ON DEFORMATIONS OF METRICS AND THEIR ASSOCIATED SPINOR STRUCTURES.

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Abstract. Smooth deformations of a Minkowski type metric in a four-dimensional space-time manifold are considered. Deformations of the basic spin-tensorial fields associated with this metric are calculated and their application to calculating the energy-momentum tensor of a massive spin 1/2 particle is shown.

1. Introduction.

Let M be a four-dimensional orientable manifold equipped with a Minkowski type metric $\mathbf{g}$ and with a polarization[1]. In general relativity such a manifold M is used as a stage for all physical phenomena. When describing the spin phenomenon M is additionally assumed to be a spin manifold. In this case it admits two spinor bundles: the bundle of Weyl spinors SM and the bundle of Dirac spinors DM (see definitions below).

Relativistic quantum particles in M are described by their fields, while fields are introduced through their contribution to the action

$$S = S_{\text{gravity}} + S_{\text{matter}}. \tag{1.1}$$

The metric $\mathbf{g}$ itself is interpreted as a gravitation field, while other fields are called matter fields. The influence of the matter upon the gravitation field is described by the following Einstein equation:

$$R_{ij} - \frac{R}{2}\,g_{ij} = \frac{8\,\pi\,G}{c^4}T_{ij}. \tag{1.2}$$

Here R_{ij} are the components of the Ricci tensor and R is the scalar curvature (both are produced from the components of the metric tensor g_{ij} and their derivatives). The right hand side of (1.2) comprises two fundamental physical constants: the Newtonian constant of gravitation G and the light velocity in vacuum c. Here are their numeric values taken from the NIST site:

$$G \approx 6.67428\cdot 10^{-8}\ cm^3\,g^{-1}\,sec^{-2}, \qquad c \approx 2.99792458\cdot 10^{10}\ cm\ \ sec^{-1}. \tag{1.3}$$

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[1] A polarization is a discrete geometric structure (like an orientation) that marks the future half light cone at each point of M (see more details in [1]).

Apart from the constants (1.3), the right hand side of the equation (1.2) comprises the components of the energy-momentum tensor T_{ij}. They are defined through the variational derivatives of the matter action in (1.1):

$$T_{ij} = 2\,c\,\frac{\delta S_{\text{matter}}}{\delta g^{ij}}. \tag{1.4}$$

The main goal of this paper is to clarify the procedure of calculating the right hand side of the formula (1.4) in the case where matter fields have spinor components.

2. The bundle of Weyl spinors.

The bundle of Weyl spinors SM is a special two-dimensional complex vector bundle over the space-time manifold M. It is a special bundle since it is related in a special way to the tangent bundle TM. This relation is based on the well-known algebraic fact — the group homomorphism

$$\phi:\ \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^{+}(1,3,\mathbb{R}). \tag{2.1}$$

The group $\mathrm{SO}^{+}(1,3,\mathbb{R})$ in the right hand side of (2.1) is the special orthochronous Lorentz group. It is linked to the metric structure of the manifold M through frames. In the context of general relativity they are also called **vierbeins**. This German word means "four feet" expressing the idea of frames in the case of a four-dimensional space-time manifold.

Definition 2.1. A frame of the tangent bundle TM for the space-time manifold M is an ordered quadruple of smooth vector fields $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ defined in some open domain $U \subset M$ and linearly independent at each point $p \in U$.

For any two frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\tilde{\boldsymbol{\Upsilon}}_0$, $\tilde{\boldsymbol{\Upsilon}}_1$, $\tilde{\boldsymbol{\Upsilon}}_2$, $\tilde{\boldsymbol{\Upsilon}}_3$ with intersecting domains the transition matrices arise. At each point $p \in U \cap \tilde{U}$ we have

$$\tilde{\boldsymbol{\Upsilon}}_i = \sum_{j=0}^{3} S^j_i\,\boldsymbol{\Upsilon}_j, \qquad\qquad \boldsymbol{\Upsilon}_i = \sum_{j=0}^{3} T^j_i\,\tilde{\boldsymbol{\Upsilon}}_j. \tag{2.2}$$

Definition 2.2. A frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ of the tangent bundle TM is called a holonomic frame if its vector fields $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ commute with each other. Otherwise it is called a non-holonomic frame:

$$[\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j] = \sum_{k=0}^{3} c^k_{ij}\,\boldsymbol{\Upsilon}_k. \tag{2.3}$$

The coefficients c^k_{ij} in (2.3) are called the commutations coefficients of this non-holonomic frame.

Note that each local chart with coordinates x^0, x^1, x^2, x^3 in M produces its associated holonomic frame of the coordinate vector fields:

$$\boldsymbol{\Upsilon}_0 = \frac{\partial}{\partial x^0}, \qquad \boldsymbol{\Upsilon}_1 = \frac{\partial}{\partial x^1}, \qquad \boldsymbol{\Upsilon}_2 = \frac{\partial}{\partial x^2}, \qquad \boldsymbol{\Upsilon}_3 = \frac{\partial}{\partial x^3}. \tag{2.4}$$

And conversely, each holonomic frame can be represented in the form of (2.4) in some neighborhood of each point where it is defined. However, in general case frames are non-holonomic.

Frames are used to represent vectorial and tensorial fields in the coordinate form. In particular, if we take the metric tensor $\mathbf{g}$, it is represented by a square matrix g_{ij}. Its dual metric tensor is represented by the inverse matrix g^{ij}.

Definition 2.3. A frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ of the tangent bundle TM is called an orthonormal frame if the components of the metric tensor $\mathbf{g}$ and its dual metric tensor are given by the standard Minkowski matrix

$$g_{ij} = g^{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{2.5}$$

Remember that we assume M to be an orientable manifold. Therefore, we can fix an orientation on it. Then all frames will subdivide into two types — right frames and left frames. If both frames in (2.2) are right or if both of them are left, then we say that they have the same orientation. In this case

$$\det S > 0, \qquad\qquad \det T > 0. \tag{2.6}$$

Otherwise we say that these frames have opposite orientations. In this case

$$\det S < 0, \qquad\qquad \det T < 0. \tag{2.7}$$

If, additionally, both frames in (2.2) are orthonormal, then S and T are Lorentzian matrices. For their determinants we have

$$\det S = \det T = \pm 1. \tag{2.8}$$

The sign in (2.8) is chosen according to (2.6) or according to (2.7) depending on the orientations of frames.

Definition 2.4. A frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ of the tangent bundle TM is called a positively polarized frame if its first vector $\boldsymbol{\Upsilon}_0$ lies inside the future half light cone, i. e. if it is a time-like vector directed to the future.

Assume that both frames in (2.2) are positively polarized right orthonormal frames. In this case both transition matrices S and T in (2.2) are Lorentzian matrices with the unit determinant such that $S^0_0 > 0$ and $T^0_0 > 0$. This means that

$$S \in \mathrm{SO}^+(1,3,\mathbb{R}), \qquad\qquad T \in \mathrm{SO}^+(1,3,\mathbb{R}). \tag{2.9}$$

Thus, having postulated the presence three geometric structures in M — the metric, the orientation, and the polarization, we have implemented the special orthochronous Lorentz group through the frame transition matrices (2.9). This means that the structural group of the tangent bundle TM reduces from $\mathrm{GL}(4,\mathbb{R})$ to $\mathrm{SO}^+(1,3,\mathbb{R})$. Generally speaking, any reduction of the structural group of TM

from $\mathrm{GL}(4,\mathbb{R})$ to some subgroup of $\mathrm{GL}(4,\mathbb{R})$ is due to the presence of some definite geometric structures in M. Some similar fact can be formulated for any other bundle over the base manifold M.

Definition 2.5. A frame of the spinor bundle SM for the space-time manifold M is an ordered pair of smooth spinor fields $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ defined in some open domain $U \subset M$ and linearly independent at each point $p \in U$.

For any two frames $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ and $\tilde{\mathbf{\Psi}}_1$, $\tilde{\mathbf{\Psi}}_2$ with intersecting domains the transition matrices arise. At each point $p \in U \cap \tilde{U}$ we have

$$\tilde{\mathbf{\Psi}}_i = \sum_{j=1}^{2} \mathfrak{S}^j_i \, \mathbf{\Psi}_j, \qquad\qquad \mathbf{\Psi}_i = \sum_{j=1}^{2} \mathfrak{T}^j_i \, \tilde{\mathbf{\Psi}}_j. \tag{2.10}$$

Definition 2.6. A two-dimensional complex vector bundle SM over the space-time manifold M is called the bundle of Weyl spinors if for any positively polarized right orthonormal frame $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$ and for any point p of its domain U some frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ of SM defined in U or in some smaller neighborhood of the point p is canonically associated to $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$ in such a way that for any two positively polarized right orthonormal frames of TM related by the matrices S and T in (2.2) their associated frames in SM are related to each other by the matrices $\mathfrak{S} \in \mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{T} \in \mathrm{SL}(2,\mathbb{C})$ in (2.10), where $S = \phi(\mathfrak{S})$ and $T = \phi(\mathfrak{T})$ and where ϕ is the group homomorphism (2.1).

According to the definition 2.6 the structural group of the spinor bundle SM reduces from $\mathrm{GL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$. Due to this reduction the bundle of Weyl spinors is equipped with the skew-symmetric spinor metric $\mathbf{d}$. It is given by the matrix

$$d_{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix} \tag{2.11}$$

in any frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ canonically associated with some positively polarized right orthonormal frame of TM. The dual metric for (2.11) in such a frame is given by the matrix inverse to the matrix (2.11):

$$d^{ij} = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}. \tag{2.12}$$

Definition 2.7. A frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ of the spinor bundle SM is called an orthonormal frame if the spinor metric $\mathbf{d}$ and its dual metric are given by the matrices (2.11) and (2.12) in this frame.

As we see, any frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ canonically associated with some positively polarized right orthonormal frame $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$ of TM is an orthonormal frame in the sense of the definition 2.7. The converse proposition is also true, i. e. any orthonormal frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ of SM is canonically associated with some positively polarized right orthonormal frame in TM. Therefore, we use the following diagram:

$$\boxed{\text{Orthonormal frames}} \rightarrow \boxed{\begin{array}{c}\text{Positively polarized}\\ \text{right orthonormal frames}\end{array}} \tag{2.13}$$

The reduction of $\mathrm{GL}(2,\mathbb{C})$ to $\mathrm{SL}(2,\mathbb{C})$ for Weyl spinors is concordant with the reduction of $\mathrm{GL}(4,\mathbb{R})$ to $\mathrm{SO}^+(1,3,\mathbb{R})$ in TM. For this reason the spinor bundle SM has one more geometric structure given by the Infeld-van der Waerden field $\mathbf{G}$. For any frame pair $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ canonically associated to each other according to the diagram (2.13) the components of the Infeld-van der Waerden field are given by the following four Pauli matrices:

$$\begin{aligned}
G^{i\bar{i}}_0 &= \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} = \sigma_0, &\qquad G^{i\bar{i}}_2 &= \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix} = \sigma_2,\\
G^{i\bar{i}}_1 &= \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} = \sigma_1, &\qquad G^{i\bar{i}}_3 &= \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} = \sigma_3.
\end{aligned}\tag{2.14}$$

The barred letter $\bar{i}$ in (2.14) is used in order to indicate that the second upper index is a conjugate spinor index. The first upper index i in (2.14) is a regular spinor index, while the lower index is a spacial one. It runs from 0 to 3 and enumerates the Pauli matrices (2.14). The following table summarize the spin tensorial types of the basic fields $\mathbf{d}$ and $\mathbf{G}$ of the bundle of Weyl spinors:

Symbol	Name	Spin-tensorial type
$\mathbf{d}$	Skew-symmetric metric tensor	$(0,2\vert 0,0\vert 0,0)$
$\mathbf{G}$	Infeld-van der Waerden field	$(1,0\vert 1,0\vert 0,1)$

(2.15)

The third column of the table (2.15) says that in the coordinate form the metric tensor $\mathbf{d}$ has two lower spinor indices, while the Infeld-van der Waerden field $\mathbf{G}$ has one upper spinor index, one upper conjugate spinor index, and one lower spacial index. These facts are in agreement with (2.11) and with (2.14).

3. Deformations of the metric.

The space-time manifold M is equipped with the metric $\mathbf{g}$. Let's change this metric replacing it by some other metric $\hat{\mathbf{g}}$ with the same signature. Locally, in the coordinate form, at each fixed point $p \in M$ the metrics $\mathbf{g}$ and $\hat{\mathbf{g}}$ are related to each other by means of the following formula:

$$g_{ij} = \sum_{p=0}^{3}\sum_{q=0}^{3} F^p_i\, F^q_j\, \hat{g}_{pq}.\tag{3.1}$$

This formula reflects the purely algebraic fact that each quadratic form can be diagonalized (see [2]). We need smooth deformations of the metric $\mathbf{g}$, i. e. smoothly depending on a point $p \in M$. For this reason we assume F^p_i and F^q_j in (3.1) to be the components of some smooth non-degenerate operator-valued field $\mathbf{F}$. We need small deformations of metric in (1.4). Therefore we assume $\mathbf{F}$ to be close to the identity operator $\mathbf{id}$. This fact is expressed by the formula

$$\mathbf{F} = \mathbf{id} + \varepsilon\, \mathbf{f}.\tag{3.2}$$

Here $\varepsilon\to 0$ is a small parameter. For F^p_i and F^q_j the formula (3.2) yields

$$F^p_i=\delta^p_i+\varepsilon\,f^p_i, \qquad\qquad F^q_j=\delta^q_j+\varepsilon\,f^q_j. \tag{3.3}$$

Substituting (3.3) into the formula (3.1), we obtain

$$g_{ij}=\hat g_{ij}+\sum_{p=0}^{3}\varepsilon\,f^p_i\,\hat g_{pj}+\sum_{q=0}^{3}\varepsilon\,\hat g_{iq}\,f^q_j+\ldots\,. \tag{3.4}$$

By dots in (3.4) we denote the higher order terms with respect to the small parameter ε. We see that $g_{ij}\sim\hat g_{ij}$ as $\varepsilon\to 0$. For this reason we can set $\hat g_{pj}=g_{pj}$ and $\hat g_{iq}=g_{iq}$ in (3.4). As a result the formula (3.4) is written as

$$\hat g_{ij}=g_{ij}-\sum_{p=0}^{3}\varepsilon\,f^p_i\,g_{pj}-\sum_{q=0}^{3}\varepsilon\,g_{iq}\,f^q_j+\ldots\,. \tag{3.5}$$

In order to simplify the formula (3.5) we denote

$$f_{ij}=\sum_{q=0}^{3}g_{iq}\,f^q_j. \tag{3.6}$$

The formula (3.6) is a particular instance of the standard index lowering procedure. It produces the twice covariant tensor field with the components f_{ij} from the operator-valued field $\mathbf{f}$ used in (3.2). Applying (3.6) to (3.5) we get

$$\hat g_{ij}=g_{ij}-\varepsilon\,(f_{ij}+f_{ji})+\ldots\,. \tag{3.7}$$

The formula (3.7) is equivalent to the following one:

$$\delta g_{ij}=-\varepsilon\,(f_{ij}+f_{ji}). \tag{3.8}$$

Thus, we see that only the $\mathbf{g}$-symmetric part of the operator field $\mathbf{f}$ is actually contribute to the variation of the metric $\mathbf{g}$ under the transformation (3.1). For this reason we choose $\mathbf{g}$-symmetric operator $\mathbf{f}$ by setting

$$f^i_j=\frac{1}{2}\sum_{p=0}^{3}g^{ip}\,h_{pj} \tag{3.9}$$

and assuming h_{ij} to be symmetric:

$$h_{ij}=h_{ji}. \tag{3.10}$$

Then from (3.8), (3.9), and (3.10) we derive

$$\delta g_{ij}=-\varepsilon\,h_{ij}. \tag{3.11}$$

Now let's consider the components of the dual metric tensor $\hat g^{ij}$. They form the matrix inverse to $\hat g_{ij}$. For the component of this inverse matrix from (3.5) we derive

$$\hat g^{ij}=g^{ij}+\sum_{q=0}^{3}\varepsilon\,f^i_q\,g^{qj}+\sum_{p=0}^{3}\varepsilon\,g^{ip}\,f^j_p+\ldots\,. \tag{3.12}$$

Like in (3.6), for the sake of brevity we denote

$$f^{ij}=\sum_{q=0}^{3}f^i_q\,g^{qj}. \tag{3.13}$$

Then, substituting (3.13) into (3.12), we transform (3.12) as follows:

$$\hat g^{ij}=g^{ij}+\varepsilon\,(f^{ij}+f^{j\,i})+\ldots\,. \tag{3.14}$$

Applying (3.9) to (3.13) and taking into account the symmetry $g^{qj}=g^{j\,q}$, we get

$$f^{ij}=\frac{1}{2}\sum_{p=0}^{3}\sum_{q=0}^{3}g^{ip}\,g^{jq}\,h_{pq}. \tag{3.15}$$

Relying on (3.14) and (3.15), it is convenient to denote

$$h^{ij}=\sum_{p=0}^{3}\sum_{q=0}^{3}g^{ip}\,g^{jq}\,h_{pq}. \tag{3.16}$$

In terms of (3.16), the formula (3.14) is written as

$$\hat g^{ij}=g^{ij}+\varepsilon\,h^{ij}+\ldots\,. \tag{3.17}$$

Due to (3.17) the variations of the components g^{ij} of the dual metric tensor under the transformation (3.1) are given by the formula

$$\delta g^{ij}=\varepsilon\,h^{ij}, \tag{3.18}$$

where h^{ij} are produced from h_{ij} used in (3.11) by means of the standard index raising procedure (3.16). For this reason h^{ij} in (3.18) are symmetric:

$$h^{ij}=h^{j\,i}. \tag{3.19}$$

The symmetry (3.19) is concordant with the symmetry of the metric tensor $\mathbf{g}$ itself.

4. Deformations of the spinor structure.

In our approach, the spinor bundle SM is introduced through the orthonormal frames (see the definition 2.6 above). Therefore it is related to the metric $\mathbf{g}$. Let's study this relation. Assume that $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ is some positively polarized right orthonormal frame of the metric $\mathbf{g}$. Then the matrix g_{ij} is diagonal and coincides with (2.5) in this frame. Using the operator $\mathbf{F}$ we define the other frame $\hat{\boldsymbol\Upsilon}_0$, $\hat{\boldsymbol\Upsilon}_1$, $\hat{\boldsymbol\Upsilon}_2$, $\hat{\boldsymbol\Upsilon}_3$ introducing it by means of the formula

$$\hat{\boldsymbol\Upsilon}_i=\sum_{j=0}^{3}F^j_i\,\boldsymbol\Upsilon_j. \tag{4.1}$$

The components of the operator $\mathbf{F}$ in (4.1) act just like the components of the transition matrix S in (2.2). The formula (4.1) is equivalent to

$$\hat{\boldsymbol\Upsilon}_i=\mathbf{F}(\boldsymbol\Upsilon_i). \tag{4.2}$$

Using (3.1), (2.2) and combining them with (4.1) or with (4.2), one easily proves that g_{ij} coincides with the matrix of the deformed metric $\hat{\mathbf{g}}$ represented in the frame $\hat{\boldsymbol\Upsilon}_0$, $\hat{\boldsymbol\Upsilon}_1$, $\hat{\boldsymbol\Upsilon}_2$, $\hat{\boldsymbol\Upsilon}_3$. Therefore the metric $\hat{\mathbf{g}}$ is diagonal in the frame $\hat{\boldsymbol\Upsilon}_0$, $\hat{\boldsymbol\Upsilon}_1$, $\hat{\boldsymbol\Upsilon}_2$, $\hat{\boldsymbol\Upsilon}_3$ and its matrix coincides with (2.5) in this frame. In other words, the frame $\hat{\boldsymbol\Upsilon}_0$, $\hat{\boldsymbol\Upsilon}_1$, $\hat{\boldsymbol\Upsilon}_2$, $\hat{\boldsymbol\Upsilon}_3$ is an orthonormal frame for the deformed metric $\hat{\mathbf{g}}$. Moreover, from (3.2) and (3.3) we derive that

$$\det\mathbf{F}\to 1,\qquad\qquad F^0_0\to 1, \tag{4.3}$$

as $\varepsilon\to 0$. The relationships (4.3) mean that for sufficiently small values of ε the determinant of the operator $\mathbf{F}$ and its component F^0_0 both are positive:

$$\det\mathbf{F}>0,\qquad\qquad F^0_0>0. \tag{4.4}$$

Due to (4.4) the deformed frame $\hat{\boldsymbol\Upsilon}_0$, $\hat{\boldsymbol\Upsilon}_1$, $\hat{\boldsymbol\Upsilon}_2$, $\hat{\boldsymbol\Upsilon}_3$ is a positively polarized and right frame. Thus, using the deformation operator $\mathbf{F}$, we can produce a positively polarized right orthonormal frame for the deformed metric $\hat{\mathbf{g}}$ from any positively polarized right orthonormal frame of the initial metric $\mathbf{g}$.

Let's recall that, according to the definition 2.6, each positively polarized right orthonormal frame $\boldsymbol\Upsilon_0$, $\boldsymbol\Upsilon_1$, $\boldsymbol\Upsilon_2$, $\boldsymbol\Upsilon_3$ is associated with some frame $\boldsymbol\Psi_1$, $\boldsymbol\Psi_2$ of the spinor bundle SM. Let's declare the deformed frame $\hat{\boldsymbol\Upsilon}_0$, $\hat{\boldsymbol\Upsilon}_1$, $\hat{\boldsymbol\Upsilon}_2$, $\hat{\boldsymbol\Upsilon}_3$ to be associated with the same spinor frame $\boldsymbol\Psi_1$, $\boldsymbol\Psi_2$. As a result we get the diagram

$$\begin{tikzcd}
\boldsymbol\Upsilon_0,\,\boldsymbol\Upsilon_1,\,\boldsymbol\Upsilon_2,\,\boldsymbol\Upsilon_3 \arrow[r, "\mathbf{g}"] \arrow[rr, bend left, "\mathbf{F}"] & \boldsymbol\Psi_1,\,\boldsymbol\Psi_2 & \hat{\boldsymbol\Upsilon}_0,\,\hat{\boldsymbol\Upsilon}_1,\,\hat{\boldsymbol\Upsilon}_2,\,\hat{\boldsymbol\Upsilon}_3 \arrow[l, red, "\hat{\mathbf{g}}"']
\end{tikzcd} \tag{4.5}$$

The arched arrow on the diagram (4.5) corresponds to the frame deformation, while the horizontal arrows are frame associations, the red arrow being our newly declared frame association. In order to define a spinor structure our newly defined frame association should be consistent with the definition 2.6. Let's prove its consistence. For this purpose we consider two positively polarized right orthonormal frames $\boldsymbol\Upsilon_0$, $\boldsymbol\Upsilon_1$, $\boldsymbol\Upsilon_2$, $\boldsymbol\Upsilon_3$ and $\tilde{\boldsymbol\Upsilon}_0$, $\tilde{\boldsymbol\Upsilon}_1$, $\tilde{\boldsymbol\Upsilon}_2$, $\tilde{\boldsymbol\Upsilon}_3$ of the metric $\mathbf{g}$ whose domains are overlapping. They are related to each other with the relationships (2.2), where S

and T are mutually inverse transition matrices. Applying the operator $\mathbf{F}$ to both sides of the relationships (2.2), we get

$$\mathbf{F}(\tilde{\boldsymbol{\Upsilon}}_i) = \sum_{j=0}^{3} S_i^j \, \mathbf{F}(\boldsymbol{\Upsilon}_j), \qquad\qquad \mathbf{F}(\boldsymbol{\Upsilon}_i) = \sum_{j=0}^{3} T_i^j \, \mathbf{F}(\tilde{\boldsymbol{\Upsilon}}_j). \tag{4.6}$$

The formulas (4.6) mean that if we produce two positively polarized right orthonormal frames $\hat{\boldsymbol{\Upsilon}}_0,\,\hat{\boldsymbol{\Upsilon}}_1,\,\hat{\boldsymbol{\Upsilon}}_2,\,\hat{\boldsymbol{\Upsilon}}_3$ and $\hat{\tilde{\boldsymbol{\Upsilon}}}_0,\,\hat{\tilde{\boldsymbol{\Upsilon}}}_1,\,\hat{\tilde{\boldsymbol{\Upsilon}}}_2,\,\hat{\tilde{\boldsymbol{\Upsilon}}}_3$ of the metric $\hat{\mathbf{g}}$ by applying the deformation operator $\mathbf{F}$ to the frames $\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3$ and $\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3$, then the deformed frames will be related to each other by the same matrices S and T in (4.6) as the original frames are related in (2.2):

$$\hat{\tilde{\boldsymbol{\Upsilon}}}_i = \sum_{j=0}^{3} S_i^j \, \hat{\boldsymbol{\Upsilon}}_j, \qquad\qquad \hat{\boldsymbol{\Upsilon}}_i = \sum_{j=0}^{3} T_i^j \, \hat{\tilde{\boldsymbol{\Upsilon}}}_j. \tag{4.7}$$

Due to (4.7) we can extend the diagram (4.5) in the following way:

$$\begin{tikzcd}
\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3 \arrow[r, "\mathbf{g}"] \arrow[d, shift right, "S"'] \arrow[rr, bend left, "\mathbf{F}"] & \boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2 \arrow[d, shift right, "\mathfrak{S}"'] & \hat{\boldsymbol{\Upsilon}}_0,\,\hat{\boldsymbol{\Upsilon}}_1,\,\hat{\boldsymbol{\Upsilon}}_2,\,\hat{\boldsymbol{\Upsilon}}_3 \arrow[l, "\hat{\mathbf{g}}"'] \arrow[d, shift right, "S"'] \\
\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3 \arrow[r, "\mathbf{g"] \arrow[u, shift right, "T"'] \arrow[rr, bend right, "\mathbf{F}"'] & \tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2 \arrow[u, shift right, "\mathfrak{T}"'] & \hat{\tilde{\boldsymbol{\Upsilon}}}_0,\,\hat{\tilde{\boldsymbol{\Upsilon}}}_1,\,\hat{\tilde{\boldsymbol{\Upsilon}}}_2,\,\hat{\tilde{\boldsymbol{\Upsilon}}}_3 \arrow[l, "\hat{\mathbf{g}}"'] \arrow[u, shift right, "T"']
\end{tikzcd} \tag{4.8}$$

Since the matrices S and T in the right hand side of the diagram (4.8) are the same as in its left hand side, the diagram is commutative in whole, provided it is commutative in the absence of the red arrows in it. The latter fact follows from the definition 2.6. Moreover, the matrices S and T are related to the matrices $\mathfrak{S}$ and $\mathfrak{T}$ through the group homomorphism (2.1):

$$S = \phi(\mathfrak{S}), \qquad\qquad T = \phi(\mathfrak{T}).$$

Therefore, the frame associations represented by the red horizontal arrows on the diagram (4.8) are consistent and define a new spinor structure. Note that the spinor frames of this new spinor structure coincide with those of the initial one. This means that the spinor structures produced by the metrics $\mathbf{g}$ and $\hat{\mathbf{g}}$ share the same spinor bundle SM.

Theorem 4.1. *The bundle of Weyl spinors SM is preserved under the metric deformations of the form* (3.1).

Note that the frames $\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2$ and $\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2$ in the diagram (4.8) are canonically associated with positively polarized right orthonormal frames for both metrics $\mathbf{g}$ and

$\hat{\mathbf{g}}$. Therefore, they are orthonormal spinor frames in the sense of the definition 2.7 for both spinor structures. This fact means that the spinor metrics $\mathbf{d}$ and $\hat{\mathbf{d}}$ of both spinor structures are represented by the same matrix (2.11) in the same set of frames whose domains cover the space-time manifold. As a conclusion we get the following theorem.

Theorem 4.2. *The skew-symmetric spinor metric* $\mathbf{d}$ *is preserved under the metric deformations of the form* (3.1)*, i. e.* $\hat{\mathbf{d}} = \mathbf{d}$.

Now let's consider the Infeld-van der Waerden field $\mathbf{G}$ for the non-deformed metric $\mathbf{g}$. It is represented by the matrices (2.14) in the frame pair

$$\boldsymbol{\Upsilon}_0,\ \boldsymbol{\Upsilon}_1,\ \boldsymbol{\Upsilon}_2,\ \boldsymbol{\Upsilon}_3 \qquad \to \qquad \boldsymbol{\Psi}_1,\ \boldsymbol{\Psi}_2. \tag{4.9}$$

Using the frame deformation formula (4.1), we can transform its components from the initial frame pair (4.9) to the deformed frame pair

$$\hat{\boldsymbol{\Upsilon}}_0,\ \hat{\boldsymbol{\Upsilon}}_1,\ \hat{\boldsymbol{\Upsilon}}_2,\ \hat{\boldsymbol{\Upsilon}}_3 \qquad \to \qquad \boldsymbol{\Psi}_1,\ \boldsymbol{\Psi}_2. \tag{4.10}$$

As a result for the components of $\mathbf{G}$ in the frame pair (4.10) we get

$$G^{i\bar{i}}_k = \sum_{q=0}^{3} F^q_k\,\sigma^{i\bar{i}}_q. \tag{4.11}$$

In the frame pair (4.10) the components of the deformed Infeld-van der Waerden field $\hat{\mathbf{G}}$ are given by the Pauli matrices: $\hat{G}^{i\bar{i}}_q = \sigma^{i\bar{i}}_q$. Applying this fact to the formula (4.11), we derive the following relationship:

$$G^{i\bar{i}}_k = \sum_{q=0}^{3} F^q_k\,\hat{G}^{i\bar{i}}_q. \tag{4.12}$$

Theorem 4.3. *The Infeld-van der Waerden field* $\mathbf{G}$ *is transformed according to the formula* (4.12) *under the metric deformations of the form* (3.1)*.*

Now we apply the expansion (3.2) to (4.12). As a result, we get

$$\hat{G}^{i\bar{i}}_k = G^{i\bar{i}}_k - \sum_{q=0}^{3} \varepsilon\,f^q_k\,G^{i\bar{i}}_q + \ldots\,. \tag{4.13}$$

Combining (4.13) with the formula (3.9) we derive the following formula for the variation of the Infeld-van der Waerden field $\mathbf{G}$:

$$\delta G^{i\bar{i}}_k = -\frac{1}{2}\sum_{q=0}^{3}\sum_{p=0}^{3} g^{pq}\,G^{i\bar{i}}_q\,\varepsilon\,h_{pk}. \tag{4.14}$$

It is preferable to express $\delta G^{i\bar{i}}_k$ through h^{ij} with upper indices. For this reason we transform the formula (4.14) as follows:

$$\delta G^{i\bar{i}}_k = -\frac{1}{2}\sum_{q=0}^{3}\sum_{p=0}^{3} g_{pk}\,G^{i\bar{i}}_q\,\varepsilon\,h^{pq}. \tag{4.15}$$

Apart from (4.15), due to the theorem 4.2 we have the formula

$$\delta d_{ij} = 0. \tag{4.16}$$

The formulas (4.15) and (4.16) describe completely the variations of the basic spin-tensorial fields (2.15) of the bundle of Weyl spinors under the deformations of metric given by the formula (3.1). According the theorem 4.1, the bundle SM itself is invariant under these metric deformations.

5. The bundle of Dirac spinors.

The bundle of Dirac spinors DM over a space-time manifold M is constructed as the direct sum of the bundle of Weyl spinors and its Hermitian conjugate bundle:

$$DM = SM \oplus S^{\dagger}M \tag{5.1}$$

(see [3] for more details). Applying the theorem 4.1 to the expansion (5.1) we immediately derive the following theorem.

Theorem 5.1. *The bundle of Dirac spinors DM is preserved under the metric deformations of the form* (3.1).

Like in the case of Weyl spinors, the structure of the Dirac bundle DM is described in terms of associated frame pairs and in terms of basic spin-tensorial fields. There are four types of associated frame pairs

$$\begin{array}{ccc}
\boxed{\begin{array}{c}\text{Canonically orthonormal}\\ \text{chiral frames}\end{array}} & \leftarrow & \boxed{\begin{array}{c}\text{Positively polarized}\\ \text{right orthonormal frames}\end{array}}\\
\boxed{\begin{array}{c}P\text{-reverse}\\ \text{antichiral frames}\end{array}} & \leftarrow & \boxed{\begin{array}{c}\text{Positively polarized}\\ \text{left orthonormal frames}\end{array}}\\
\boxed{\begin{array}{c}T\text{-reverse}\\ \text{antichiral frames}\end{array}} & \leftarrow & \boxed{\begin{array}{c}\text{Negatively polarized}\\ \text{right orthonormal frames}\end{array}}\\
\boxed{\begin{array}{c}PT\text{-reverse}\\ \text{chiral frames}\end{array}} & \leftarrow & \boxed{\begin{array}{c}\text{Negatively polarized}\\ \text{left orthonormal frames}\end{array}}
\end{array} \tag{5.2}$$

and there are four basic spin-tensorial fields

Symbol	Name	Spin-tensorial type
$\mathbf{d}$	Skew-symmetric metric tensor	$(0, 2\|0, 0\|0, 0)$
$\mathbf{H}$	Chirality operator	$(1, 1\|0, 0\|0, 0)$
$\mathbf{D}$	Dirac form	$(0, 1\|0, 1\|0, 0)$
$\boldsymbol{\gamma}$	Dirac γ-field	$(1, 1\|0, 0\|0, 1)$

(5.3)

in the bundle of Dirac spinors DM. Associated frame pairs of the first type in (5.2) are produced directly from associated frame pairs of the bundle of Weyl spinors in

(2.13). They are sufficient for our purposes in this paper. Associated frame pairs of the three other types are produced from associated frame pairs of the first type by means of the P and T-reversion procedures. We do not consider them here referring the reader to the paper [3].

Let's consider some associated frame pair of the first type for the bundle of Dirac spinors DM. It is composed of two frames:

$$\boldsymbol\Upsilon_0,\,\boldsymbol\Upsilon_1,\,\boldsymbol\Upsilon_2,\,\boldsymbol\Upsilon_3 \qquad\to\qquad \boldsymbol\Psi_1,\,\boldsymbol\Psi_2,\,\boldsymbol\Psi_3,\,\boldsymbol\Psi_4. \tag{5.4}$$

Here $\boldsymbol\Upsilon_0,\,\boldsymbol\Upsilon_1,\,\boldsymbol\Upsilon_2,\,\boldsymbol\Upsilon_3$ is a positively polarized right orthonormal frame of the tangent bundle TM. The second frame $\boldsymbol\Psi_1,\,\boldsymbol\Psi_2,\,\boldsymbol\Psi_3,\,\boldsymbol\Psi_4$ in (5.4) is a canonically orthonormal chiral frame of DM. Its first two spinor fields $\boldsymbol\Psi_1$ and $\boldsymbol\Psi_2$ form an orthonormal frame of the bundle of Weyl spinors SM (see (4.9) and the definition 2.7 above). The second two fields $\boldsymbol\Psi_3$ and $\boldsymbol\Psi_4$ belong to $S^\dagger M$ in (5.1). They are semilinear functional dual to $\boldsymbol\Psi_1$ and $\boldsymbol\Psi_2$ in the sense of the following relationships:

$$\begin{aligned}
&\boldsymbol\Psi_3(\boldsymbol\Psi_1)=1, &&\boldsymbol\Psi_3(\boldsymbol\Psi_2)=0,\\
&\boldsymbol\Psi_4(\boldsymbol\Psi_1)=0, &&\boldsymbol\Psi_4(\boldsymbol\Psi_2)=1.
\end{aligned}\tag{5.5}$$

The relationships (5.5) fix $\boldsymbol\Psi_3$ and $\boldsymbol\Psi_4$ uniquely, provided $\boldsymbol\Psi_1$ and $\boldsymbol\Psi_2$ are fixed. For this reason we can transform the diagram (4.5) as follows:

$$\begin{tikzcd}
\boldsymbol\Upsilon_0,\,\boldsymbol\Upsilon_1,\,\boldsymbol\Upsilon_2,\,\boldsymbol\Upsilon_3 \arrow[r, "\mathbf g"] \arrow[rr, bend left, "\mathbf F"] & \boldsymbol\Psi_1,\,\boldsymbol\Psi_2,\,\boldsymbol\Psi_3,\,\boldsymbol\Psi_4 & \hat{\boldsymbol\Upsilon}_0,\,\hat{\boldsymbol\Upsilon}_1,\,\hat{\boldsymbol\Upsilon}_2,\,\hat{\boldsymbol\Upsilon}_3 \arrow[l, "\hat{\mathbf g}"']
\end{tikzcd}\tag{5.6}$$

Once some associated frame pair (5.4) of the first type is fixed, the basic spin tensorial fields $\mathbf d$, $\mathbf H$, $\mathbf D$, and γ (5.3) are introduced by the following definitions.

Definition 5.1. The skew-symmetric metric tensor $\mathbf d$ is a spin-tensorial field of the type $(0,2|0,0|0,0)$ given by the matrix

$$d_{ij}=\begin{Vmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\end{Vmatrix}\tag{5.7}$$

in any canonically orthonormal chiral frame of the Dirac bundle DM.

Definition 5.2. The chirality operator $\mathbf H$ is a spin-tensorial field of the type $(1,1|0,0|0,0)$ given by the matrix

$$H^i_j=\begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{Vmatrix}\tag{5.8}$$

in any canonically orthonormal chiral frame of the Dirac bundle DM.

Definition 5.3. The Dirac form $\mathbf{D}$ is a spin-tensorial field of the type $(0,1|0,1|0,0)$ given by the matrix

$$D_{i\bar{j}} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix} \tag{5.9}$$

in any canonically orthonormal chiral frame of the Dirac bundle DM.

Definition 5.4. The Dirac γ-filed is a spin-tensorial field of the type $(1,1|0,0|0,1)$ given by the Dirac matrices

$$\begin{aligned} &\gamma^a_{b0} = m^a_{b0} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}, &&\gamma^a_{b1} = m^a_{b1} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{Vmatrix}, \\ &\gamma^a_{b2} = m^a_{b2} = \begin{Vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{Vmatrix}, &&\gamma^a_{b3} = m^a_{b3} = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix} \end{aligned} \tag{5.10}$$

in any frame pair composed by a positively polarized right orthonormal frame in TM and its associated canonically orthonormal chiral frame in DM.

6. Deformation of the basic spin-tensorial fields.

Note that two positively polarized right orthonormal frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\hat{\boldsymbol{\Upsilon}}_0$, $\hat{\boldsymbol{\Upsilon}}_1$, $\hat{\boldsymbol{\Upsilon}}_2$, $\hat{\boldsymbol{\Upsilon}}_3$ for the initial and deformed metrics on the diagram (5.6) share the same canonically orthonormal chiral frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$. Note also that the spin-tensorial fields $\mathbf{d}$, $\mathbf{H}$, and $\mathbf{D}$ are defined through spinor frames only (see definitions 5.1, 5.2, and 5.3 above). Therefore, the components of $\mathbf{d}$, $\mathbf{H}$, and $\mathbf{D}$ coincide with the components of $\hat{\mathbf{d}}$, $\hat{\mathbf{H}}$, and $\hat{\mathbf{D}}$ in the spinor frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ shared by two frame pairs on the diagram (5.6). They are given by the matrices (5.7), (5.8), and (5.9) respectively. This result is expressed by the equalities

$$\mathbf{d} = \hat{\mathbf{d}}, \qquad \mathbf{H} = \hat{\mathbf{H}}, \qquad \mathbf{D} = \hat{\mathbf{D}} \tag{6.1}$$

and formulated verbally in the following theorem.

Theorem 6.1. *The skew-symmetric spinor metric* $\mathbf{d}$*, the chirality operator* $\mathbf{H}$ *and the Dirac form* $\mathbf{D}$ *are preserved under the metric deformations of the form* (3.1).

The formulas (6.1) and the theorem 6.1 can be expressed in terms of variations:

$$\delta d_{ij} = 0, \qquad \delta H^i_j = 0, \qquad \delta D_{i\bar{j}} = 0. \tag{6.2}$$

Now let's proceed to the Dirac γ-field. In this case $\boldsymbol{\gamma}$ and $\hat{\boldsymbol{\gamma}}$ are different. The γ-field for the non-deformed metric $\mathbf{g}$ is given by the m-matrices (5.10) in the non-deformed frame pair $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3 \to \boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ shown by the black right arrow on the diagram (5.6). Using the transition formulas (4.1), we can calculate its components in the deformed frame pair shown by the red arrow:

$$\gamma^a_{bk} = \sum_{q=0}^{3} F^q_k \, m^a_{bq}. \tag{6.3}$$

This formula is analogous to (4.11). Now if we remember that the deformed γ-field is given by the m-matrices (5.10) in the deformed frame pair, we can transform (6.3) to the formula analogous to (4.12):

$$\gamma^a_{bk} = \sum_{q=0}^{3} F^q_k \, \hat{\gamma}^a_{bq}. \tag{6.4}$$

Theorem 6.2. *The Dirac γ-field is transformed according to the formula (6.4) under the metric deformations of the form (3.1).*

Let's apply the expansion (3.2) to F^q_k in (6.4). As a result we obtain

$$\hat{\gamma}^a_{bk} = \gamma^a_{bk} - \sum_{q=0}^{3} \varepsilon \, f^q_k \, \gamma^a_{bq} + \ldots\,. \tag{6.5}$$

The formula (6.5) is analogous to (4.13). Applying (3.9) to (6.5), we get

$$\delta\gamma^a_{bk} = -\frac{1}{2} \sum_{q=0}^{3} \sum_{p=0}^{3} g^{pq} \, \gamma^a_{bq} \, \varepsilon \, h_{pk}. \tag{6.6}$$

Raising indices of h_{pk} in (6.6), we can transform this formula as follows:

$$\delta\gamma^a_{bk} = -\frac{1}{2} \sum_{q=0}^{3} \sum_{p=0}^{3} g_{pk} \, \gamma^a_{bq} \, \varepsilon \, h^{pq}. \tag{6.7}$$

The formulas (6.6) and (6.7) are analogous to the formulas (4.14) and (4.15) respectively. The formula (6.7) is complementary to (6.2).

7. Deformation of the metric connection.

Each metric $\mathbf{g}$ of the space-time manifold M produces the torsion-free metric connection Γ in TM. This connection has extensions to the spinor bundles SM and DM. In this paper we consider the extension $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the metric connection Γ to the bundle of Dirac spinors DM only because it is preferably used in particle physics. Once some frame pair $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ is chosen, the components of the metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ are given by explicit formulas. For its Γ-components we use the formula

$$\begin{aligned} \Gamma^k_{ij} &= \sum_{r=0}^{3} \frac{g^{kr}}{2} \left(L_{\boldsymbol{\Upsilon}_i}(g_{rj}) + L_{\boldsymbol{\Upsilon}_j}(g_{ir}) - L_{\boldsymbol{\Upsilon}_r}(g_{ij})\right) + \\ &+ \frac{c^k_{ij}}{2} - \sum_{r=0}^{3} \sum_{s=0}^{3} \frac{c^s_{ir}}{2} \, g^{kr} \, g_{sj} - \sum_{r=0}^{3} \sum_{s=0}^{3} \frac{c^s_{jr}}{2} \, g^{kr} \, g_{si} \end{aligned} \tag{7.1}$$

taken from [4]. Here $L_{\boldsymbol{\Upsilon}_i}$, $L_{\boldsymbol{\Upsilon}_j}$, and $L_{\boldsymbol{\Upsilon}_r}$ are the derivatives along the vectors $\boldsymbol{\Upsilon}_i$, $\boldsymbol{\Upsilon}_j$, and $\boldsymbol{\Upsilon}_r$ respectively, while c_{ij}^k are the commutation coefficients taken from the commutation relationships (2.3).

According to the theorem 4.1, the bundle of Dirac spinors DM is preserved under the deformations of metric. For this reason we fix a pair of frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ and assume these frames to be unchanged under the deformation of metric. Moreover, we assume these frames to be canonically associated to each other according to the first line in the diagram (5.2). This means that $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ is a positively polarized right orthonormal frame with respect to the initial metric $\mathbf{g}$, while $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ is its associated canonically orthonormal chiral frame in DM. Under these assumptions we have

$$\delta c_{ij}^k = 0. \tag{7.2}$$

Moreover, under these assumptions the components of the initial metric $\mathbf{g}$ are constants, i. e. g_{ij} in (3.7) are constants taken from the matrix (2.5), while $\hat{g}_{ij} \neq \mathrm{const}$ and δg_{ij} in (3.11) are also not constants. From $g_{ij} = \mathrm{const}$ we derive

$$L_{\boldsymbol{\Upsilon}_k}(g_{ij}) = 0. \tag{7.3}$$

Applying (7.3) to the formula (7.1), we simplify it as follows:

$$\Gamma_{ij}^k = \frac{c_{ij}^k}{2} - \sum_{r=0}^{3}\sum_{s=0}^{3} \frac{c_{ir}^s}{2}\, g^{kr}\, g_{sj} - \sum_{r=0}^{3}\sum_{s=0}^{3} \frac{c_{jr}^s}{2}\, g^{kr}\, g_{si}. \tag{7.4}$$

Then we apply (3.11), (3.18), and (7.2) to the formula (7.1). As a result we get

$$\begin{aligned} \delta\Gamma_{ij}^k = &-\sum_{r=0}^{3} \varepsilon\, \frac{g^{kr}}{2} \bigl(L_{\boldsymbol{\Upsilon}_i}(h_{rj}) + L_{\boldsymbol{\Upsilon}_j}(h_{ir}) - L_{\boldsymbol{\Upsilon}_r}(h_{ij})\bigr) - \\ &-\sum_{r=0}^{3}\sum_{s=0}^{3} \varepsilon\, \frac{c_{ir}^s}{2}\, h^{kr}\, g_{sj} - \sum_{r=0}^{3}\sum_{s=0}^{3} \varepsilon\, \frac{c_{jr}^s}{2}\, h^{kr}\, g_{si} + \\ &+\sum_{r=0}^{3}\sum_{s=0}^{3} \varepsilon\, \frac{c_{ir}^s}{2}\, g^{kr}\, h_{sj} + \sum_{r=0}^{3}\sum_{s=0}^{3} \varepsilon\, \frac{c_{jr}^s}{2}\, g^{kr}\, h_{si}. \end{aligned} \tag{7.5}$$

The derivatives $L_{\boldsymbol{\Upsilon}_i}(h_{rj})$, $L_{\boldsymbol{\Upsilon}_j}(h_{ir})$, and $L_{\boldsymbol{\Upsilon}_r}(h_{ij})$ in (7.5) should be expressed through covariant derivatives. For this purpose we use the formula

$$\nabla_i h_{jk} = L_{\boldsymbol{\Upsilon}_i}(h_{jk}) - \sum_{s=0}^{3} \Gamma_{ij}^s\, h_{sk} - \sum_{s=0}^{3} \Gamma_{ik}^s\, h_{js}. \tag{7.6}$$

Applying (7.6) to (7.5) we obtain the following formula:

$$\delta\Gamma_{ij}^k = -\sum_{r=0}^{3} \varepsilon\, \frac{g^{kr}}{2} \left(\nabla_i h_{rj} + \nabla_j h_{ir} - \nabla_r h_{ij}\right). \tag{7.7}$$

This formula is well-known. Another proof of the formula (7.7) can be found in [1].

Now let's consider the A-components of the metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$. They are given by the following formula taken from [5]:

$$\begin{aligned}\mathrm{A}^a_{ib} &= \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta\alpha}}{8}\,\delta^a_b - \sum_{\alpha=1}^{4}\sum_{\beta=1}^{4}\sum_{d=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(d_{\alpha\beta})\,d^{\beta d}\,H^\alpha_d}{8}\,H^a_b\,-\\ &-\sum_{c=1}^{4}\sum_{d=1}^{4}\sum_{r=1}^{4}\frac{d_{bc}\,L_{\boldsymbol{\Upsilon}_i}(H^c_d)\,H^d_r\,d^{ra}}{4} + \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(\gamma^\alpha_{bm}\,g^{mn})}{4}\,\times\\ &\times\gamma^a_{\alpha n} + \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\frac{\gamma^\alpha_{bm}\,\Gamma^n_{is}\,g^{ms}\,\gamma^a_{\alpha n}}{4}.\end{aligned}\tag{7.8}$$

Due to our special choice of frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$ the quantities $d_{\alpha\beta}$, H^c_d, γ^α_{bm}, and g^{mn} are constants. Their values are given by the formulas (5.7), (5.8), (5.10), and (2.5). For this reason the formula (7.8) reduces to

$$\mathrm{A}^a_{ib} = \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\frac{\gamma^\alpha_{bm}\,\Gamma^n_{is}\,g^{ms}\,\gamma^a_{\alpha n}}{4}.\tag{7.9}$$

The quantities Γ^n_{is} in (7.9) are given by (7.4). The formula (7.9) is applicable only for the case of the initial non-deformed metric $\mathbf{g}$. Passing to its deformations (3.1), we should use the formula (7.8) again. Taking into account (6.2), we get

$$\begin{aligned}\delta\mathrm{A}^a_{ib} &= \sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_i}(\delta\gamma^\alpha_{bm}\,g^{mn} + \gamma^\alpha_{bm}\,\delta g^{mn})}{4}\,\gamma^a_{\alpha n}\,+\\ &+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\frac{\Gamma^n_{is}\,(\delta\gamma^\alpha_{bm}\,g^{ms} + \gamma^\alpha_{bm}\,\delta g^{ms})\,\gamma^a_{\alpha n}}{4}\,+\\ &+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\frac{\gamma^\alpha_{bm}\,\delta\Gamma^n_{is}\,g^{ms}\,\gamma^a_{\alpha n} + \gamma^\alpha_{bm}\,\Gamma^n_{is}\,g^{ms}\,\delta\gamma^a_{\alpha n}}{4}.\end{aligned}\tag{7.10}$$

Before transforming the formula (7.10) in whole we consider some smaller subexpression in the right hand side of this formula. Applying the formulas (3.18) and (6.7) to this subexpression, we get

$$\begin{aligned}\sum_{m=0}^{3}(\delta\gamma^\alpha_{bm}\,g^{mn} + \gamma^\alpha_{bm}\,\delta g^{mn}) &= -\frac{1}{2}\sum_{q=0}^{3}\sum_{p=0}^{3}\sum_{m=0}^{3} g_{pm}\,\gamma^\alpha_{bq}\,\varepsilon\,h^{pq}\,g^{mn}\,+\\ &+\sum_{m=0}^{3}\gamma^\alpha_{bm}\,\varepsilon\,h^{mn} = \frac{1}{2}\sum_{m=0}^{3}\gamma^\alpha_{bm}\,\varepsilon\,h^{mn}.\end{aligned}\tag{7.11}$$

The subexpression (7.11) is differentiated in (7.10). Before substituting (7.11) back into (7.10) we express the Lie derivative $L_{\boldsymbol{\Upsilon}_i}$ of it through its covariant derivative:

$$\begin{gathered}\sum_{m=0}^{3} L_{\boldsymbol{\Upsilon}_i}(\delta\gamma^{\alpha}_{bm}\,g^{mn}+\gamma^{\alpha}_{bm}\,\delta g^{mn})=\frac{\varepsilon}{2}\sum_{m=0}^{3} L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm}\,h^{mn})=\\ =\frac{1}{2}\sum_{m=0}^{3}\nabla_i(\gamma^{\alpha}_{bm}\,\varepsilon\,h^{mn})-\frac{1}{2}\sum_{m=0}^{3}\sum_{\theta=1}^{4}A^{\alpha}_{i\theta}\,\gamma^{\theta}_{bm}\,\varepsilon\,h^{mn}+\\ +\frac{1}{2}\sum_{m=0}^{3}\sum_{\theta=1}^{4}A^{\theta}_{ib}\,\gamma^{\alpha}_{\theta m}\,\varepsilon\,h^{mn}-\frac{1}{2}\sum_{m=0}^{3}\sum_{s=0}^{3}\Gamma^{n}_{is}\,\gamma^{\alpha}_{bm}\,\varepsilon\,h^{ms}.\end{gathered}\tag{7.12}$$

Now, substituting (7.12) into (7.10), we recall the following identities:

$$\nabla\mathbf{d}=0,\qquad\nabla\mathbf{H}=0,\qquad\nabla\mathbf{D}=0,\qquad\nabla\boldsymbol{\gamma}=0,\qquad\nabla\mathbf{g}=0.\tag{7.13}$$

The identities (7.13) are known as the concordance conditions for the metric $\mathbf{g}$ and its metric connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$. The spin-tensorial fields $\mathbf{d}$, $\mathbf{H}$, $\mathbf{D}$, and $\boldsymbol{\gamma}$ are treated as attributes of the metric $\mathbf{g}$. Thus, applying the identities (7.13) to (7.12) and substituting (7.12) back into the formula (7.10), we get

$$\begin{gathered}\delta\mathrm{A}^{a}_{ib}=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\left(\frac{\gamma^{\alpha}_{bm}\,\varepsilon\,\nabla_i h^{mn}}{8}-\sum_{\theta=1}^{4}\frac{A^{\alpha}_{i\theta}\,\gamma^{\theta}_{bm}\,\varepsilon\,h^{mn}}{8}+\right.\\ \left.+\sum_{\theta=1}^{4}\frac{A^{\theta}_{ib}\,\gamma^{\alpha}_{\theta m}\,\varepsilon\,h^{mn}}{8}\right)\gamma^{a}_{\alpha n}+\\ +\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\frac{\gamma^{\alpha}_{bm}\,\delta\Gamma^{n}_{is}\,g^{ms}\,\gamma^{a}_{\alpha n}+\gamma^{\alpha}_{bm}\,\Gamma^{n}_{is}\,g^{ms}\,\delta\gamma^{a}_{\alpha n}}{4}.\end{gathered}\tag{7.14}$$

In the next step we apply the formula (7.7) for $\delta\Gamma^{n}_{is}$ and the formula (6.7) for $\delta\gamma^{a}_{\alpha n}$ in order to transform the formula (7.14). As a result we get

$$\begin{gathered}\delta\mathrm{A}^{a}_{ib}=\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{\theta=1}^{4}\left(\frac{A^{\theta}_{ib}\,\gamma^{\alpha}_{\theta m}}{8}-\frac{A^{\alpha}_{i\theta}\,\gamma^{\theta}_{bm}}{8}\right)\gamma^{a}_{\alpha n}\,\varepsilon\,h^{mn}+\\ +\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{r=0}^{3}\sum_{s=0}^{3}\frac{g_{in}\,\gamma^{\alpha}_{bm}\,g^{rs}\,\gamma^{a}_{\alpha s}\,\varepsilon\,\nabla_r h^{mn}}{8}-\\ -\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{\alpha=1}^{4}\sum_{r=0}^{3}\sum_{s=0}^{3}\frac{g_{in}\,\gamma^{\alpha}_{bs}\,g^{rs}\,\gamma^{a}_{\alpha m}\,\varepsilon\,\nabla_r h^{mn}}{8}-\\ -\sum_{n=0}^{3}\sum_{m=0}^{3}\sum_{p=0}^{3}\sum_{q=0}^{3}\sum_{\alpha=1}^{4}\sum_{s=0}^{3}\frac{\gamma^{\alpha}_{bp}\,\Gamma^{q}_{is}\,g^{ps}\,g_{mq}\,\gamma^{a}_{\alpha n}\,\varepsilon\,h^{mn}}{8}.\end{gathered}\tag{7.15}$$

Now let's remember that $\nabla_i\gamma^{\alpha}_{bm}=0$ and $\nabla_i\,g^{pq}=0$. These identities are the coordinate forms of $\nabla\boldsymbol{\gamma}$ and $\nabla\mathbf{g}$ from (7.13). Expanding them, we get

$$\begin{aligned}
&\nabla_i\gamma^{\alpha}_{bm}=L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm})+\sum^{4}_{\theta=1}A^{\alpha}_{i\theta}\,\gamma^{\theta}_{bm}-\sum^{4}_{\theta=1}A^{\theta}_{ib}\,\gamma^{\alpha}_{\theta m}-\sum^{3}_{s=0}\Gamma^{s}_{im}\,\gamma^{\alpha}_{\theta s}=0,\\
&\nabla_i\,g^{pq}=L_{\boldsymbol{\Upsilon}_i}(g^{pq})+\sum^{3}_{s=0}\Gamma^{p}_{is}\,g^{sq}+\sum^{3}_{s=0}\Gamma^{q}_{is}\,g^{ps}=0.
\end{aligned}$$

Due to our special choice of frames $\boldsymbol{\Upsilon}_0,\ \boldsymbol{\Upsilon}_1,\ \boldsymbol{\Upsilon}_2,\ \boldsymbol{\Upsilon}_3$ and $\boldsymbol{\Psi}_1,\ \boldsymbol{\Psi}_2,\ \boldsymbol{\Psi}_3,\ \boldsymbol{\Psi}_4$ the quantities γ^{α}_{bm} and g^{pq} are constants. Therefore $L_{\boldsymbol{\Upsilon}_i}(\gamma^{\alpha}_{bm})=0$ and $L_{\boldsymbol{\Upsilon}_i}(g^{pq})=0$. As a result we obtain the following identities:

$$\sum^{4}_{\theta=1}A^{\theta}_{ib}\,\gamma^{\alpha}_{\theta m}-\sum^{4}_{\theta=1}A^{\alpha}_{i\theta}\,\gamma^{\theta}_{bm}=-\sum^{3}_{s=0}\Gamma^{s}_{im}\,\gamma^{\alpha}_{\theta s},\tag{7.16}$$

$$\sum^{3}_{s=0}\Gamma^{q}_{is}\,g^{ps}=-\sum^{3}_{s=0}\Gamma^{p}_{is}\,g^{sq}.\tag{7.17}$$

Applying (7.16) to the first term in (7.15) and applying (7.17) to the last term in (7.15), we find that these terms cancel each other. This yields

$$\begin{aligned}
\delta\mathrm{A}^{a}_{ib}=&\sum^{3}_{m=0}\sum^{3}_{n=0}\sum^{4}_{\alpha=1}\sum^{3}_{r=0}\sum^{3}_{s=0}\frac{g_{in}\,\gamma^{\alpha}_{bm}\,g^{rs}\,\gamma^{a}_{\alpha s}\,\varepsilon\,\nabla_r h^{mn}}{8}\,-\\
&-\sum^{3}_{m=0}\sum^{3}_{n=0}\sum^{4}_{\alpha=1}\sum^{3}_{r=0}\sum^{3}_{s=0}\frac{g_{in}\,\gamma^{\alpha}_{bs}\,g^{rs}\,\gamma^{a}_{\alpha m}\,\varepsilon\,\nabla_r h^{mn}}{8}.
\end{aligned}\tag{7.18}$$

Passing to the barred A components of the metric connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ remember that they are obtained as the complex conjugates of A components:

$$\bar{\mathrm{A}}^{\bar{a}}_{i\bar{b}}=\overline{\mathrm{A}^{\bar{a}}_{i\bar{b}}}.\tag{7.19}$$

From (7.19) we immediately derive the formula

$$\delta\bar{\mathrm{A}}^{\bar{a}}_{i\bar{b}}=\overline{\delta\,\mathrm{A}^{\bar{a}}_{i\bar{b}}}.\tag{7.20}$$

The formulas (7.7), (7.18), and (7.20) describe completely the variation of the metric connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ under the metric deformations of the form (3.1).

8. Energy-momentum tensor of the massive spin 1/2 particle.

A single spin 1/2 particle with the mass m in a space-time manifold M is described by a spinor-valued $\boldsymbol{\psi}$-function satisfying the Dirac equation:

$$i\,\hbar\sum^{4}_{b=1}\sum^{3}_{p=0}\sum^{3}_{q=0}\gamma^{a}_{bp}\,g^{pq}\,\nabla_q\psi^b-m\,c\;\psi^a=0.\tag{8.1}$$

The Dirac equation (8.1) is derived from the following action functional:

$$S_{\text{matter}}=i\,\hbar\int\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}\sum_{p=0}^{3}\sum_{q=0}^{3}D_{a\bar a}\,\gamma^a_{bp}\,g^{pq}\,\frac{\overline{\psi^{\bar a}}\ \nabla_q\psi^b-\psi^b\ \nabla_q\overline{\psi^{\bar a}}}{2}\,dV\,-\\ -\,m\,c\int\sum_{a=1}^{4}\sum_{\bar a=1}^{4}D_{a\bar a}\,\overline{\psi^{\bar a}}\,\psi^a\,dV \tag{8.2}$$

(see [6]). Our goal is to substitute the action functional (8.2) into the formula (1.4). For this purpose we write the action (8.2) formally as

$$S_{\text{matter}}=\int L\,dV. \tag{8.3}$$

The real scalar quantity L in (8.3) is called the Lagrangian density. The variation of the action integral (8.3) is written as follows:

$$\delta S_{\text{matter}}=\int\delta L\,dV-\frac{1}{2}\int\sum_{i=0}^{3}\sum_{j=0}^{3}L\,g_{ij}\,\varepsilon\,h^{ij}\,dV. \tag{8.4}$$

The second integral in (8.4) arises because the volume element dV depends on the metric $\mathbf{g}$ and changes under the metric deformations of the form (3.1) (see more details in [1]).

Due to the formula (8.4) we need to calculate δL. For this purpose we subdivide L into subexpressions and calculate their variations separately. According to (8.2), the Lagrangian density L is a sum of two terms:

$$L=L_{\text{kinetic}}+L_{\text{massive}}. \tag{8.5}$$

The massive term in (8.5) is given by the formula

$$L_{\text{massive}}=-m\,c\sum_{a=1}^{4}\sum_{\bar a=1}^{4}D_{a\bar a}\,\overline{\psi^{\bar a}}\,\psi^a. \tag{8.6}$$

The variation of the massive term (8.6) is equal to zero. Indeed, we have

$$\delta\psi^a=0,\qquad\qquad \delta\overline{\psi^{\bar a}}=0 \tag{8.7}$$

since $\boldsymbol{\psi}$-function is treated as an independent parameter under the metric deformations (3.1). As for $D_{a\bar a}$ in (8.6), for these parameters we have

$$\delta D_{a\bar a}=0. \tag{8.8}$$

The equality (8.8) is derived from (7.13). The equalities (8.7) and (8.8) lead to

$$\delta L_{\text{massive}}=0. \tag{8.9}$$

Let's proceed to the kinetic term of the Lagrangian density (8.5). According to (8.2), this term is given by the formula

$$L_{\text{kinetic}} = i\,\hbar\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}\sum_{p=0}^{3}\sum_{q=0}^{3} D_{a\bar a}\,\gamma^{a}_{bp}\,g^{pq}\,\frac{\overline{\psi^{\bar a}}\ \nabla_{q}\psi^{b}-\psi^{b}\ \nabla_{q}\overline{\psi^{\bar a}}}{2}. \tag{8.10}$$

Though due to (8.7) ψ^{b} and $\overline{\psi^{\bar a}}$ in (8.10) are not sensitive to the metric deformations (3.1), their covariant derivatives are sensitive. Indeed, we have

$$\begin{aligned}\nabla_{q}\psi^{b} &= L_{\boldsymbol{\Upsilon}_{q}}(\psi^{b})+\sum_{\theta=1}^{4}\mathrm{A}^{b}_{q\theta}\ \psi^{\theta},\\ \nabla_{q}\overline{\psi^{\bar a}} &= L_{\boldsymbol{\Upsilon}_{q}}(\overline{\psi^{\bar a}})+\sum_{\theta=1}^{4}\bar{\mathrm{A}}^{\bar a}_{q\theta}\ \overline{\psi^{\theta}}.\end{aligned} \tag{8.11}$$

From the equalities (8.11) we immediately derive

$$\delta\nabla_{q}\psi^{b}=\sum_{\theta=1}^{4}\delta\mathrm{A}^{b}_{q\theta}\ \psi^{\theta}, \qquad\qquad \delta\nabla_{q}\overline{\psi^{\bar a}}=\sum_{\theta=1}^{4}\delta\bar{\mathrm{A}}^{\bar a}_{q\theta}\ \overline{\psi^{\theta}}. \tag{8.12}$$

Apart from (8.11), there are two other terms γ^{a}_{bp} and g^{pq} in the formula (8.10) which are sensitive to the metric deformations (3.1). Their variations are given by the formulas (6.7) and (3.18).

Now let's calculate the variation of the kinetic term of the Lagrangian density (8.5). From the formula (8.10), applying (8.12), we derive

$$\begin{gathered}\delta L_{\text{kinetic}} = i\,\hbar\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}\sum_{p=0}^{3}\sum_{q=0}^{3} D_{a\bar a}\,(\delta\gamma^{a}_{bp}\,g^{pq}+\gamma^{a}_{bp}\,\delta g^{pq})\,\times\\ \times\,\frac{\overline{\psi^{\bar a}}\,\nabla_{q}\psi^{b}-\psi^{b}\ \nabla_{q}\overline{\psi^{\bar a}}}{2}+i\,\hbar\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}\sum_{p=0}^{3}\sum_{q=0}^{3} D_{a\bar a}\,\gamma^{a}_{bp}\,g^{pq}\,\times\\ \times\,\frac{1}{2}\sum_{\theta=1}^{4}\left(\overline{\psi^{\bar a}}\,\delta\mathrm{A}^{b}_{q\theta}\ \psi^{\theta}-\psi^{b}\,\delta\bar{\mathrm{A}}^{\bar a}_{q\theta}\,\overline{\psi^{\theta}}\right).\end{gathered} \tag{8.13}$$

Applying (7.18) and (7.20) to (8.13), we find that second term in (8.13) is identically equal to zero. Therefore, applying (7.11) to (8.13), we obtain

$$\begin{gathered}\delta L_{\text{kinetic}} = i\,\hbar\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}\sum_{p=0}^{3}\sum_{q=0}^{3} D_{a\bar a}\,\gamma^{a}_{bp}\,\times\\ \times\,\frac{\overline{\psi^{\bar a}}\ \nabla_{q}\psi^{b}-\psi^{b}\ \nabla_{q}\overline{\psi^{\bar a}}}{4}\ \varepsilon\,h^{pq}.\end{gathered} \tag{8.14}$$

Let's remember that h^{pq} is symmetric with respect to the indices p and q. Our next goal is to make other terms in (8.14) symmetric with respect to these indices.

Applying the symmetrization procedure to (8.14), we remember (8.9) which means that $\delta L=\delta L_{\text{kinetic}}$. Then the formula (8.14) yields

$$\begin{aligned}\delta L=i\,\hbar\sum_{p=0}^{3}\sum_{q=0}^{3}\Biggl(\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}D_{a\bar a}\,\frac{\gamma^a_{bp}\ \overline{\psi^{\bar a}}\ \nabla_{\!q}\psi^b+\gamma^a_{bq}\ \overline{\psi^{\bar a}}\ \nabla_{\!p}\psi^b}{8}\,-\\ -\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}D_{a\bar a}\,\frac{\gamma^a_{bp}\ \psi^b\ \nabla_{\!q}\overline{\psi^{\bar a}}+\gamma^a_{bq}\ \psi^b\ \nabla_{\!p}\overline{\psi^{\bar a}}}{8}\Biggr)\,\varepsilon\,h^{pq}.\end{aligned}\tag{8.15}$$

Now, substituting (8.15) into (8.4) and applying the formula (1.4), we derive the explicit formula for the components of the energy-momentum tensor

$$\begin{aligned}T_{ij}=i\hbar\,c\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}D_{a\bar a}\,\frac{\gamma^a_{bi}\ \overline{\psi^{\bar a}}\ \nabla_{\!j}\psi^b+\gamma^a_{bj}\ \overline{\psi^{\bar a}}\ \nabla_{\!i}\psi^b}{4}\,-\\ -\,i\hbar\,c\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}D_{a\bar a}\,\frac{\gamma^a_{bi}\ \psi^b\ \nabla_{\!j}\overline{\psi^{\bar a}}+\gamma^a_{bj}\ \psi^b\ \nabla_{\!i}\overline{\psi^{\bar a}}}{4}\,+\\ +\,i\hbar\,c\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}\sum_{p=0}^{3}\sum_{q=0}^{3}D_{a\bar a}\,\frac{\gamma^a_{bp}\,g^{pq}\,\psi^b\ \nabla_{\!q}\overline{\psi^{\bar a}}}{2}\,g_{ij}\,-\\ -\,i\hbar\,c\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\sum_{b=1}^{4}\sum_{p=0}^{3}\sum_{q=0}^{3}D_{a\bar a}\,\frac{\gamma^a_{bp}\,g^{pq}\,\overline{\psi^{\bar a}}\ \nabla_{\!q}\psi^b}{2}\,g_{ij}\,+\\ +\,m\,c^2\sum_{a=1}^{4}\sum_{\bar a=1}^{4}D_{a\bar a}\,\overline{\psi^{\bar a}}\,\psi^a\,g_{ij}.\end{aligned}\tag{8.16}$$

Though the $\boldsymbol{\psi}$-function is a complex-valued spinor function, the components of the energy-momentum tensor (8.16) are real. This fact is proved on the base of the following identity relating the components of the fields $\mathbf{D}$ and $\boldsymbol{\gamma}$:

$$\sum_{a=1}^{4}\overline{D_{a\bar a}}\ \overline{\gamma^a_{bp}}=\sum_{a=1}^{4}D_{ab}\ \gamma^a_{\bar a p}\,.\tag{8.17}$$

Due to the same identity (8.17) the Lagrangian density L itself is a real-valued scalar field in M.

The formula (8.16) is not new. There are some other papers, where the energy-momentum tensor for a spinor field is calculated, e. g. [7]. The formula (8.16) is quite similar to the formula (3.15) in [7]. These formulas should coincide upon passing from SGS units, which are used in present paper, to special units, where $\hbar=c=1$ instead of (1.3).

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