

ON KILLING VECTOR FIELDS OF A HOMOGENEOUS AND ISOTROPIC UNIVERSE IN CLOSED MODEL.

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ABSTRACT. Killing vector fields of a closed homogeneous and isotropic universe are studied. It is shown that in general case there is no time-like Killing vector fields in such a universe. Two exceptional cases are revealed.

1. INTRODUCTION.

Killing vector fields (infinitesimal isometries) are used in building vacuum states for quantum fields in a curved space-time (see [1] and [2]). We study a homogeneous and isotropic universe as an example of such a curved space-time.

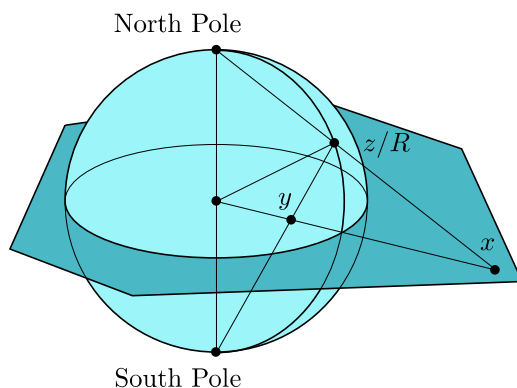


Fig. 1.1

This universe is diffeomorphic to the Cartesian product $M = \mathbb{R} \times S^3$ (see § 111 and § 112 in [3]). Its spinor structure was studied in [4]. We use some technique from [4] in present paper. In particular, we use the stereographic coordinates x^0, x^1, x^2, x^3 and y^0, y^1, y^2, y^3 as two local charts covering the whole universe. We call them *North Pole stereographic coordinates* and *South Pole stereographic coordinates* respectively. The domain of the North Pole stereographic coordinates x^0, x^1, x^2, x^3 is the whole sphere S^3 except for one point, which is called the North Pole. Similarly, the South Pole stereographic coordinates are defined on the whole sphere S^3 except for the diametrically opposite point, which is called the South Pole. Below are the transition functions relating the North Pole and South Pole stereographic coordinates in the intersection of their domains:

$$\begin{pmatrix} y^0 \\ y^1 \\ y^2 \\ y^3 \end{pmatrix} = \frac{1}{|x|^2} \begin{pmatrix} |x|^2 x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \frac{1}{|y|^2} \begin{pmatrix} |y|^2 y^0 \\ y^1 \\ y^2 \\ y^3 \end{pmatrix}. \quad (1.1)$$

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Here $|x|^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$ and $|y|^2 = (y^1)^2 + (y^2)^2 + (y^3)^2$. The Minkowski type metric \mathbf{g} in M is given by the following formulas:

$$ds^2 = R^2 (dx^0)^2 - \frac{4R^2 (dx^1)^2 + 4R^2 (dx^2)^2 + 4R^2 (dx^3)^2}{(|x|^2 + 1)^2}, \quad (1.2)$$

$$ds^2 = R^2 (dy^0)^2 - \frac{4R^2 (dy^1)^2 + 4R^2 (dy^2)^2 + 4R^2 (dy^3)^2}{(|y|^2 + 1)^2}. \quad (1.3)$$

Looking at (1.2) and (1.3), we see that the formulas for metric in two different stereographic coordinates are very similar. Therefore, we can derive some formulas in North Pole stereographic coordinates and then transform them to South Pole coordinates by substituting y^0, y^1, y^2, y^3 for x^0, x^1, x^2, x^3 without use of the transition functions (1.1). The parameter R in the formulas (1.2) and (1.3) is interpreted as the radius of the sphere S^3 in its realization as a hypersurface in the Euclidean space \mathbb{R}^4 . This parameter is not a constant:

$$R = R(x^0) = R(y^0). \quad (1.4)$$

According to [3], the time variable t is introduced through the following formula:

$$R dx^0 = R dy^0 = c dt \quad (c \text{ is the light velocity}). \quad (1.5)$$

Then we can write (1.4) as $R = R(t)$. If $R(t)$ is constant, we say that the universe is stable, if $R(t)$ is an increasing function, we say that the universe is expanding, and if $R(t)$ is a decreasing function, we say that the universe is contracting. Oscillatory regimes are also possible. The main goal in this paper is to study under which conditions for the function (1.4) the universe $M = \mathbb{R} \times S^3$ has at least one time-like Killing vector field.

2. CONNECTION COMPONENTS AND THE CURVATURE TENSOR.

The metric tensor \mathbf{g} is determined by a diagonal matrix g_{ij} in North Pole stereographic coordinates. Its components are determined by the formula (1.2):

$$g_{00} = R^2, \quad g_{11} = g_{22} = g_{33} = -\frac{4R^2}{(|x|^2 + 1)^2}. \quad (2.1)$$

The dual metric tensor \mathbf{g} is also given by a diagonal matrix:

$$g^{00} = \frac{1}{R^2}, \quad g^{11} = g^{22} = g^{33} = -\frac{(|x|^2 + 1)^2}{4R^2}. \quad (2.2)$$

Now we choose the following well-known formula in order to calculate the components of the symmetric Levi-Civita connection:

$$\Gamma_{ij}^k = \sum_{s=0}^3 \frac{g^{ks}}{2} \left(\frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s} \right). \quad (2.3)$$

Note that, unlike [4], here we do not use non-holonomic frames since we do not need to deal with spinors in this paper. Applying (2.3) to (2.1) and (2.2), we get the following complete list of the nonzero components of Γ_{ij}^k :

$$\begin{aligned}
\Gamma_{11}^0 &= \frac{4R'}{R(|x|^2+1)^2}, & \Gamma_{22}^0 &= \frac{4R'}{R(|x|^2+1)^2}, & \Gamma_{33}^0 &= \frac{4R'}{R(|x|^2+1)^2}, \\
\Gamma_{11}^1 &= -\frac{2(x^1)}{|x|^2+1}, & \Gamma_{22}^1 &= \frac{2(x^1)}{|x|^2+1}, & \Gamma_{33}^1 &= \frac{2(x^1)}{|x|^2+1}, \\
\Gamma_{11}^2 &= \frac{2(x^2)}{|x|^2+1}, & \Gamma_{22}^2 &= -\frac{2(x^2)}{|x|^2+1}, & \Gamma_{33}^2 &= \frac{2(x^2)}{|x|^2+1}, \\
\Gamma_{11}^3 &= \frac{2(x^3)}{|x|^2+1}, & \Gamma_{22}^3 &= \frac{2(x^3)}{|x|^2+1}, & \Gamma_{33}^3 &= -\frac{2(x^3)}{|x|^2+1},
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
\Gamma_{12}^2 &= \Gamma_{21}^2 = -\frac{2(x^1)}{|x|^2+1}, & \Gamma_{13}^3 &= \Gamma_{31}^3 = -\frac{2(x^1)}{|x|^2+1}, \\
\Gamma_{23}^3 &= \Gamma_{32}^3 = -\frac{2(x^2)}{|x|^2+1}, & \Gamma_{21}^1 &= \Gamma_{12}^1 = -\frac{2(x^2)}{|x|^2+1}, \\
\Gamma_{31}^1 &= \Gamma_{13}^1 = -\frac{2(x^3)}{|x|^2+1}, & \Gamma_{32}^2 &= \Gamma_{23}^2 = -\frac{2(x^3)}{|x|^2+1}, \\
\Gamma_{00}^0 &= \frac{R'}{R}, & \Gamma_{01}^1 &= \Gamma_{10}^1 = \frac{R'}{R}, \\
\Gamma_{02}^2 &= \Gamma_{20}^2 = \frac{R'}{R}, & \Gamma_{03}^3 &= \Gamma_{30}^3 = \frac{R'}{R}.
\end{aligned} \tag{2.5}$$

Note that here we have 31 nonzero connection components, while in [4] in the case of a non-holonomic frame we had 18.

The next step is to calculate the components of the Riemannian curvature tensor \mathbf{R} . They are given by the well-known formula

$$R_{qij}^p = \frac{\partial \Gamma_{jq}^p}{\partial x^i} - \frac{\partial \Gamma_{iq}^p}{\partial x^j} + \sum_{h=0}^3 \left(\Gamma_{ih}^p \Gamma_{jq}^h - \Gamma_{jh}^p \Gamma_{iq}^h \right). \tag{2.6}$$

Applying (2.6) to (2.4) and (2.5), we derive the following expressions

$$\begin{aligned}
R_{101}^0 &= -R_{110}^0 = 4 \frac{R''R - (R')^2}{R^2(|x|^2+1)^2}, & R_{001}^1 &= -R_{010}^1 = \frac{R''R - (R')^2}{R^2}, \\
R_{202}^0 &= -R_{220}^0 = 4 \frac{R''R - (R')^2}{R^2(|x|^2+1)^2}, & R_{002}^2 &= -R_{020}^2 = \frac{R''R - (R')^2}{R^2}, \\
R_{303}^0 &= -R_{330}^0 = 4 \frac{R''R - (R')^2}{R^2(|x|^2+1)^2}, & R_{003}^3 &= -R_{030}^3 = \frac{R''R - (R')^2}{R^2},
\end{aligned} \tag{2.7}$$

$$\begin{aligned}
R_{212}^1 &= -R_{221}^1 = R_{121}^2 = -R_{112}^2 = 4 \frac{(R')^2 + R^2}{R^2 (|x|^2 + 1)^2}, \\
R_{323}^2 &= -R_{332}^2 = R_{232}^3 = -R_{223}^3 = 4 \frac{(R')^2 + R^2}{R^2 (|x|^2 + 1)^2}, \\
R_{131}^3 &= -R_{113}^3 = R_{313}^1 = -R_{331}^1 = 4 \frac{(R')^2 + R^2}{R^2 (|x|^2 + 1)^2}.
\end{aligned} \tag{2.8}$$

Using the above formulas (2.7) and (2.8), we can calculate the components of the Ricci tensor. Here is the list of nonzero ones of them:

$$\begin{aligned}
R_{00} &= \frac{3(R')^2 - 3RR''}{R^2}, & R_{11} &= \frac{8R^2 + 4(R')^2 + 4RR''}{R^2 (|x|^2 + 1)^2}, \\
R_{22} &= \frac{8R^2 + 4(R')^2 + 4RR''}{R^2 (|x|^2 + 1)^2}, & R_{33} &= \frac{8R^2 + 4(R')^2 + 4RR''}{R^2 (|x|^2 + 1)^2}.
\end{aligned} \tag{2.9}$$

And finally, using (2.9), we calculate the scalar curvature:

$$R_{\text{scalar}} = -\frac{6}{R^2} - \frac{6R''}{R^3}. \tag{2.10}$$

As we see, the scalar curvature given by the formula (2.10) coincides with the scalar curvature calculated in [1] and [4] for this particular model of the universe.

3. DIFFERENTIAL EQUATIONS FOR KILLING VECTOR FIELDS.

Killing vector fields are also known as infinitesimal isometries. Local one-parametric diffeomorphism groups generated by these vector fields are composed by isometries — they preserve the metric \mathbf{g} . Therefore, if \mathbf{X} is a Killing vector field in M , then the Lie derivative $L_{\mathbf{X}}$, when applied to \mathbf{g} , yields zero:

$$L_{\mathbf{X}}(\mathbf{g}) = 0. \tag{3.1}$$

In the coordinate form the equation (3.1) is written as follows:

$$\sum_{s=0}^3 X^s \frac{\partial g_{ij}}{\partial x^s} + \sum_{s=0}^3 g_{sj} \frac{\partial X^s}{\partial x^i} + \sum_{s=0}^3 g_{is} \frac{\partial X^s}{\partial x^j} = 0. \tag{3.2}$$

Let's replace the partial derivatives in (3.2) with the covariant derivatives:

$$\frac{\partial g_{ij}}{\partial x^s} = \nabla_s g_{ij} + \sum_{k=0}^3 \Gamma_{si}^k g_{kj} + \sum_{k=0}^3 \Gamma_{sj}^k g_{ik}, \tag{3.3}$$

$$\frac{\partial X^s}{\partial x^i} = \nabla_i X^s - \sum_{k=0}^3 \Gamma_{ik}^s X^k, \tag{3.4}$$

$$\frac{\partial X^s}{\partial x^j} = \nabla_j X^s - \sum_{k=0}^3 \Gamma_{jk}^s X^k. \tag{3.5}$$

Substituting (3.3), (3.4), and (3.5) into (3.2) and taking into account the symmetry of g_{ij} and Γ_{ij}^k with respect to the indices i and j , we get

$$\nabla_s g_{ij} + \sum_{s=0}^3 g_{sj} \nabla_i X^s + \sum_{s=0}^3 g_{is} \nabla_j X^s = 0. \quad (3.6)$$

Now let's remember that the metric \mathbf{g} is concordant with its metric connection Γ , i. e. $\nabla \mathbf{g} = 0$. As a result the equation (3.6) is reduced to

$$\sum_{s=0}^3 g_{sj} \nabla_i X^s + \sum_{s=0}^3 g_{is} \nabla_j X^s = 0. \quad (3.7)$$

In a metric manifold each vector field \mathbf{X} is associated with some unique covector field. This covector field is usually denoted by the same symbol \mathbf{X} . The components of such two associated vectorial and covectorial fields are related to each other through the index lowering and index raising procedures:

$$X_i = \sum_{j=0}^3 g_{ij} X^j, \quad X^i = \sum_{j=0}^3 g^{ij} X_j. \quad (3.8)$$

Applying (3.8) to (3.7) and taking into account that $\nabla \mathbf{g} = 0$, we derive

$$\nabla_i X_j + \nabla_j X_i = 0. \quad (3.9)$$

The equation (3.9) is a basic equation for Killing vector fields we are going to study in this paper. It is written in the covectorial form. Let's denote

$$\nabla_i X_j = Y_{ij} \quad \text{for } i < j. \quad (3.10)$$

In terms of (3.10) the equation (3.9) can be rewritten in the following form:

$$\nabla_i X_j = \begin{cases} Y_{ij} & \text{for } i < j, \\ 0 & \text{for } i = j, \\ -Y_{ji} & \text{for } i > j. \end{cases} \quad (3.11)$$

The equations (3.11) look like a Pfaff system of first order PDE's if we treat Y_{ij} as new undetermined functions. However, in this case we need to write the differential equations for these functions. For this purpose let's differentiate (3.9):

$$\nabla_k \nabla_i X_j + \nabla_k \nabla_j X_i = 0. \quad (3.12)$$

Then we triplicate the equations (3.12) by means of the cyclic transposition of indices: $i \rightarrow j \rightarrow k \rightarrow i$. As a result we get other two copies of the equation (3.12):

$$\nabla_i \nabla_j X_k + \nabla_i \nabla_k X_j = 0, \quad (3.13)$$

$$\nabla_j \nabla_k X_i + \nabla_j \nabla_i X_k = 0. \quad (3.14)$$

Now let's add (3.13) and (3.14), then subtract (3.12) from them. As a result we get

$$\nabla_i \nabla_j X_k + \nabla_j \nabla_i X_k = (\nabla_k \nabla_i - \nabla_i \nabla_k) X_j + (\nabla_k \nabla_j - \nabla_j \nabla_k) X_i. \quad (3.15)$$

In order to transform the equality (3.15) we use the following well-known identity:

$$(\nabla_i \nabla_j - \nabla_j \nabla_i) X_k = - \sum_{s=0}^3 R_{kij}^s X_s. \quad (3.16)$$

Here R_{kij}^s are the components of the Riemannian curvature tensor \mathbf{R} (see (2.6), (2.7), and (2.8) above). Applying (3.16) to (3.15), we derive

$$2 \nabla_i \nabla_j X_k = - \sum_{s=0}^3 R_{kij}^s X_s - \sum_{s=0}^3 R_{jki}^s X_s - \sum_{s=0}^3 R_{ikj}^s X_s. \quad (3.17)$$

Now let's recall the following identities:

$$R_{ijk}^s + R_{ikj}^s = 0, \quad R_{ijk}^s + R_{kij}^s + R_{jki}^s = 0. \quad (3.18)$$

These are the well-known identities for the components of the curvature tensor. Their proof can be found in [5]. Applying (3.18) to (3.17), we get

$$\nabla_i \nabla_j X_k = \sum_{s=0}^3 R_{ijk}^s X_s. \quad (3.19)$$

If we remember the notations (3.10), then (3.19) can be rewritten as

$$\nabla_i Y_{jk} = \sum_{s=0}^3 R_{ijk}^s X_s. \quad (3.20)$$

Both (3.11) and (3.20) form a complete system of Pfaff equations for ten functions $X_0, X_1, X_2, X_3, Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}$. The following theorem is an immediate consequence of this observation.

Theorem 3.1. *A four-dimensional space-time manifold M can have at most ten linearly independent Killing vector fields.*

The actual number of isometries depends on the so-called compatibility conditions for the Pfaff equations (3.11) and (3.20). In order to derive these compatibility conditions, let's calculate $\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k$ and $\nabla_i \nabla_j Y_{pq} - \nabla_j \nabla_i Y_{pq}$ on the base of the Pfaff equations (3.11) and (3.20). The inequalities in (3.11) produce many special cases that should be studied separately. In order to avoid this inconvenience we extend the definition of Y_{ij} . In (3.10) they are defined for $i < j$. Let's set

$$Y_{ij} = \begin{cases} 0 & \text{for } i = j; \\ -Y_{ji} & \text{for } i > j. \end{cases} \quad (3.21)$$

Due to the extension (3.21) of (3.10) we can write (3.11) as

$$\nabla_i X_j = Y_{ij} \quad (3.22)$$

for all i and j , but we should keep in mind that only 6 of 16 components of the skew-symmetric tensorial field \mathbf{Y} are independent. Due to the skew symmetry $R_{ijk}^s = -R_{ikj}^s$ the extension (3.21) of (3.10) and the extension (3.22) of (3.11) are compatible with (3.20). Now for $\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k$ we have

$$\nabla_i \nabla_j X_k - \nabla_j \nabla_i X_k = \nabla_i Y_{jk} - \nabla_j Y_{ik}. \quad (3.23)$$

Applying the equation (3.20) to the right hand side of (3.23) and applying the identity (3.16) to its left hand side, we derive

$$-\sum_{s=0}^3 R_{kij}^s X_s = \sum_{s=0}^3 R_{ijk}^s X_s - \sum_{s=0}^3 R_{jik}^s X_s. \quad (3.24)$$

It is easy to see that the compatibility condition (3.24) is fulfilled identically due to the properties (3.18) of the curvature tensor \mathbf{R} .

Now we proceed to the compatibility conditions derived from $\nabla_i \nabla_j Y_{pq} - \nabla_j \nabla_i Y_{pq}$. In this case, applying the equation (3.20), we get

$$\nabla_i \nabla_j Y_{pq} - \nabla_j \nabla_i Y_{pq} = \sum_{s=0}^3 \nabla_i (R_{j pq}^s X_s) - \sum_{s=0}^3 \nabla_j (R_{i pq}^s X_s). \quad (3.25)$$

The left hand side of the equality (3.25) is transformed by means of the identity

$$\nabla_i \nabla_j Y_{pq} - \nabla_j \nabla_i Y_{pq} = -\sum_{s=0}^3 R_{pij}^s Y_{sq} - \sum_{s=0}^3 R_{qij}^s Y_{ps}. \quad (3.26)$$

The identity (3.26) is a tensorial generalization of (3.16). Applying (3.26) to (3.25) and taking into account (3.22), we transform (3.25) as follows:

$$\begin{aligned} & -\sum_{s=0}^3 R_{pij}^s Y_{sq} - \sum_{s=0}^3 R_{qij}^s Y_{ps} = \sum_{s=0}^3 \nabla_i R_{j pq}^s X_s + \\ & + \sum_{s=0}^3 R_{j pq}^s Y_{is} - \sum_{s=0}^3 \nabla_j R_{i pq}^s X_s - \sum_{s=0}^3 R_{i pq}^s Y_{js}. \end{aligned} \quad (3.27)$$

The equality (3.27) is a non-trivial compatibility condition for the system of Pfaff equations (3.22) and (3.20). In the next section we shall study this equality for our particular case, where $M = \mathbb{R} \times S^3$.

4. SIMPLIFYING THE COMPATIBILITY CONDITIONS.

Note that the compatibility equations (3.27) contain the covariant derivatives of the curvature tensor. Therefore, we begin our study of (3.27) with calculating these covariant derivatives. They are given by the formula:

$$\nabla_s R_{qij}^p = \frac{R_{qij}^p}{\partial x^s} + \sum_{h=0}^3 \Gamma_{sh}^p R_{qij}^h - \sum_{h=0}^3 \Gamma_{sq}^h R_{hij}^p - \sum_{h=0}^3 \Gamma_{si}^h R_{qjh}^p - \sum_{h=0}^3 \Gamma_{sj}^h R_{qih}^p.$$

We substitute (2.4), (2.5), (2.7), and (2.8) into this formula and get the following list of nonzero components $\nabla_s R_{qij}^p$ in North Pole stereographic coordinates:

$$\begin{aligned} \nabla_0 R_{101}^0 &= -\nabla_0 R_{110}^0 = \nabla_0 R_{202}^0 = \\ &= -\nabla_0 R_{220}^0 = \nabla_0 R_{303}^0 = -\nabla_0 R_{330}^0 = \\ &= \frac{16 (R')^3 - 20 R'' R' R + 4 R''' R^2}{R^3 (|x|^2 + 1)^2}, \end{aligned} \quad (4.1)$$

$$\begin{aligned} \nabla_0 R_{001}^1 &= -\nabla_0 R_{010}^1 = \nabla_0 R_{002}^2 = \\ &= -\nabla_0 R_{020}^2 = \nabla_0 R_{003}^3 = -\nabla_0 R_{030}^3 = \\ &= \frac{4 (R')^3 - 5 R'' R' R + R''' R^2}{R^3}, \end{aligned} \quad (4.2)$$

$$\begin{aligned} \nabla_0 R_{212}^1 &= -\nabla_0 R_{221}^1 = \nabla_0 R_{121}^2 = -\nabla_0 R_{112}^2 = \\ \nabla_0 R_{323}^2 &= -\nabla_0 R_{332}^2 = \nabla_0 R_{232}^3 = -\nabla_0 R_{223}^3 = \\ \nabla_0 R_{131}^3 &= -\nabla_0 R_{113}^3 = \nabla_0 R_{313}^1 = -\nabla_0 R_{331}^1 = \\ &= \frac{-16 (R')^3 + 8 R'' R' R - 8 R' R^2}{R^3 (|x|^2 + 1)^2}, \end{aligned} \quad (4.3)$$

$$\begin{aligned} \nabla_1 R_{212}^0 &= -\nabla_1 R_{221}^0 = \nabla_2 R_{323}^0 = -\nabla_2 R_{332}^0 = \\ \nabla_3 R_{131}^0 &= -\nabla_3 R_{113}^0 = \nabla_1 R_{313}^0 = -\nabla_1 R_{331}^0 = \\ \nabla_2 R_{121}^0 &= -\nabla_2 R_{112}^0 = \nabla_3 R_{232}^0 = -\nabla_3 R_{223}^0 = \\ &= \frac{32 (R')^3 - 16 R'' R' R + 16 R' R^2}{R^3 (|x|^2 + 1)^4}, \end{aligned} \quad (4.4)$$

$$\begin{aligned} \nabla_1 R_{012}^2 &= -\nabla_1 R_{021}^2 = \nabla_2 R_{023}^3 = -\nabla_2 R_{032}^3 = \\ \nabla_3 R_{031}^1 &= -\nabla_3 R_{013}^1 = \nabla_1 R_{013}^3 = -\nabla_1 R_{031}^3 = \\ \nabla_2 R_{021}^1 &= -\nabla_2 R_{012}^1 = \nabla_3 R_{032}^2 = -\nabla_3 R_{023}^2 = \\ &= \frac{8 (R')^3 - 4 R'' R' R + 4 R' R^2}{R^3 (|x|^2 + 1)^2}, \end{aligned} \quad (4.5)$$

$$\begin{aligned} \nabla_1 R_{220}^1 &= -\nabla_1 R_{202}^1 = \nabla_2 R_{330}^2 = -\nabla_2 R_{303}^2 = \\ \nabla_3 R_{110}^3 &= -\nabla_3 R_{101}^3 = \nabla_1 R_{330}^1 = -\nabla_1 R_{303}^1 = \\ \nabla_2 R_{110}^2 &= -\nabla_2 R_{101}^2 = \nabla_3 R_{320}^3 = -\nabla_3 R_{202}^3 = \\ &= \frac{8 (R')^3 - 4 R'' R' R + 4 R' R^2}{R^3 (|x|^2 + 1)^2}, \end{aligned} \quad (4.6)$$

$$\begin{aligned}
\nabla_1 R_{102}^2 &= -\nabla_1 R_{120}^2 = \nabla_2 R_{203}^3 = -\nabla_2 R_{230}^3 = \\
\nabla_3 R_{301}^1 &= -\nabla_3 R_{310}^1 = \nabla_1 R_{103}^3 = -\nabla_1 R_{130}^3 = \\
\nabla_2 R_{201}^1 &= -\nabla_2 R_{210}^1 = \nabla_3 R_{302}^2 = -\nabla_3 R_{320}^2 = \\
&= \frac{8(R')^3 - 4R''R'R + 4R'R^2}{R^3(|x|^2 + 1)^2}.
\end{aligned} \tag{4.7}$$

Now we substitute (4.1), (4.2), (4.3), (4.4), (4.5), (4.6), and (4.7) into (3.27). We also substitute (2.7) and (2.8) into (3.27). As a result we obtain a series of linear algebraic equations for the functions $X_0, X_1, X_2, X_3, Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}$. Since the expressions in both sides of (3.27) are skew-symmetric with respect to i and j and with respect to p and q , we could have at most 36 mutually independent equations. However, in our particular case the number of mutually independent equations is 5. Here is the list of these five equations:

$$\begin{cases} (4(R')^3 - 5R''R'R + R'''R^2)X_0 = 0, \\ R'(2(R')^2 - R''R + R^2)X_0 = 0, \end{cases} \tag{4.8}$$

$$\begin{cases} (2(R')^2 - R''R + R^2)(R'X_1 - RY_{01}) = 0, \\ (2(R')^2 - R''R + R^2)(R'X_2 - RY_{02}) = 0, \\ (2(R')^2 - R''R + R^2)(R'X_3 - RY_{03}) = 0. \end{cases} \tag{4.9}$$

As we see the compatibility equations (4.8) and (4.9) depend essentially on the function (1.4) and its derivatives. These simplified equations will be studied in the next two sections.

5. SPACIAL ROTATIONS.

Note that the functions Y_{12}, Y_{13}, Y_{23} are not presented in the equations (4.8) and (4.9). This fact reflects the spherical symmetry of our universe $M = \mathbb{R} \times S^3$. It is known that the sphere S^3 has a 6-parametric group of isometries. These isometries produce 6 linearly independent Killing vector fields corresponding to 3 meridional and 3 equatorial rotations. Now we write these vector fields explicitly.

Meridional rotation in the plane z^1Oz^4 . This rotation induces a Killing vector field \mathbf{X} in M expressed by the formula

$$\mathbf{X} = \frac{2(x^1)^2 - |x|^2 + 1}{2} \frac{\partial}{\partial x^1} + (x^1)(x^2) \frac{\partial}{\partial x^2} + (x^1)(x^3) \frac{\partial}{\partial x^3} \tag{5.1}$$

in the North Pole stereographic coordinates. Applying the index lowering procedure (3.8) to the components of (5.1), we get the covectorial components of \mathbf{X} :

$$\begin{aligned}
X_0 &= 0, & X_1 &= -\frac{4(x^1)^2 - 2|x|^2 + 2}{(|x|^2 + 1)^2} R^2, \\
X_2 &= -\frac{4(x^1)(x^2)}{(|x|^2 + 1)^2} R^2, & X_3 &= -\frac{4(x^1)(x^3)}{(|x|^2 + 1)^2} R^2.
\end{aligned} \tag{5.2}$$

Then, using the formula (3.10), we calculate the functions $Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}$ associated with the Killing vector field (5.1):

$$\begin{aligned} Y_{01} &= -\frac{2RR'(2(x^1)^2 - |x|^2 + 1)}{(|x|^2 + 1)^2}, & Y_{02} &= -\frac{4RR'(x^1)(x^2)}{(|x|^2 + 1)^2}, \\ Y_{03} &= -\frac{4RR'(x^1)(x^3)}{(|x|^2 + 1)^2}, & Y_{12} &= -\frac{8R^2(x^2)}{(|x|^2 + 1)^3}, \\ Y_{23} &= 0, & Y_{13} &= -\frac{8R^2(x^3)}{(|x|^2 + 1)^3}. \end{aligned} \quad (5.3)$$

Let's substitute $x^1 = x^2 = x^3 = 0$ into (5.2) and (5.3). As a result we get

$$\begin{aligned} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ -2R^2 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{01} \\ Y_{02} \\ Y_{03} \end{pmatrix} &= \begin{pmatrix} -2RR' \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{12} \\ Y_{13} \\ Y_{23} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (5.4)$$

The quantities listed in the formulas (5.4) can be treated as initial data for the Pfaff equations (3.20) and (3.22).

Meridional rotation in the plane z^2Oz^4 . This case is very similar to the previous one. Here is the formula for the Killing vector field in this case:

$$\mathbf{X} = (x^2)(x^1) \frac{\partial}{\partial x^1} + \frac{2(x^2)^2 - |x|^2 + 1}{2} \frac{\partial}{\partial x^2} + (x^2)(x^3) \frac{\partial}{\partial x^3}. \quad (5.5)$$

Below are the covariant components of the vector (5.5)

$$\begin{aligned} X_0 &= 0, & X_1 &= -\frac{4(x^2)(x^1)}{(|x|^2 + 1)^2} R^2, \\ X_2 &= -\frac{4(x^2)^2 - 2|x|^2 + 2}{(|x|^2 + 1)^2} R^2, & X_3 &= -\frac{4(x^2)(x^3)}{(|x|^2 + 1)^2} R^2. \end{aligned} \quad (5.6)$$

Substituting (5.6) into (3.10) we obtain the functions $Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}$:

$$\begin{aligned} Y_{01} &= -\frac{4RR'(x^2)(x^1)}{(|x|^2 + 1)^2}, & Y_{02} &= -\frac{2RR'(2(x^2)^2 - |x|^2 + 1)}{(|x|^2 + 1)^2}, \\ Y_{03} &= -\frac{4RR'(x^2)(x^3)}{(|x|^2 + 1)^2}, & Y_{12} &= \frac{8R^2(x^1)}{(|x|^2 + 1)^3}, \\ Y_{23} &= -\frac{8R^2(x^3)}{(|x|^2 + 1)^3}, & Y_{13} &= 0. \end{aligned} \quad (5.7)$$

By setting $x^1 = x^2 = x^3 = 0$ in (5.6) and (5.7) we derive

$$\begin{aligned} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ -2R^2 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{01} \\ Y_{02} \\ Y_{03} \end{pmatrix} &= \begin{pmatrix} 0 \\ -2RR' \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{12} \\ Y_{13} \\ Y_{23} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (5.8)$$

Like (5.4), the quantities (5.8) are initial data for the equations (3.20) and (3.22).

Meridional rotation in the plane z^3Oz^4 . This case is also very similar to the previous cases. Here is the formula for the Killing vector field in this case:

$$\mathbf{X} = (x^3)(x^1) \frac{\partial}{\partial x^1} + (x^3)(x^2) \frac{\partial}{\partial x^2} + \frac{2(x^3)^2 - |x|^2 + 1}{2} \frac{\partial}{\partial x^3}. \quad (5.9)$$

Below are the covariant components of the vector (5.9):

$$\begin{aligned} X_0 &= 0, & X_1 &= -\frac{4(x^3)(x^1)}{(|x|^2 + 1)^2} R^2, \\ X_2 &= -\frac{4(x^3)(x^2)}{(|x|^2 + 1)^2} R^2, & X_3 &= -\frac{4(x^3)^2 - 2|x|^2 + 2}{(|x|^2 + 1)^2} R^2. \end{aligned} \quad (5.10)$$

Now, substituting (5.10) into (3.10), we find $Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}$:

$$\begin{aligned} Y_{01} &= -\frac{4RR'(x^3)(x^1)}{(|x|^2 + 1)^2}, & Y_{02} &= -\frac{4RR'(x^3)(x^2)}{(|x|^2 + 1)^2}, \\ Y_{03} &= -\frac{2RR'(2(x^3)^2 - |x|^2 + 1)}{(|x|^2 + 1)^2}, & Y_{12} &= 0, \\ Y_{23} &= \frac{8R^2(x^2)}{(|x|^2 + 1)^3}, & Y_{13} &= \frac{8R^2(x^1)}{(|x|^2 + 1)^3}. \end{aligned} \quad (5.11)$$

By setting $x^1 = x^2 = x^3 = 0$ in (5.10) and (5.11) we obtain

$$\begin{aligned} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2R^2 \end{pmatrix}, & \begin{pmatrix} Y_{01} \\ Y_{02} \\ Y_{03} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ -2RR' \end{pmatrix}, & \begin{pmatrix} Y_{12} \\ Y_{13} \\ Y_{23} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (5.12)$$

The quantities (5.12) are initial data for the equations (3.20) and (3.22).

The next three cases are produced by the equatorial rotations. They are somewhat different from meridional ones.

Equatorial rotation in the plane z^1Oz^2 . The Killing vector field in this case is given by the following formula:

$$\mathbf{X} = (x^2) \frac{\partial}{\partial x^1} - (x^1) \frac{\partial}{\partial x^2}. \quad (5.13)$$

This formula is more simple than (5.1), (5.5), or (5.9). Here are the covariant components of the vector field given by the formula (5.13):

$$\begin{aligned} X_0 &= 0, & X_1 &= -\frac{4R^2(x^2)}{(|x|^2 + 1)^2} R^2, \\ X_2 &= \frac{4R^2(x^1)}{(|x|^2 + 1)^2} R^2, & X_3 &= 0. \end{aligned} \quad (5.14)$$

Substituting (5.14) into (3.10), we calculate the functions $Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}$ for the vector field given by the formula (5.13):

$$\begin{aligned} Y_{01} &= -\frac{4RR'(x^2)}{(|x|^2+1)^2}, & Y_{02} &= \frac{4RR'(x^1)}{(|x|^2+1)^2}, \\ Y_{03} &= 0, & Y_{12} &= \frac{4R^2(2(x^3)^2-|x|^2+1)}{(|x|^2+1)^3}, \\ Y_{23} &= \frac{8R^2(x^3)(x^1)}{(|x|^2+1)^3}, & Y_{13} &= -\frac{8R^2(x^3)(x^2)}{(|x|^2+1)^3}. \end{aligned} \quad (5.15)$$

By setting $x^1 = x^2 = x^3 = 0$ in (5.14) and (5.15), we obtain

$$\begin{aligned} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{01} \\ Y_{02} \\ Y_{03} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{12} \\ Y_{13} \\ Y_{23} \end{pmatrix} &= \begin{pmatrix} 4R^2 \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (5.16)$$

The quantities (5.16) are initial data for the equations (3.20) and (3.22).

Equatorial rotation in the plane z^2Oz^3 . This case is very similar to the previous one. Here is the formula for the Killing vector field in this case:

$$\mathbf{X} = (x^3) \frac{\partial}{\partial x^2} - (x^2) \frac{\partial}{\partial x^3}. \quad (5.17)$$

Below are the covariant components of the vector (5.17):

$$\begin{aligned} X_0 &= 0, & X_1 &= 0, \\ X_2 &= -\frac{4R^2(x^3)}{(|x|^2+1)^2} R^2, & X_3 &= \frac{4R^2(x^2)}{(|x|^2+1)^2} R^2. \end{aligned} \quad (5.18)$$

Substituting (5.18) into (3.10), we calculate $Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}$:

$$\begin{aligned} Y_{01} &= 0, & Y_{02} &= -\frac{4RR'(x^3)}{(|x|^2+1)^2}, \\ Y_{03} &= \frac{4RR'(x^2)}{(|x|^2+1)^2}, & Y_{12} &= \frac{8R^2(x^1)(x^3)}{(|x|^2+1)^3}, \\ Y_{23} &= \frac{4R^2(2(x^1)^2-|x|^2+1)}{(|x|^2+1)^3}, & Y_{13} &= -\frac{8R^2(x^1)(x^2)}{(|x|^2+1)^3}. \end{aligned} \quad (5.19)$$

By setting $x^1 = x^2 = x^3 = 0$ in (5.18) and (5.19), we obtain

$$\begin{aligned} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{01} \\ Y_{02} \\ Y_{03} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{12} \\ Y_{13} \\ Y_{23} \end{pmatrix} &= \begin{pmatrix} 0 \\ 4R^2 \\ 0 \end{pmatrix}. \end{aligned} \quad (5.20)$$

The quantities (5.20) are initial data for the equations (3.20) and (3.22).

Equatorial rotation in the plane $z^3 O z^1$. The Killing vector field in this last case is given by the following formula:

$$\mathbf{X} = -(x^3) \frac{\partial}{\partial x^1} + (x^1) \frac{\partial}{\partial x^3}. \quad (5.21)$$

Below are the covariant components of the vector (5.21):

$$\begin{aligned} X_0 &= 0, & X_1 &= \frac{4R^2(x^3)}{(|x|^2+1)^2} R^2, \\ X_2 &= 0, & X_3 &= -\frac{4R^2(x^2)}{(|x|^2+1)^2} R^2. \end{aligned} \quad (5.22)$$

Substituting (5.22) into (3.10), we find the functions $Y_{01}, Y_{02}, Y_{03}, Y_{12}, Y_{13}, Y_{23}$:

$$\begin{aligned} Y_{01} &= \frac{4RR'(x^3)}{(|x|^2+1)^2}, & Y_{02} &= 0, \\ Y_{03} &= -\frac{4RR'(x^1)}{(|x|^2+1)^2}, & Y_{12} &= \frac{8R^2(x^2)(x^3)}{(|x|^2+1)^3}, \\ Y_{23} &= -\frac{8R^2(x^2)(x^1)}{(|x|^2+1)^3}, & Y_{13} &= -\frac{4R^2(2(x^2)^2 - |x|^2 + 1)}{(|x|^2+1)^3}. \end{aligned} \quad (5.23)$$

By setting $x^1 = x^2 = x^3 = 0$ in (5.12) and (5.23), we obtain

$$\begin{aligned} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{01} \\ Y_{02} \\ Y_{03} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} Y_{12} \\ Y_{13} \\ Y_{23} \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \\ -4R^2 \end{pmatrix}. \end{aligned} \quad (5.24)$$

The quantities (5.24) are initial data for the equations (3.20) and (3.22).

6. ANALYSIS OF THE COMPATIBILITY CONDITIONS.

Six linearly independent Killing vector fields (5.1), (5.5), (5.9), (5.13), (5.17), and (5.21) do always exist regardless to the function (1.4). However, all of them are space-like vector fields since $X_0 = 0$ for them. It is known that time-like Killing vector fields are more important for quantum field theories. For this reason we look for the solutions of the equations (4.8) and (4.9) with

$$X_0 \neq 0. \quad (6.1)$$

Under the assumption (6.1) the second equation in (4.8) produces two mutually exclusive options for the function $R = R(x^0)$ in (1.4):

$$R' = 0 \quad \text{or} \quad 2(R')^2 - R''R + R^2 = 0. \quad (6.2)$$

We study these two options in (6.2) as two different cases.

The first case: $R' = 0$. In this case $R = \text{const}$ and $R > 0$, i. e. R is a positive constant. Applying the condition $R' = 0$ to (4.8), we find that both of the equations (4.8) are fulfilled identically in this case. As for (4.9), here we have

$$2(R')^2 - R''R + R^2 = R^2 \neq 0. \quad (6.3)$$

Due to (6.3) and due to $R' = 0$ from (4.9) we derive

$$Y_{01} = 0, \quad Y_{02} = 0, \quad Y_{03} = 0. \quad (6.4)$$

Now let's write the differential equations (3.22), taking into account (6.4). For the function X_0 we get the following equations:

$$\frac{\partial X_0}{\partial x^0} = 0, \quad \frac{\partial X_0}{\partial x^1} = 0, \quad \frac{\partial X_0}{\partial x^2} = 0, \quad \frac{\partial X_0}{\partial x^3} = 0. \quad (6.5)$$

The equations (6.5) mean that X_0 is a constant function:

$$X_0 = \text{const}.$$

Moreover, taking into account (6.4), from (3.20) and (3.22) we derive:

$$\frac{\partial X_1}{\partial x^0} = 0, \quad \frac{\partial X_2}{\partial x^0} = 0, \quad \frac{\partial X_3}{\partial x^0} = 0, \quad (6.6)$$

$$\frac{\partial Y_{12}}{\partial x^0} = 0, \quad \frac{\partial Y_{23}}{\partial x^0} = 0, \quad \frac{\partial Y_{13}}{\partial x^0} = 0. \quad (6.7)$$

The equations (6.6) and (6.7) mean that the functions $X_1, X_2, X_3, Y_{12}, Y_{13}, Y_{23}$ actually do not depend on the variable x^0 . As appears, other equations in (3.20) and (3.22) in the case of $R' = 0$ do not contain X_0 and form complete system of Pfaff equations for six functions $X_1, X_2, X_3, Y_{12}, Y_{13}, Y_{23}$ with respect to three variables x^1, x^2, x^3 . These equations have at most six linearly independent solutions. These solutions are exhausted by six Killing vector field considered in section 5.

Theorem 6.1. *In the case of $R' = 0$ the spherical universe $M = \mathbb{R} \times S^3$ with the metric (1.2) admits exactly one linearly independent time-like Killing vector field*

$$\mathbf{X} = \frac{\partial}{\partial x^0} \quad (6.8)$$

in addition to six space-like Killing vector fields (5.1), (5.5), (5.9), (5.13), (5.17), (5.21) produced by the rotations of the sphere S^3 .

The vector field (6.8) is orthogonal to the sphere S^3 in M . It commutes with other six Killing vector fields, which are tangent to S^3 .

The second case. In contrast to (6.3), in this case we have the following equality for the function $R = R(x^0)$ in (1.4):

$$2(R')^2 - R''R + R^2 = 0. \quad (6.9)$$

Due to the equality (6.9) the compatibility conditions (4.9) and the first compatibility equation (4.8) are fulfilled identically. Moreover, we have

$$\begin{aligned} -(2(R')^2 - R''R + R^2)' + 2R'(2(R')^2 - R''R + R^2) = \\ = 4(R')^3 - 5R''R' + R'''R^2. \end{aligned} \quad (6.10)$$

Due to (6.10) the first compatibility equation in (4.8) is also fulfilled identically. Thus, the equation (6.9) is the only compatibility condition derived from (3.27) in the second case.

Note that the equation (6.9) can be integrated up to the first order differential equation. Indeed, since $R \neq 0$, it can be written as follows:

$$\left(\frac{R'}{R^2}\right)' = \frac{1}{R} \quad (6.11)$$

Let's multiply both sides of (6.11) by the fraction

$$\frac{R'}{R^2} \quad (6.12)$$

As a result we get the equation with the pure derivatives in both sides:

$$\left(\frac{1}{2}\left(\frac{R'}{R^2}\right)^2\right)' = \frac{R'}{R^3} = \left(-\frac{1}{2}\frac{1}{R^2}\right)'. \quad (6.13)$$

Integrating the equality (6.13), we derive

$$(R')^2 = CR^4 - R^2, \quad (6.14)$$

where C is a constant of integration. Note that R' can vanish at some points, it is not identically zero in this case. The same is true for the fraction (6.12). Therefore the equation (6.14) is equivalent to the initial equation (6.9) at all point except for those, where $R' = 0$.

It is clear that C in (6.14) is a positive constant. Let's denote $C = 1/a^2$. Then R is a function with the values ranging in the interval

$$R \in [a, +\infty).$$

The equation (6.14) itself can be written as follows

$$\left(\left(\frac{1}{R}\right)'\right)^2 = \frac{1}{a^2} - \frac{1}{R^2}. \quad (6.15)$$

Let's denote $u = 1/R$ for a while. Then we transform (6.15) to

$$(u')^2 = 1/a^2 - u^2 \quad (6.16)$$

The equation (6.16) can be integrated. Its general solution looks like

$$u(x^0) = \frac{\cos(x^0 + b)}{a}, \quad (6.17)$$

where b is a constant of integration. Without loss of generality we can take $b = 0$. Then from (6.17) we derive the following formula

$$R(x^0) = \frac{a}{\cos(x^0)}. \quad (6.18)$$

Having defined the function (1.4) by means of the formula (6.18), now let's define the time variable t by means of the differential equation (1.5):

$$\frac{dx^0}{\cos(x^0)} = \frac{c dt}{a}. \quad (6.19)$$

Integrating both sides of the equality (6.19), we obtain:

$$\ln \left(\frac{1 + \sin(x^0)}{\cos(x^0)} \right) = \frac{c t}{a}. \quad (6.20)$$

Transforming (6.20), we pass from logarithms to exponentials. As a result we get:

$$\frac{1 + \sin(x^0)}{\cos(x^0)} = e^{\frac{c t}{a}}. \quad (6.21)$$

Now we square both sides of the equality (6.21). This yields

$$\frac{1 + 2 \sin(x^0) + \sin^2(x^0)}{\cos^2(x^0)} = \frac{2 + 2 \sin(x^0) - \cos^2(x^0)}{\cos^2(x^0)} = e^{\frac{2c t}{a}}. \quad (6.22)$$

The equality (6.22) can be transformed to the following one:

$$1 + \sin(x^0) = \frac{1 + e^{\frac{2c t}{a}}}{2} \cos^2(x^0). \quad (6.23)$$

Note that the left hand side of (6.23) coincides with the numerator of the fraction in the left hand side of (6.21). Substituting (6.23) back into (6.21), we get

$$\cos(x^0) = \frac{2 e^{\frac{c t}{a}}}{1 + e^{\frac{2c t}{a}}} = \frac{1}{\cosh\left(\frac{c t}{a}\right)}. \quad (6.24)$$

Substituting (6.24) into (6.18) we find the dependence of R on the time variable t :

$$R(t) = a \cosh\left(\frac{c t}{a}\right).$$

Moreover, substituting (6.24) into (6.21), we derive the following formula:

$$\sin(x^0) = \frac{\sinh\left(\frac{c t}{a}\right)}{\cosh\left(\frac{c t}{a}\right)} = \tanh\left(\frac{c t}{a}\right). \quad (6.25)$$

Now, relying on the above calculations, we introduce the modified stereographic coordinates especially for this particular case:

$$u^0 = \frac{c t}{a}, \quad u^1 = x^1, \quad u^2 = x^2, \quad u^3 = x^3. \quad (6.26)$$

The metric (1.2) in these coordinates u^0, u^1, u^2, u^3 is written as follows:

$$ds^2 = a^2 (du^0)^2 - 4a^2 \cosh^2(u^0) \frac{(du^1)^2 + (du^2)^2 + (du^3)^2}{(|u|^2 + 1)^2}, \quad (6.27)$$

where $|u|^2 = (u^1)^2 + (u^2)^2 + (u^3)^2$. Let's consider the following four functions of the modified stereographic coordinates u^0, u^1, u^2, u^3 :

$$\begin{aligned} z^1 &= A \cosh(u^0) \frac{2u^1}{|u| + 1}, & z^2 &= A \cosh(u^0) \frac{2u^2}{|u| + 1}, \\ z^3 &= A \cosh(u^0) \frac{2u^3}{|u| + 1}, & z^4 &= A \cosh(u^0) \frac{|u| - 1}{|u| + 1}. \end{aligned} \quad (6.28)$$

As it was shown in [4], the functions (6.28) determine an embedding of the sphere S^3 into the four-dimensional Euclidean space \mathbb{R}^4 . Let's complement the functions (6.28) with one additional function

$$z^0 = A \sinh(u^0). \quad (6.29)$$

Then the functions (6.28) and (6.29) taken together determine an embedding of our universe $M = \mathbb{R} \times S^3$ into the five-dimensional space \mathbb{R}^5 . If we equip this space with the sign-indefinite metric

$$ds^2 = (dz^0)^2 - (dz^1)^2 - (dz^2)^2 - (dz^3)^2 - (dz^4)^2, \quad (6.30)$$

then we find that the metric (6.30) induces the metric (6.27) in M under the embedding given by the functions (6.28) and (6.29).

Calculating the curvature tensor for the metric (6.27), we find that our universe $M = \mathbb{R} \times S^3$ in this case is a manifold of constant negative sectional curvature

$$K = -\frac{1}{a^2}.$$

It is known that any four-dimensional constant curvature manifold has exactly ten linear independent Killing vector fields (compare this fact with the theorem 3.1). Six of them are given by the formulas (5.1), (5.5), (5.9), (5.13), (5.17), (5.21). These fields are associated with meridional and equatorial rotations of the sphere S^3 . Other four fields are determined by the hyperbolic rotations of M itself.

Hyperbolic rotation in the plane $z^1 O z^0$. This rotation induces the Killing vector field \mathbf{X} in M expressed by the formula

$$\begin{aligned} \mathbf{X} &= \frac{2(u^1)}{|u|^2 + 1} \frac{\partial}{\partial u^0} + \frac{|u|^2 - 2(u^1)^2 + 1}{2} \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^1} - \\ &- (u^1)(u^2) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^2} - (u^1)(u^3) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^3}. \end{aligned} \quad (6.31)$$

Hyperbolic rotation in the plane $z^2 O z^0$. This rotation induces a Killing vector field in M very similar to the previous one. The Killing vector field \mathbf{X} for this case

is expressed by the following formula:

$$\begin{aligned} \mathbf{X} = & \frac{2(u^2)}{|u|^2+1} \frac{\partial}{\partial u^0} - (u^2)(u^1) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^1} + \\ & + \frac{|u|^2 - 2(u^2)^2 + 1}{2} \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^2} - (u^2)(u^3) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^3}. \end{aligned} \quad (6.32)$$

Hyperbolic rotation in the plane z^3Oz^0 . This rotation induces the Killing vector field \mathbf{X} in M expressed by the formula

$$\begin{aligned} \mathbf{X} = & \frac{2(u^3)}{|u|^2+1} \frac{\partial}{\partial u^0} - (u^3)(u^1) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^1} - \\ & - (u^3)(u^2) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^2} + \frac{|u|^2 - 2(u^3)^2 + 1}{2} \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^3}. \end{aligned} \quad (6.33)$$

Hyperbolic rotation in the plane z^4Oz^0 . This rotation produces the Killing vector field \mathbf{X} in M expressed by the formula

$$\begin{aligned} \mathbf{X} = & \frac{|u|^2 - 1}{|u|^2 + 1} \frac{\partial}{\partial u^0} + (u^1) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^1} + \\ & + (u^2) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^2} + (u^3) \frac{\sinh(u^0)}{\cosh(u^0)} \frac{\partial}{\partial u^3}. \end{aligned} \quad (6.34)$$

Using (6.26), (6.20), (6.24), and (6.25), one can easily transform the above four vector fields to the initial North Pole stereographic coordinates x^0, x^1, x^2, x^3 . Note that none of the vector fields (6.31), (6.32), (6.33), (6.34) is purely time-like. They are build by time-like vectors at some points of M and by space-like vectors at some other points.

Theorem 6.2. *In the case of $2(R')^2 - R''R + R^2 = 0$ the spherical universe $M = \mathbb{R} \times S^3$ with the metric (1.2) admits four linearly independent Killing vector fields (6.31), (6.32), (6.33), (6.34) in addition to six space-like Killing vector fields (5.1), (5.5), (5.9), (5.13), (5.17), (5.21). Neither of these ten Killing vector fields, nor any linear combination of them with constant coefficients is a purely time-like vector field in M .*

7. CONCLUSIONS.

The main result of this paper is that in general case a homogeneous and isotropic closed universe $M = \mathbb{R} \times S^3$ has no time-like Killing vector fields at all, i. e. it is non-stationary in the sense of [1] and [2]. For this reason it is a good model for to study various quantization procedures for the matter fields in the presence of a non-stationary gravitation field as a background. The theorems 6.1 and 6.2 specify two exceptional cases. In the first of them the universe $M = \mathbb{R} \times S^3$ is stationary in whole, while in the second case is is piecewise stationary, i. e. it is broken into stationary and non-stationary fragments.

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