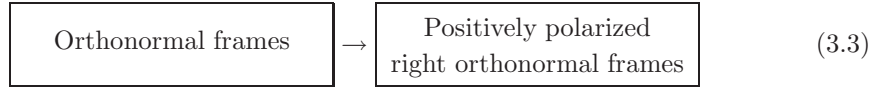
In each canonically associated pair of frames the components of the Infeld-van der Waerden field $G_q^{i\bar{i}}$ are presented by the Pauli matrices:

$$G_0^{i\bar{i}} = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, \qquad\qquad G_2^{i\bar{i}} = \left\| \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right\|,$$

$$G_1^{i\bar{i}} = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|, \qquad\qquad G_3^{i\bar{i}} = \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|. \tag{3.4}$$

The lower spacial index $q = 0$, 1, 2, 3 enumerates the matrices in (3.4). The spinor index i and the conjugate spinor index $\bar{i}$ determine the position of the component $G_q^{i\bar{i}}$ within one of these matrices. In section 2 above we have constructed the following two pairs of associated frames of SM and TM:

$$\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2 \qquad \rightarrow \qquad \mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \tag{3.5}$$

$$\tilde{\boldsymbol{\Phi}}_1, \tilde{\boldsymbol{\Phi}}_2 \qquad \rightarrow \qquad \tilde{\mathbf{Y}}_0, \tilde{\mathbf{Y}}_1, \tilde{\mathbf{Y}}_2, \tilde{\mathbf{Y}}_3. \tag{3.6}$$

Once the spinor bundle SM is constructed, the choice of associated frame pairs is not obligatory, e. g. we can choose the frames

$$\tilde{\boldsymbol{\Phi}}_1, \tilde{\boldsymbol{\Phi}}_2 \qquad - \qquad \mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3. \tag{3.7}$$

These frames are not canonically associated. However, using (1.26) and (3.4), we easily calculate the Infeld-van der Waerden symbols $G_q^{i\bar{i}}$ in this frame pair:

$$G_0^{i\bar{i}} = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, \qquad\qquad G_2^{i\bar{i}} = - \left\| \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right\|,$$

$$G_1^{i\bar{i}} = - \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\|, \qquad\qquad G_3^{i\bar{i}} = - \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\|.$$

Similarly, one can combine any frame of SM with any frame of TM into a frame pair and then calculate the components of $\mathbf{G}$ in such a non-canonical frame pair.

4. The bundle of Dirac spinors and its basic fields.

The bundle of Dirac spinors DM is constructed as the direct sum of the bundle of Weyl spinors SM and its Hermitian conjugate bundle $S^\dagger M$:

$$DM = SM \oplus S^\dagger M. \tag{4.1}$$

It does exist provided SM does. The bundles SM and $S^\dagger M$ are called chiral and antichiral components of the expansion (4.1). The Dirac bundle DM has more basic spin-tensorial fields as compared to SM:

Symbol	Name	Spin-tensorial type
$\mathbf{d}$	Skew-symmetric metric tensor	$(0, 2|0, 0|0, 0)$
$\mathbf{H}$	Chirality operator	$(1, 1|0, 0|0, 0)$
$\mathbf{D}$	Dirac form	$(0, 1|0, 1|0, 0)$
$\boldsymbol{\gamma}$	Dirac γ-field	$(1, 1|0, 0|0, 1)$

(4.2)

Let $\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2$ be an orthonormal frame of the bundle SM and let $\overline{\boldsymbol{\vartheta}}^1, \overline{\boldsymbol{\vartheta}}^2$ be its conjugate frame in $S^\dagger M$. At each point p off the space-time manifold $\overline{\boldsymbol{\vartheta}}^1$ and $\overline{\boldsymbol{\vartheta}}^2$ are semilinear functionals in the fiber $S_p(M)$ such that

$$\overline{\boldsymbol{\vartheta}}^i(\boldsymbol{\Psi}_1) = \delta_j^i,$$

where δ_j^i is the Kronecker delta-symbol. Denoting $\boldsymbol{\Psi}_3 = \overline{\boldsymbol{\vartheta}}^{\,1}$ and $\boldsymbol{\Psi}_4 = \overline{\boldsymbol{\vartheta}}^{\,2}$, we get a frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ of the Dirac bundle (4.1) (see more details in [3]). Such a frame of DM is called a canonically orthonormal chiral frame. There are four types of special frames in DM. Each special frame is canonically associated with some frame in tangent bundle TM according to the following diagram:

$$
\begin{array}{ccc}
\boxed{\begin{array}{c}\text{Canonically orthonormal}\\\text{chiral frames}\end{array}} & \rightarrow & \boxed{\begin{array}{c}\text{Positively polarized}\\\text{right orthonormal frames}\end{array}} \\[2ex]
\boxed{\begin{array}{c}P\text{-reverse}\\\text{antichiral frames}\end{array}} & \rightarrow & \boxed{\begin{array}{c}\text{Positively polarized}\\\text{left orthonormal frames}\end{array}} \\[2ex]
\boxed{\begin{array}{c}T\text{-reverse}\\\text{antichiral frames}\end{array}} & \rightarrow & \boxed{\begin{array}{c}\text{Negatively polarized}\\\text{right orthonormal frames}\end{array}} \\[2ex]
\boxed{\begin{array}{c}PT\text{-reverse}\\\text{chiral frames}\end{array}} & \rightarrow & \boxed{\begin{array}{c}\text{Negatively polarized}\\\text{left orthonormal frames}\end{array}}
\end{array}
\tag{4.3}
$$

Definition 4.1. A frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ of the Dirac bundle is called an orthonormal frame if the metric tensor $\mathbf{d}$ is given by the matrix

$$
d_{ij} = \left\| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right\|
\tag{4.4}
$$

in this frame. This frame is called an anti-orthonormal frame if the metric tensor $\mathbf{d}$ is given by the opposite matrix in this frame:

$$
d_{ij} = - \left\| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right\|.
\tag{4.5}
$$

Definition 4.2. A frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ of the Dirac bundle is called a chiral frame if the chirality operator $\mathbf{H}$ is given by the matrix

$$
H_j^i = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right\|
\tag{4.6}
$$

in this frame. This frame is called an antichiral frame if the chirality operator $\mathbf{H}$ is given by the opposite matrix in this frame:

$$
H_j^i = - \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right\|.
\tag{4.7}
$$

Definition 4.3. A frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ of the Dirac bundle is called a self-adjoint frame if the Dirac form $\mathbf{D}$ is given by the matrix

$$D_{i\bar{j}} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix} \tag{4.8}$$

in this frame. This frame is called an anti-self-adjoint frame if the Dirac form $\mathbf{D}$ is given by the opposite matrix in this frame:

$$D_{i\bar{j}} = -\begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}. \tag{4.9}$$

Definition 4.4. A canonically orthonormal chiral frame of the Dirac bundle is a frame which is orthonormal, chiral, and self-adjoint simultaneously.

Definition 4.5. A P-reverse antichiral frame of the Dirac bundle is a frame which is anti-orthonormal, antichiral, and self-adjoint simultaneously.

Definition 4.6. A T-reverse antichiral frame of the Dirac bundle is a frame which is orthonormal, antichiral, and anti-self-adjoint simultaneously.

Definition 4.7. A PT-reverse chiral frame of the Dirac bundle is a frame which is anti-orthonormal, chiral, and anti-self-adjoint simultaneously.

The definitions 4.4, 4.5, 4.6, 4.7 and the formulas (4.4), (4.5), (4.6), (4.7), (4.8), (4.9) describe the coordinate presentation of the basic field $\mathbf{d}$, $\mathbf{H}$, and $\mathbf{D}$ in special frames whose types are listed in the diagram (4.3).

According to the table (4.2), the Dirac γ-field has one lower spacial index. Therefore its coordinate presentation depends not only on the choice of a spinor frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ in the bundle DM, but on the choice of a tangent frame $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ in TM too. Assume that

$$\boldsymbol{\Psi}_1, \boldsymbol{\Psi}_2, \boldsymbol{\Psi}_3, \boldsymbol{\Psi}_4 \qquad \rightarrow \qquad \mathbf{X}_0, \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$$

is a pair of canonically associated frames belonging to any one of the four types specified in the diagram (4.3). Then the components γ^a_{bq} of the Dirac γ-field $\boldsymbol{\gamma}$ are presented by the following four Dirac matrices in this frame pair:

$$\gamma^a_{b0} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}, \qquad \gamma^a_{b1} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{Vmatrix},$$

$$\gamma^a_{b2} = \begin{Vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{Vmatrix}, \qquad \gamma^a_{b3} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}. \tag{4.10}$$

The spacial index $q = 0,\,1,\,2,\,3$ enumerates the matrices in (4.10). The spinor indices a and b determine the position of the component γ^a_{bq} within one of these four matrices (a is the row number and b is the column number).

Now let's return to the frame pairs (3.5) and (3.6). The frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ and $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$ of the bundle SM can be extended up to frames of the Dirac bundle DM. As a result we get two pairs of associated frames

$$\boldsymbol{\Psi}_1,\ \boldsymbol{\Psi}_2,\ \boldsymbol{\Psi}_3,\ \boldsymbol{\Psi}_4 \qquad \rightarrow \qquad \mathbf{X}_0,\ \mathbf{X}_1,\ \mathbf{X}_2,\ \mathbf{X}_3, \tag{4.11}$$

$$\tilde{\boldsymbol{\Phi}}_1,\ \tilde{\boldsymbol{\Phi}}_2,\ \tilde{\boldsymbol{\Phi}}_3,\ \tilde{\boldsymbol{\Phi}}_4 \qquad \rightarrow \qquad \tilde{\mathbf{Y}}_0,\ \tilde{\mathbf{Y}}_1,\ \tilde{\mathbf{Y}}_2,\ \tilde{\mathbf{Y}}_3. \tag{4.12}$$

Both frame pairs (4.11) and (4.12) belong to the first type in the diagram (4.3).

The frames (3.7) are not canonically associated. Therefore they have no extension to a canonically associated pair. However, note that $\mathbf{Y}_0$, $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ is a positively polarized left orthonormal frame of TM related to the frame $\tilde{\mathbf{Y}}_0$, $\tilde{\mathbf{Y}}_1$, $\tilde{\mathbf{Y}}_2$, $\tilde{\mathbf{Y}}_3$ according to the formula (1.26), i. e. through the inversion of the space-like vectors $\tilde{\mathbf{Y}}_1$, $\tilde{\mathbf{Y}}_2$, $\tilde{\mathbf{Y}}_3$. Therefore it has an associated frame $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, $\boldsymbol{\Phi}_3$, $\boldsymbol{\Phi}_4$:

$$\boldsymbol{\Phi}_1,\ \boldsymbol{\Phi}_2,\ \boldsymbol{\Phi}_3,\ \boldsymbol{\Phi}_4 \qquad \rightarrow \qquad \mathbf{Y}_0,\ \mathbf{Y}_1,\ \mathbf{Y}_2,\ \mathbf{Y}_3. \tag{4.13}$$

The frame $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, $\boldsymbol{\Phi}_3$, $\boldsymbol{\Phi}_4$ in (4.13) is produced from the frame $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$, $\tilde{\boldsymbol{\Phi}}_3$, $\tilde{\boldsymbol{\Phi}}_4$ by means of the so called P-reversion procedure (see [3]):

$$\boldsymbol{\Phi}_1 = \tilde{\boldsymbol{\Phi}}_3, \qquad \boldsymbol{\Phi}_2 = \tilde{\boldsymbol{\Phi}}_4, \qquad \boldsymbol{\Phi}_3 = \tilde{\boldsymbol{\Phi}}_1, \qquad \boldsymbol{\Phi}_4 = \tilde{\boldsymbol{\Phi}}_2. \tag{4.14}$$

The frame $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, $\boldsymbol{\Phi}_3$, $\boldsymbol{\Phi}_4$ produced from $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$, $\tilde{\boldsymbol{\Phi}}_3$, $\tilde{\boldsymbol{\Phi}}_4$ by means of the formulas (4.14) is a P-reverse antichiral frame, i. e. the pair (4.13) belongs to the second type of associated frame pairs in the diagram (4.3).

The frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ and $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$, $\tilde{\boldsymbol{\Phi}}_3$, $\tilde{\boldsymbol{\Phi}}_4$ in (4.11) and (4.12) are related to each other by means of the formulas

$$\boldsymbol{\Psi}_i = \sum_{j=1}^{4} \tilde{\mathfrak{T}}^j_i\, \tilde{\boldsymbol{\Phi}}_j, \qquad\qquad \tilde{\boldsymbol{\Phi}}_i = \sum_{j=1}^{4} \tilde{\mathfrak{S}}^j_i\, \boldsymbol{\Psi}_j. \tag{4.15}$$

The formula (4.15) is analogous to the formula (2.10). The 4×4 matrices $\tilde{\mathfrak{S}}$ and $\tilde{\mathfrak{T}}$ in (4.15) are produced from the matrices $\tilde{\mathfrak{S}}$ and $\tilde{\mathfrak{T}}$ in (2.10) in some special way (see formula (2.19) in [3]). In our particular case they are are block-diagonal extensions of the matrices (2.6) and (2.7):

$$\tilde{\mathfrak{S}} = \frac{1}{|y|} \left\|\begin{array}{cccc} i\,y^3 & i\,y^1 + y^2 & 0 & 0 \\[6pt] i\,y^1 - y^2 & -i\,y^3 & 0 & 0 \\[6pt] 0 & 0 & i\,y^3 & i\,y^1 + y^2 \\[6pt] 0 & 0 & i\,y^1 - y^2 & -i\,y^3 \end{array}\right\|, \tag{4.16}$$

$$\tilde{\mathfrak{T}} = \frac{1}{|x|} \begin{Vmatrix} i\,x^3 & i\,x^1 + x^2 & 0 & 0 \\ i\,x^1 - x^2 & -i\,x^3 & 0 & 0 \\ 0 & 0 & i\,x^3 & i\,x^1 + x^2 \\ 0 & 0 & i\,x^1 - x^2 & -i\,x^3 \end{Vmatrix}. \tag{4.17}$$

The frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ and $\boldsymbol{\Phi}_1$, $\boldsymbol{\Phi}_2$, $\boldsymbol{\Phi}_3$, $\boldsymbol{\Phi}_4$ in (4.11) and (4.13) are also related to each other by means of the formulas similar to (4.15):

$$\boldsymbol{\Psi}_i = \sum_{j=1}^{4} \mathfrak{T}_i^j\, \boldsymbol{\Phi}_j, \qquad\qquad \boldsymbol{\Phi}_i = \sum_{j=1}^{4} \mathfrak{S}_i^j\, \boldsymbol{\Psi}_j. \tag{4.18}$$

The matrices $\mathfrak{S}$ and $\mathfrak{T}$ in (4.18) are given by the formulas

$$\mathfrak{S} = \frac{1}{|y|} \begin{Vmatrix} 0 & 0 & i\,y^3 & i\,y^1 + y^2 \\ 0 & 0 & i\,y^1 - y^2 & -i\,y^3 \\ i\,y^3 & i\,y^1 + y^2 & 0 & 0 \\ i\,y^1 - y^2 & -i\,y^3 & 0 & 0 \end{Vmatrix}, \tag{4.19}$$

$$\mathfrak{T} = -\frac{1}{|x|} \begin{Vmatrix} 0 & 0 & i\,x^3 & i\,x^1 + x^2 \\ 0 & 0 & i\,x^1 - x^2 & -i\,x^3 \\ i\,x^3 & i\,x^1 + x^2 & 0 & 0 \\ i\,x^1 - x^2 & -i\,x^3 & 0 & 0 \end{Vmatrix}. \tag{4.20}$$

The matrices (4.19) and (4.20) are similar to (4.17) and (4.18). However, unlike (4.17) and (4.18), they are not block-diagonal. Hence, they mix chiral and antichiral subbundles in the expansion (4.1).

5. Spherical coordinates.

The stereographic projections (1.2) and (1.3) and the local charts x^0, x^1, x^2, x^3 and y^0, y^1, y^2, y^3 introduced through them are not very popular in physical literature, e. g. in [1] the spherical coordinates are used. Therefore we consider the third chart of spherical coordinates in $M = \mathbb{R} \times S^3$. The initial coordinate η^0 of these spherical coordinates coincides with x^0 and y^0:

$$x^0 = y^0 = \eta^0 = \eta. \tag{5.1}$$

Other three coordinates η^1, η^2, η^3 are angular variables:

$$\eta^1 = \chi, \qquad\qquad \eta^2 = \theta, \qquad\qquad \eta^3 = \varphi. \qquad (5.2)$$

They are introduced through the following formulas:

$$
\begin{aligned}
z^1 &= R \sin\chi \sin\theta \sin\varphi, \\
z^2 &= R \sin\chi \sin\theta \cos\varphi, \\
z^3 &= R \sin\chi \cos\theta, \\
z^4 &= R \cos\chi.
\end{aligned}
\qquad (5.3)
$$

The formulas relating η^0, η^1, η^2, η^3 with x^0, x^1, x^2, x^3 and y^0, y^1, y^2, y^3 are derived from (5.1) and (5.3) by inverting (1.2) and (1.3):

$$
\begin{cases}
x^0 = \eta \\
x^1 = \dfrac{\sin\chi \sin\theta \sin\varphi}{1 - \cos\chi}, \\
x^2 = \dfrac{\sin\chi \sin\theta \cos\varphi}{1 - \cos\chi}, \\
x^3 = \dfrac{\sin\chi \cos\theta}{1 - \cos\chi},
\end{cases}
\qquad
\begin{cases}
y^0 = \eta \\
y^1 = \dfrac{\sin\chi \sin\theta \sin\varphi}{1 + \cos\chi}, \\
y^2 = \dfrac{\sin\chi \sin\theta \cos\varphi}{1 + \cos\chi}, \\
y^3 = \dfrac{\sin\chi \cos\theta}{1 + \cos\chi}.
\end{cases}
\qquad (5.4)
$$

Differentiating the formulas (5.4) and substituting them into (1.8) and (1.9), we derive the formula for the metric in the local coordinates η^0, η^1, η^2, η^3:

$$ds^2 = R^2 \, (d\eta^0)^2 - R^2 \, (d\eta^1)^2 - R^2 \sin^2\chi \, (d\eta^2)^2 - R^2 \sin^2\chi \, \sin^2\theta \, (d\eta^3)^2. \qquad (5.5)$$

The formula (5.5) means that the holonomic coordinate frame

$$\frac{\partial}{\partial\eta^0}, \qquad\qquad \frac{\partial}{\partial\eta^1}, \qquad\qquad \frac{\partial}{\partial\eta^2}, \qquad\qquad \frac{\partial}{\partial\eta^3} \qquad (5.6)$$

is orthogonal, but it is not an orthonormal frame. For this reason, instead of (5.6), we use the following non-holonomic orthonormal frame:

$$
\begin{aligned}
\mathbf{E}_0 &= \frac{1}{R} \frac{\partial}{\partial\eta^0}, &\qquad \mathbf{E}_1 &= \frac{1}{R} \frac{\partial}{\partial\eta^1}, \\
\mathbf{E}_2 &= \frac{1}{R \sin\chi} \frac{\partial}{\partial\eta^2}, &\qquad \mathbf{E}_3 &= \frac{1}{R \sin\chi \sin\theta} \frac{\partial}{\partial\eta^3}.
\end{aligned}
\qquad (5.7)
$$

Like in (1.16) and (1.18), we have nontrivial commutation relationships for the frame vectors fields $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ defined by the formulas (5.7):

$$[\mathbf{E}_i, \mathbf{E}_j] = \sum_{k=0}^{3} c_{ij}^k \, \mathbf{E}_k. \qquad (5.8)$$

Most of the coefficients c^k_{ij} in (5.8) are zero. Here is the list of nonzero ones:

$$c^1_{01} = -c^1_{10} = c^2_{02} = -c^2_{20} = c^3_{03} = -c^3_{30} = -\frac{R'}{R^2},$$

$$c^2_{12} = -c^2_{21} = c^3_{13} = -c^3_{31} = -\frac{\cos\chi}{R\sin\chi}, \tag{5.9}$$

$$c^3_{23} = -c^3_{32} = -\frac{\cos\theta}{R\sin\chi\sin\theta}.$$

Now let's study how the frame (5.7) is related to the frames (1.13) and (1.26):

$$\mathbf{E}_i = \sum_{j=0}^{3} \hat{S}^j_i \, \mathbf{X}_j, \qquad\qquad \mathbf{E}_i = \sum_{j=0}^{3} \check{S}^j_i \, \tilde{\mathbf{Y}}_j. \tag{5.10}$$

By means of direct calculations, using (5.7), (1.13), (1.14), and (1.26), we derive the following formulas for the matrices $\hat{S}$ and $\check{S}$ in (5.10):

$$\hat{S} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\sin\varphi\sin\theta & \sin\varphi\cos\theta & \cos\varphi \\ 0 & -\cos\varphi\sin\theta & \cos\varphi\cos\theta & -\sin\varphi \\ 0 & -\cos\theta & -\sin\theta & 0 \end{Vmatrix}, \tag{5.11}$$

$$\check{S} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -\sin\varphi\sin\theta & -\sin\varphi\cos\theta & -\cos\varphi \\ 0 & -\cos\varphi\sin\theta & -\cos\varphi\cos\theta & \sin\varphi \\ 0 & -\cos\theta & \sin\theta & 0 \end{Vmatrix}. \tag{5.12}$$

As appears, both matrices (5.11) and (5.12) belong to the special orthochronous Lorentz group $\mathrm{SO}^+(1,3,\mathbb{R})$. For this reason they are related with two matrices $\hat{\mathfrak{S}}$ and $\check{\mathfrak{S}}$ belonging to the group $\mathrm{SL}(2,\mathbb{C})$ through the group homomorphism (2.1):

$$\hat{S} = \phi(\hat{\mathfrak{S}}), \qquad\qquad \check{S} = \phi(\check{\mathfrak{S}}). \tag{5.13}$$

These two matrices $\hat{\mathfrak{S}}$ and $\check{\mathfrak{S}}$ satisfying (5.13) can be found in explicit form:

$$\hat{\mathfrak{S}} = \frac{1}{\sqrt{2}} \begin{Vmatrix} \exp\!\left(\dfrac{i\,\varphi + i\,\theta}{2}\right) & -\exp\!\left(\dfrac{i\,\varphi - i\,\theta}{2}\right) \\[2mm] \exp\!\left(\dfrac{i\,\theta - i\,\varphi}{2}\right) & \exp\!\left(\dfrac{-i\,\theta - i\,\varphi}{2}\right) \end{Vmatrix} \tag{5.14}$$

$$\check{\mathfrak{S}} = \frac{i}{\sqrt{2}} \begin{Vmatrix} -\exp\!\left(\dfrac{i\,\varphi - i\,\theta}{2}\right) & \exp\!\left(\dfrac{i\,\varphi + i\,\theta}{2}\right) \\[2mm] \exp\!\left(\dfrac{-i\,\theta - i\,\varphi}{2}\right) & \exp\!\left(\dfrac{i\,\theta - i\,\varphi}{2}\right) \end{Vmatrix} \tag{5.15}$$

If we express the matrix (2.6) through the spherical coordinates (5.2), then we get

$$\tilde{\mathbb{S}} = \left\| \begin{array}{cc} i\cos\theta & \sin\theta\,\exp(i\varphi) \\ -\sin\theta\,\exp(-i\varphi) & -i\cos\theta \end{array} \right\|. \tag{5.16}$$

The matrices (5.14), (5.15), and (5.16) are related to each other as follows:

$$\hat{\mathbb{S}} = \tilde{\mathbb{S}}\,\breve{\tilde{\mathbb{S}}}. \tag{5.17}$$

Applying the group homomorphism (2.1) to (5.17), we get:

$$\hat{S} = \tilde{S}\,\breve{\tilde{S}}. \tag{5.18}$$

The relationship (5.18) can be verified directly using the formulas (5.11), (5.12), and the the formula for $\hat{S}$ in section 1. Now we define a spinor frame $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$ of the bundle SM associated with the tangent frame $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$. We do it by setting

$$\boldsymbol{\Xi}_i = \sum_{j=1}^{2} \hat{\mathbb{S}}_i^{\,j}\,\boldsymbol{\Psi}_j, \qquad\qquad \boldsymbol{\Xi}_i = \sum_{j=1}^{2} \breve{\tilde{\mathbb{S}}}_i^{\,j}\,\tilde{\boldsymbol{\Phi}}_j. \tag{5.19}$$

Due to (5.17), (5.10), and (2.10) the formulas (5.19) are consistent with each other. The frame $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$ determined by any one of these two formulas is an orthonormal frame of the bundle of Weyl spinors. So we have the frame association

$$\boldsymbol{\Xi}_1,\ \boldsymbol{\Xi}_2 \qquad \rightarrow \qquad \mathbf{E}_0,\ \mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3 \tag{5.20}$$

analogous to (3.5) and (3.6). This frame association (5.20) is a special case of the general scheme presented by the diagram (3.3).

The orthonormal frame $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$ of the bundle of Weyl spinors SM has a unique extension $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$, $\boldsymbol{\Xi}_3$, $\boldsymbol{\Xi}_4$ to the Dirac bundle DM. This extension is a canonically orthonormal chiral frame associated with the frame $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$:

$$\boldsymbol{\Xi}_1,\ \boldsymbol{\Xi}_2,\ \boldsymbol{\Xi}_3,\ \boldsymbol{\Xi}_4 \qquad \rightarrow \qquad \mathbf{E}_0,\ \mathbf{E}_1,\ \mathbf{E}_2,\ \mathbf{E}_3. \tag{5.21}$$

The frame association (5.21) is analogous to (4.11) and (4.12). Note that the frame $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ in (5.20) and (5.21) is a positively polarized right orthonormal frame. This fact follows from $\hat{S} \in \mathrm{SO}^+(1,3,\mathbb{R})$ and from (5.10). Therefore the frame association (5.21) is a special case for the scheme presented by the first line in the diagram (4.3).

The extended frame $\boldsymbol{\Xi}_1$, $\boldsymbol{\Xi}_2$, $\boldsymbol{\Xi}_3$, $\boldsymbol{\Xi}_4$ is related to the previously defined frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ and $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$, $\tilde{\boldsymbol{\Phi}}_3$, $\tilde{\boldsymbol{\Phi}}_4$ as follows

$$\boldsymbol{\Xi}_i = \sum_{j=1}^{4} \hat{\mathbb{S}}_i^{\,j}\,\boldsymbol{\Psi}_j, \qquad\qquad \boldsymbol{\Xi}_i = \sum_{j=1}^{4} \breve{\tilde{\mathbb{S}}}_i^{\,j}\,\tilde{\boldsymbol{\Phi}}_j. \tag{5.22}$$

The matrices $\hat{\mathfrak{S}}$ and $\check{\tilde{\mathfrak{S}}}$ in (5.22) are four-dimensional extensions of the matrices (5.14) and (5.14). They are block-diagonal matrices constructed with the use of the two-dimensional matrices $\hat{\mathfrak{S}}$ and $\tilde{\mathfrak{S}}$. Here are the explicit formulas for them:

$$
\hat{\mathfrak{S}} = \frac{1}{\sqrt{2}} \left\|
\begin{array}{cccc}
e^{\frac{i\varphi+i\theta}{2}} & -e^{\frac{i\varphi-i\theta}{2}} & 0 & 0 \\[2mm]
e^{\frac{i\theta-i\varphi}{2}} & e^{\frac{-i\theta-i\varphi}{2}} & 0 & 0 \\[2mm]
0 & 0 & e^{\frac{i\varphi+i\theta}{2}} & -e^{\frac{i\varphi-i\theta}{2}} \\[2mm]
0 & 0 & e^{\frac{i\theta-i\varphi}{2}} & e^{\frac{-i\theta-i\varphi}{2}}
\end{array}
\right\|,
\tag{5.23}
$$

$$
\check{\tilde{\mathfrak{S}}} = \frac{i}{\sqrt{2}} \left\|
\begin{array}{cccc}
-e^{\frac{i\varphi-i\theta}{2}} & e^{\frac{i\varphi+i\theta}{2}} & 0 & 0 \\[2mm]
e^{\frac{-i\theta-i\varphi}{2}} & e^{\frac{i\theta-i\varphi}{2}} & 0 & 0 \\[2mm]
0 & 0 & -e^{\frac{i\varphi-i\theta}{2}} & e^{\frac{i\varphi+i\theta}{2}} \\[2mm]
0 & 0 & e^{\frac{-i\theta-i\varphi}{2}} & e^{\frac{i\theta-i\varphi}{2}}
\end{array}
\right\|.
\tag{5.24}
$$

The matrix (5.16) has an analogous four-dimensional extension. Here is the explicit formula for such an extension:

$$
\tilde{\mathfrak{S}} = \left\|
\begin{array}{cccc}
i\cos\theta & \sin\theta\,e^{i\varphi} & 0 & 0 \\[2mm]
-\sin\theta\,e^{-i\varphi} & -i\cos\theta & 0 & 0 \\[2mm]
0 & 0 & i\cos\theta & \sin\theta\,e^{i\varphi} \\[2mm]
0 & 0 & -\sin\theta\,e^{-i\varphi} & -i\cos\theta
\end{array}
\right\|.
\tag{5.25}
$$

The matrix (5.25) is also a block-diagonal matrix. It is used in the following transition formula relating the frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ and $\tilde{\boldsymbol{\Phi}}_1$, $\tilde{\boldsymbol{\Phi}}_2$, $\tilde{\boldsymbol{\Phi}}_3$, $\tilde{\boldsymbol{\Phi}}_4$:

$$
\tilde{\boldsymbol{\Phi}}_i = \sum_{j=1}^{4} \tilde{\mathfrak{S}}_i^j \, \boldsymbol{\Psi}_j.
\tag{5.26}
$$

The formula (5.26) is an extension for the second formula (2.10). It coincides with the second formula (4.15). As for the matrices (5.23), (5.24), and (5.25), being extensions of the matrices (5.14), (5.15), and (5.16), they satisfy the relationship

(5.17). The matrix (5.25) can be produced from (4.16) by passing to the angular variables (5.2).

Let's recall that the frame $\mathbf{Y}_0$, $\mathbf{Y}_1$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ has no canonically associated spinor frame in SM (since it is left oriented), but has an associated frame $\mathbf{\Phi}_1$, $\mathbf{\Phi}_2$, $\mathbf{\Phi}_3$, $\mathbf{\Phi}_4$ in DM (see (4.13) above). For this reason we can write

$$\mathbf{\Xi}_i = \sum_{j=1}^{4} \check{\mathfrak{S}}_i^j \, \mathbf{\Phi}_j. \tag{5.27}$$

The matrix $\check{\mathfrak{S}}$ in (5.27) is given by the following formula:

$$\check{\mathfrak{S}} = \frac{i}{\sqrt{2}} \left\| \begin{array}{cccc} 0 & 0 & -e^{\frac{i\varphi - i\theta}{2}} & e^{\frac{i\varphi + i\theta}{2}} \\[2ex] 0 & 0 & e^{\frac{-i\theta - i\varphi}{2}} & e^{\frac{i\theta - i\varphi}{2}} \\[2ex] -e^{\frac{i\varphi - i\theta}{2}} & e^{\frac{i\varphi + i\theta}{2}} & 0 & 0 \\[2ex] e^{\frac{-i\theta - i\varphi}{2}} & e^{\frac{i\theta - i\varphi}{2}} & 0 & 0 \end{array} \right\|. \tag{5.28}$$

As we see, the matrix (5.28) is not block-diagonal. It is because the frame pair (4.13) corresponds to the second line in the diagram (4.3).

6. Metric connection and its spinor components.

In previous sections we considered three local chart in the spherical universe $M = \mathbb{R} \times S^3$ — two charts with stereographic coordinates and one chart with spherical coordinates. Considering Dirac spinors, we have equipped these local charts with three associated frame pairs (see (4.11), (4.13), and (5.21) above). Our next goal is to calculate the components of the metric connection $(\Gamma, A, \bar{A})$ in each of these three frame pairs.

Spinor extension of the metric connection has three groups of components: spacial components Γ_{ij}^k, spinor components A_{ia}^b, and conjugate spinor components $\bar{A}_{i\bar{b}}^{\bar{a}}$. Assume for a while that we have some arbitrary frame pair

$$\mathbf{\Psi}_1, \, \mathbf{\Psi}_2, \, \mathbf{\Psi}_3, \, \mathbf{\Psi}_4 \qquad - \qquad \mathbf{X}_0, \, \mathbf{X}_1, \, \mathbf{X}_2, \, \mathbf{X}_3, \tag{6.1}$$

where $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$, $\mathbf{\Psi}_3$, $\mathbf{\Psi}_4$ is a spinor frame in DM and $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ is a spacial frame in TM. They can be either associated or non-associated frames. In any case the conjugate spinor components $\bar{A}_{i\bar{b}}^{\bar{a}}$ are expressed through spinor components in an elementary way through complex conjugation:

$$\bar{A}_{i\bar{b}}^{\bar{a}} = \overline{A_{i\,b}^{\,a}}. \tag{6.2}$$

Due to the formula (6.2) it is sufficient to know Γ_{ij}^k and $A_{i\,a}^{\,b}$ for to describe the metric connection completely. The spacial components Γ_{ij}^k correspond to the well-

known Levi-Civita connection in TM. They are given by the formula

$$\Gamma^k_{ij} = \sum_{r=0}^{3} \frac{g^{kr}}{2}\left(L_{\mathbf{X}_i}(g_{rj}) + L_{\mathbf{X}_j}(g_{ir}) - L_{\mathbf{X}_r}(g_{ij})\right) +$$

$$+ \frac{c^k_{ij}}{2} - \sum_{r=0}^{3}\sum_{s=0}^{3} \frac{c^s_{ir}}{2}\, g^{kr}\, g_{sj} - \sum_{r=0}^{3}\sum_{s=0}^{3} \frac{c^s_{jr}}{2}\, g^{kr}\, g_{si} \tag{6.3}$$

(see [4]). The Levi-Civita connection is a torsion-free connection. Nevertheless, its components Γ^k_{ij} in (6.3) are not symmetric: $\Gamma^k_{ij} - \Gamma^k_{ji} = c^k_{ij}$. It is because in general case the tangent frame $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ in the pair (6.1) is not commutative (see (1.16) above).

Now let's proceed to the spinor components $A^b_{i\,a}$ of our metric connection. These components are given by the formula

$$A^a_{ib} = \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4} \frac{L_{\mathbf{X}_i}(d_{\alpha\beta})\,d^{\beta\alpha}}{8}\,\delta^a_b - \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{d=1}^{4} \frac{L_{\mathbf{X}_i}(d_{\alpha\beta})\,d^{\beta d}\,H^\alpha_d}{8}\,H^a_b -$$

$$- \sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4} \frac{d_{bc}\,L_{\mathbf{X}_i}(H^c_d)\,H^d_r\,d^{ra}}{4} + \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4} \frac{L_{\mathbf{X}_i}(\gamma^\alpha_{bm}\,g^{mn})}{4} \times \tag{6.4}$$

$$\times \gamma^a_{\alpha n} + \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3} \frac{\gamma^\alpha_{bm}\,\Gamma^n_{is}\,g^{ms}\,\gamma^a_{\alpha n}}{4}$$

(see [4] for more details). This is a general formula applicable to an arbitrary frame pair (6.1). In our particular case all of our three frame pairs, which we study in this section, are special ones. They are canonically associated frame pairs described by the first and second lines in the diagram (4.3). In each such frame pair the components of the basic fields $\mathbf{g}$, $\mathbf{d}$, $\mathbf{H}$, $\mathbf{D}$, and $\boldsymbol\gamma$ are constants (see formulas (1.15), (4.4), (4.5), (4.6), (4.7), (4.8), (4.9), and (4.10)). Therefore, their derivatives $L_{\mathbf{X}_i}$ are identically zero. As a result the formulas (6.3) and (6.4) are reduced to

$$\Gamma^k_{ij} = \frac{c^k_{ij}}{2} - \sum_{r=0}^{3}\sum_{s=0}^{3} \frac{c^s_{ir}}{2}\, g^{kr}\, g_{sj} - \sum_{r=0}^{3}\sum_{s=0}^{3} \frac{c^s_{jr}}{2}\, g^{kr}\, g_{si}, \tag{6.5}$$

$$A^a_{ib} = \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3} \frac{\gamma^\alpha_{bm}\,\Gamma^n_{is}\,g^{ms}\,\gamma^a_{\alpha n}}{4}. \tag{6.6}$$

The metric connection with the components (6.5) and (6.6) produces two curvature tensors. Here are the formulas for their components:

$$R^p_{qij} = L_{\mathbf{X}_i}(\Gamma^p_{j\,q}) - L_{\mathbf{X}_j}(\Gamma^p_{i\,q}) + \sum_{h=0}^{3}\left(\Gamma^p_{i\,h}\,\Gamma^h_{j\,q} - \Gamma^p_{j\,h}\,\Gamma^h_{i\,q}\right) - \sum_{k=0}^{3} c^k_{ij}\,\Gamma^p_{kq}, \tag{6.7}$$

$$\mathfrak{R}^p_{qij} = L_{\mathbf{X}_i}(A^p_{j\,q}) - L_{\mathbf{X}_j}(A^p_{i\,q}) + \sum_{h=1}^{4}\left(A^p_{i\,h}\,A^h_{j\,q} - A^p_{j\,h}\,A^h_{i\,q}\right) - \sum_{k=0}^{3} c^k_{ij}\,A^p_{kq}. \tag{6.8}$$

The formula (6.7) yields the components of the well-known Riemannian curvature tensor, while (6.8) are the components of its spinor extension. As appears, the tensors $\mathbf{R}$ and $\mathfrak{R}$ are related to each other as follows:

$$\mathfrak{R}^p_{qij} = \frac{1}{4} \sum_{m=0}^{3} \sum_{n=0}^{3} \sum_{r=0}^{3} \sum_{\alpha=1}^{4} R^r_{mij} \, \gamma^\alpha_{qn} \, g^{mn} \, \gamma^p_{\alpha r} \tag{6.9}$$

(see proof of the formula (6.9) in [4]). Unlike (6.5) and (6.6), the formulas (6.7), (6.8), and (6.9) are applicable to an arbitrary frame pair (6.1).

The next step now is to apply the above formulas to our three local charts and three frame pairs and get some formulas specific to the homogeneous and isotropic spherical universe $M = \mathbb{R} \times S^3$.

1. **Stereographic coordinates** in projection from the **North Pole**. We denote these coordinates with x^0, x^1, x^2, x^3. Their domain is the whole sphere S^3 except for the North Pole itself. The frames (4.11) are defined and smooth in this domain. This frame pair is linked to the coordinates x^0, x^1, x^2, x^3. The commutation coefficients for the frame $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ in this pair are given by the formulas (1.17). Now we substitute these coefficients into the formula (6.5). As a result we get the following complete list of nonzero Γ-components of the metric connection in the non-holonomic frame $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$:

$$
\begin{aligned}
&\Gamma^0_{11} = \frac{R'}{R^2}, &\qquad &\Gamma^0_{22} = \frac{R'}{R^2}, &\qquad &\Gamma^0_{33} = \frac{R'}{R^2}, \\[2mm]
&\Gamma^1_{10} = \frac{R'}{R^2}, &\qquad &\Gamma^2_{20} = \frac{R'}{R^2}, &\qquad &\Gamma^3_{30} = \frac{R'}{R^2}, \\[2mm]
&\Gamma^1_{22} = \frac{(x^1)}{R^2}, &\qquad &\Gamma^2_{33} = \frac{(x^2)}{R^2}, &\qquad &\Gamma^3_{11} = \frac{(x^3)}{R^2}, \\[2mm]
&\Gamma^2_{11} = \frac{(x^2)}{R^2}, &\qquad &\Gamma^3_{22} = \frac{(x^3)}{R^2}, &\qquad &\Gamma^1_{33} = \frac{(x^1)}{R^2}, \\[2mm]
&\Gamma^1_{12} = -\frac{(x^2)}{R^2}, &\qquad &\Gamma^2_{23} = -\frac{(x^3)}{R^2}, &\qquad &\Gamma^3_{31} = -\frac{(x^1)}{R^2}, \\[2mm]
&\Gamma^2_{21} = -\frac{(x^1)}{R^2}, &\qquad &\Gamma^3_{32} = -\frac{(x^2)}{R^2}, &\qquad &\Gamma^1_{13} = -\frac{(x^2)}{R^2}.
\end{aligned}
\tag{6.10}
$$

Here R' is the derivative of the function (1.10). Substituting (6.10) into (6.6), we derive the explicit formulas for A-components of the metric connection:

$$
\begin{aligned}
&A^1_{11} = -\frac{i\,(x^2)}{2\,R}, &\qquad &A^1_{21} = \frac{i\,(x^1)}{2\,R}, &\qquad &A^1_{31} = \frac{R'}{2\,R^2}, \\[2mm]
&A^2_{12} = \frac{i\,(x^2)}{2\,R}, &\qquad &A^2_{22} = -\frac{i\,(x^1)}{2\,R}, &\qquad &A^2_{32} = -\frac{R'}{2\,R^2}, \\[2mm]
&A^3_{13} = -\frac{i\,(x^2)}{2\,R}, &\qquad &A^3_{23} = \frac{i\,(x^1)}{2\,R}, &\qquad &A^3_{33} = -\frac{R'}{2\,R^2}, \\[2mm]
&A^4_{14} = \frac{i\,(x^2)}{2\,R}, &\qquad &A^4_{24} = -\frac{i\,(x^1)}{2\,R}, &\qquad &A^4_{34} = \frac{R'}{2\,R^2},
\end{aligned}
\tag{6.11}
$$

$$A^1_{12} = \frac{R'}{2\,R^2} + \frac{(x^3)}{2\,R}, \qquad\qquad A^2_{11} = \frac{R'}{R^2} - \frac{(x^3)}{2\,R},$$

$$A^3_{14} = -\frac{R'}{2\,R^2} + \frac{(x^3)}{2\,R}, \qquad\qquad A^4_{13} = -\frac{R'}{2\,R^2} - \frac{(x^3)}{2\,R},$$

$$A^1_{22} = -\frac{i\,R'}{2\,R^2} - \frac{i\,(x^3)}{2\,R}, \qquad\qquad A^2_{21} = \frac{i\,R'}{2\,R^2} - \frac{i\,(x^3)}{2\,R},$$

$$A^3_{24} = \frac{i\,R'}{2\,R^2} - \frac{i\,(x^3)}{2\,R}, \qquad\qquad A^4_{23} = -\frac{i\,R'}{2\,R^2} - \frac{i\,(x^3)}{2\,R}, \qquad (6.12\ )$$

$$A^1_{32} = -\frac{(x^1)}{2\,R} + \frac{i\,(x^2)}{2\,R}, \qquad\qquad A^2_{31} = \frac{(x^1)}{2\,R} + \frac{i\,(x^2)}{2\,R},$$

$$A^3_{34} = -\frac{(x^1)}{2\,R} + \frac{i\,(x^2)}{2\,R}, \qquad\qquad A^4_{33} = \frac{(x^1)}{2\,R} + \frac{i\,(x^2)}{2\,R}.$$

Substituting (6.10) into (6.7), we find the components of the Riemannian curvature tensor **R**. Its nonzero components are listed here:

$$R^0_{101} = -R^0_{110} = -\frac{(R')^2}{R^4} + \frac{R''}{R^3}, \qquad R^1_{001} = -R^1_{010} = -\frac{(R')^2}{R^4} + \frac{R''}{R^3},$$

$$R^0_{202} = -R^0_{220} = -\frac{(R')^2}{R^4} + \frac{R''}{R^3}, \qquad R^2_{002} = -R^2_{020} = -\frac{(R')^2}{R^4} + \frac{R''}{R^3},$$

$$R^0_{303} = -R^0_{330} = -\frac{(R')^2}{R^4} + \frac{R''}{R^3}, \qquad R^3_{003} = -R^3_{030} = -\frac{(R')^2}{R^4} + \frac{R''}{R^3}, \qquad (6.13)$$

$$R^1_{212} = -R^1_{221} = \frac{1}{R^2} + \frac{(R')^2}{R^4}, \qquad R^2_{112} = -R^2_{121} = -\frac{1}{R^2} - \frac{(R')^2}{R^4},$$

$$R^2_{323} = -R^2_{333} = \frac{1}{R^2} + \frac{(R')^2}{R^4}, \qquad R^3_{223} = -R^3_{232} = -\frac{1}{R^2} - \frac{(R')^2}{R^4},$$

$$R^3_{131} = -R^3_{113} = \frac{1}{R^2} + \frac{(R')^2}{R^4}, \qquad R^1_{331} = -R^1_{313} = -\frac{1}{R^2} - \frac{(R')^2}{R^4}.$$

The next step is to substitute (6.11) and (6.12) into (6.8). As a result we derive the explicit formulas for the components of the spinor curvature tensor $\mathfrak{R}$. Below is the list of all its nonzero components:

$$\mathfrak{R}^1_{201} = -\mathfrak{R}^1_{210} = -\frac{(R')^2}{2\,R^4} + \frac{R''}{2\,R^3}, \qquad \mathfrak{R}^2_{101} = -\mathfrak{R}^2_{110} = -\frac{(R')^2}{2\,R^4} + \frac{R''}{2\,R^3},$$

$$\mathfrak{R}^3_{401} = -\mathfrak{R}^3_{410} = \frac{(R')^2}{2\,R^4} - \frac{R''}{2\,R^3}, \qquad \mathfrak{R}^4_{301} = -\mathfrak{R}^4_{310} = -\frac{(R')^2}{2\,R^4} + \frac{R''}{2\,R^3},$$

$$\mathfrak{R}^1_{202} = -\mathfrak{R}^1_{220} = \frac{i\,(R')^2}{2\,R^4} - \frac{i\,R''}{2\,R^3}, \qquad \mathfrak{R}^2_{102} = -\mathfrak{R}^2_{120} = -\frac{i\,(R')^2}{2\,R^4} + \frac{i\,R''}{2\,R^3}, \quad (6.14)$$

$$\mathfrak{R}^3_{402} = -\mathfrak{R}^3_{420} = -\frac{i\,(R')^2}{2\,R^4} + \frac{i\,R''}{2\,R^3}, \qquad \mathfrak{R}^4_{302} = -\mathfrak{R}^4_{320} = \frac{i\,(R')^2}{2\,R^4} - \frac{i\,R''}{2\,R^3},$$

$$\mathfrak{R}^1_{103} = -\mathfrak{R}^1_{130} = -\frac{(R')^2}{2\,R^4} + \frac{R''}{2\,R^3}, \qquad \mathfrak{R}^2_{203} = -\mathfrak{R}^2_{230} = \frac{(R')^2}{2\,R^4} - \frac{R''}{2\,R^3},$$

$$\mathfrak{R}^3_{303} = -\mathfrak{R}^3_{330} = \frac{(R')^2}{2\,R^4} - \frac{R''}{2\,R^3}, \qquad \mathfrak{R}^4_{403} = -\mathfrak{R}^4_{430} = -\frac{(R')^2}{2\,R^4} + \frac{R''}{2\,R^3},$$

$$\mathfrak{R}^1_{112} = -\mathfrak{R}^1_{121} = \frac{i}{2\,R^2} + \frac{i\,(R')^2}{2\,R^4}, \qquad \mathfrak{R}^2_{212} = -\mathfrak{R}^2_{221} = -\frac{i}{2\,R^2} - \frac{i\,(R')^2}{2\,R^4},$$

$$\mathfrak{R}^3_{312} = -\mathfrak{R}^3_{321} = \frac{i}{2\,R^2} + \frac{i\,(R')^2}{2\,R^4}, \qquad \mathfrak{R}^4_{412} = -\mathfrak{R}^4_{421} = -\frac{i}{2\,R^2} - \frac{i\,(R')^2}{2\,R^4},$$

$$\mathfrak{R}^1_{223} = -\mathfrak{R}^1_{232} = \frac{i}{2\,R^2} + \frac{i\,(R')^2}{2\,R^4}, \qquad \mathfrak{R}^2_{123} = -\mathfrak{R}^2_{132} = \frac{i}{2\,R^2} + \frac{i\,(R')^2}{2\,R^4}, \qquad (6.15)$$

$$\mathfrak{R}^3_{423} = -\mathfrak{R}^3_{432} = \frac{i}{2\,R^2} + \frac{i\,(R')^2}{2\,R^4}, \qquad \mathfrak{R}^4_{323} = -\mathfrak{R}^4_{332} = \frac{i}{2\,R^2} + \frac{i\,(R')^2}{2\,R^4},$$

$$\mathfrak{R}^1_{231} = -\mathfrak{R}^1_{213} = \frac{1}{2\,R^2} + \frac{(R')^2}{2\,R^4}, \qquad \mathfrak{R}^2_{131} = -\mathfrak{R}^2_{131} = -\frac{1}{2\,R^2} - \frac{(R')^2}{2\,R^4},$$

$$\mathfrak{R}^3_{431} = -\mathfrak{R}^3_{413} = \frac{1}{2\,R^2} + \frac{(R')^2}{2\,R^4}, \qquad \mathfrak{R}^4_{331} = -\mathfrak{R}^4_{331} = -\frac{1}{2\,R^2} - \frac{(R')^2}{2\,R^4}.$$

One can verify the relationship (6.9) by direct calculations as a test for consistency of (6.13), (6.14), and (6.15). The Ricci tensor is an important object in general relativity. It is used in Einstein's equation (see [1] or [5]). Using (6.13), we can calculate the components of the Ricci tensor in the frame $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$. As appears, the matrix of the Ricci tensor is diagonal. Here are its diagonal elements

$$R_{00} = \frac{3\,(R')^2}{R^4} - \frac{3\,R''}{R^3}, \qquad\qquad R_{11} = \frac{2}{R^2} + \frac{(R')^2}{R^4} + \frac{R''}{R^3},$$

$$R_{22} = \frac{2}{R^2} + \frac{(R')^2}{R^4} + \frac{R''}{R^3}, \qquad\qquad R_{33} = \frac{2}{R^2} + \frac{(R')^2}{R^4} + \frac{R''}{R^3}. \qquad (6.16)$$

And finally, using (6.16), we find the scalar curvature:

$$R_{\text{scalar}} = -\frac{6}{R^2} - \frac{6\,R''}{R^3}. \qquad (6.17)$$

This formula (6.17) coincides with the formula for the scalar curvature derived in § 112 of the book [1].

2. **Stereographic coordinates** in projection from the **South Pole**. We denote these coordinates with y^0, y^1, y^2, y^3. Their domain is the whole sphere S^3 except for the South Pole. The frames (4.13) are defined and smooth in this domain. This frame pair is linked to the coordinates y^0, y^1, y^2, y^3. The commutation coefficients for the frame $\mathbf{Y}_0$, $\mathbf{Y}$, $\mathbf{Y}_2$, $\mathbf{Y}_3$ in this pair are given by the formulas (1.19). Other formulas (6.10), (6.11), (6.12), (6.13), (6.14), (6.15), (6.16), and (6.17) are valid in this case upon changing x^0, x^1, x^2, x^3 for y^0, y^1, y^2, y^3 in them.

3. **Spherical coordinates**. We denote these coordinates with η^0, η^1, η^2, η^3 and also use special notations (5.1) and (5.2) for separate coordinates. The domain of spherical coordinates is the whole sphere S^3 except for both poles North and South. The frame pair (5.21) is used for spherical coordinates. The frames of this pair are defined and smooth in the domain of the spherical coordinates. The commutation coefficients for the frame $\mathbf{E}_0$, $\mathbf{E}$, $\mathbf{E}_2$, $\mathbf{E}_3$ are given by the formulas (5.9).

Substituting them into (6.5) we find the Γ-components of the metric connection in the frame $\mathbf{E}_0$, $\mathbf{E}$, $\mathbf{E}_2$, $\mathbf{E}_3$. Here is the list of nonzero ones of them:

$$\Gamma^0_{11} = \frac{R'}{R^2}, \qquad \Gamma^0_{22} = \frac{R'}{R^2}, \qquad \Gamma^0_{33} = \frac{R'}{R^2},$$

$$\Gamma^1_{10} = \frac{R'}{R^2}, \qquad \Gamma^2_{20} = \frac{R'}{R^2}, \qquad \Gamma^3_{30} = \frac{R'}{R^2},$$

$$\Gamma^1_{22} = -\frac{\cos\chi}{R\,\sin\chi}, \qquad \Gamma^2_{33} = -\frac{\cos\theta}{R\,\sin\chi\,\sin\theta}, \qquad \Gamma^2_{21} = \frac{\cos\chi}{R\,\sin\chi}, \qquad (6.18)$$

$$\Gamma^1_{33} = -\frac{\cos\chi}{R\,\sin\chi}, \qquad \Gamma^3_{32} = \frac{\cos\theta}{R\,\sin\chi\,\sin\theta}, \qquad \Gamma^3_{31} = \frac{\cos\chi}{R\,\sin\chi}.$$

Now we substitute (6.18) into (6.6) in order to get the A-components of the metric connection. Below is the list of nonzero ones of these components:

$$A^1_{12} = \frac{R'}{2\,R^2}, \qquad\qquad A^2_{11} = \frac{R'}{2\,R^2},$$

$$A^3_{14} = -\frac{R'}{2\,R^2}, \qquad\qquad A^4_{13} = -\frac{R'}{2\,R^2},$$

$$A^1_{21} = -\frac{i\,\cos\chi}{2\,R\,\sin\chi}, \qquad\qquad A^2_{22} = \frac{i\,\cos\chi}{2\,R\,\sin\chi},$$

$$A^3_{23} = -\frac{i\,\cos\chi}{2\,R\,\sin\chi}, \qquad\qquad A^4_{24} = \frac{i\,\cos\chi}{2\,R\,\sin\chi},$$

$$A^1_{22} = -\frac{i\,R'}{2\,R^2}, \qquad\qquad A^2_{21} = \frac{i\,R'}{2\,R^2}, \qquad\qquad (6.19)$$

$$A^3_{24} = \frac{i\,R'}{2\,R^2}, \qquad\qquad A^4_{23} = -\frac{i\,R'}{2\,R^2},$$

$$A^1_{31} = \frac{R'}{2\,R^2}, \qquad\qquad A^2_{32} = -\frac{R'}{2\,R^2},$$

$$A^3_{33} = -\frac{R'}{2\,R^2}, \qquad\qquad A^4_{34} = \frac{R'}{2\,R^2},$$

$$A^1_{32} = \frac{\cos\chi}{2\,R\,\sin\chi} - \frac{i\,\cos\theta}{2\,R\,\sin\chi\,\sin\theta},$$

$$A^2_{31} = -\frac{\cos\chi}{2\,R\,\sin\chi} - \frac{i\,\cos\theta}{2\,R\,\sin\chi\,\sin\theta},$$

$$A^3_{34} = \frac{\cos\chi}{2\,R\,\sin\chi} - \frac{i\,\cos\theta}{2\,R\,\sin\chi\,\sin\theta}, \qquad (6.20)$$

$$A^4_{33} = -\frac{\cos\chi}{2\,R\,\sin\chi} - \frac{i\,\cos\theta}{2\,R\,\sin\chi\,\sin\theta}.$$

The formulas (6.19) and (6.20) are analogs of (6.11) and (6.12) Using the Γ-components of the metric connection (6.18) and applying the formula (6.7) to them,

we derive the explicit formulas for the components of the Riemannian curvature tensor. Here is the list of nonzero components of $\mathbf{R}$:

$$R_{101}^0 = -R_{110}^0 = -\frac{(R')^2}{R^4} + \frac{R''}{R^3}, \qquad R_{001}^1 = -R_{010}^1 = -\frac{(R')^2}{R^4} + \frac{R''}{R^3},$$

$$R_{202}^0 = -R_{220}^0 = -\frac{(R')^2}{R^4} + \frac{R''}{R^3}, \qquad R_{002}^2 = -R_{020}^2 = -\frac{(R')^2}{R^4} + \frac{R''}{R^3},$$

$$R_{303}^0 = -R_{330}^0 = -\frac{(R')^2}{R^4} + \frac{R''}{R^3}, \qquad R_{003}^3 = -R_{030}^3 = -\frac{(R')^2}{R^4} + \frac{R''}{R^3}, \tag{6.21}$$

$$R_{212}^1 = -R_{221}^1 = \frac{1}{R^2} + \frac{(R')^2}{R^4}, \qquad R_{112}^2 = -R_{121}^2 = -\frac{1}{R^2} - \frac{(R')^2}{R^4},$$

$$R_{323}^2 = -R_{333}^2 = \frac{1}{R^2} + \frac{(R')^2}{R^4}, \qquad R_{223}^3 = -R_{232}^3 = -\frac{1}{R^2} - \frac{(R')^2}{R^4},$$

$$R_{131}^3 = -R_{113}^3 = \frac{1}{R^2} + \frac{(R')^2}{R^4}, \qquad R_{331}^1 = -R_{313}^1 = -\frac{1}{R^2} - \frac{(R')^2}{R^4}.$$

Comparing (6.21) with (6.13), we find that these formulas are identical. Though written for two different frames $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$, the components of the curvature tensor $\mathbf{R}$ do coincide. Then the same is true for the components of the Ricci tensor and the scalar curvature, i. e. they are given by the formulas (6.16) and (6.17) in spherical coordinates[1]. Applying the formula (6.9), we conclude that the components of the spinor curvature tensor $\mathfrak{R}$ in spherical coordinates should coincide with those calculated for stereographic coordinates. They are given by the formulas (6.14) and (6.15).

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[1] We should warn here that typically the spherical coordinates are equipped with their canonical holonomic frame (5.6). We equip them with the non-holonomic frame (5.7).