

ON OPERATOR FIELDS IN THE BUNDLE OF DIRAC SPINORS.

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ABSTRACT. Operator fields in the bundle of Dirac spinors and their conversion to spatial fields are considered. Some commutator equations are studied with the use of the conversion technique.

1. INTRODUCTION.

The bundle of Dirac spinors is used for describing particles with half-integer spin in general relativity and in quantum field theory. It is a special four-dimensional complex vector-bundle over the space-time manifold M . Let's remind that the space-time manifold M itself is a four-dimensional real manifold equipped with a Minkowski type metric \mathbf{g} of the signature $(+, -, -, -)$. Apart from \mathbf{g} , the space-time manifold M is equipped with two other geometric structures — the orientation and the polarization. The orientation distinguishes right quadruples of tangent vectors from left ones, while the polarization distinguishes future and past half light cones in tangent spaces at each point of M .

The bundle of Dirac spinors is denoted DM . It is equipped with four basic spin-tensorial fields in addition to \mathbf{g} . They are presented in the following table.

Symbol	Name	Spin-tensorial type
d	Skew-symmetric metric tensor	$(0, 2 0, 0 0, 0)$
H	Chirality operator	$(1, 1 0, 0 0, 0)$
D	Dirac form	$(0, 1 0, 1 0, 0)$
γ	Dirac γ -field	$(1, 1 0, 0 1, 0)$

The metric tensor \mathbf{g} itself is interpreted as a spin-tensorial field of the spin-tensorial type $(0, 0|0, 0|0, 2)$.

In this paper, saying an operator field, we assume a spin-tensorial field \mathbf{F} of the spin-tensorial type $(1, 1|0, 0|0, 0)$. In the coordinate form it is presented by a matrix

2000 *Mathematics Subject Classification.* 53B30, 81T20, 81R25.

F_b^a , where a and b are two spinor indices. Each operator field \mathbf{F} in the bundle of Dirac spinors has a unique presentation of the following form:

$$F_b^a = u \delta_b^a + v H_b^a + \sum_{k=0}^3 \gamma_b^{ak} u_k + \sum_{k=0}^3 \sum_{c=1}^4 H_c^a \gamma_b^{ck} v_k + \sum_{p=0}^3 \sum_{q=0}^3 \sum_{c=1}^4 \gamma_c^{ap} \gamma_b^{cq} w_{pq}. \quad (1.1)$$

Here u and v are two scalar fields, u_k and v_k are the components of two covectorial fields \mathbf{u} and \mathbf{v} , and w_{pq} are the components of a skew-symmetric tensorial field \mathbf{w} . Through H_b^a in (1.1) we denote the components of the chirality operator \mathbf{H} , while γ_b^{ak} are the components of the Dirac γ -filed γ .

The formula (1.1) is a conversion formula associating the spin-tensorial operator field \mathbf{F} with the purely tensorial fields u , v , \mathbf{u} , \mathbf{v} , \mathbf{w} . The presentation (1.1) is well-known (see §28 in [1]). The main goal of this paper is to study some special commutator equations for operator fields in terms of their associated tensorial fields.

2. FRAMES AND COORDINATE PRESENTATIONS OF THE BASIC FIELDS.

Definition 2.1. A *spatial frame* is a quadruple of vector fields $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ defined in some open domain of the space-time manifold M and linearly independent at each point of their domain.

Definition 2.2. A spatial frame $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ is called a *right frame* if at each point of its domain its vectors $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ form a right quadruple in the sense of the orientation in M .

Definition 2.3. A spatial frame $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ is called an *orthonormal frame* if the metric tensor \mathbf{g} is presented by the standard Minkowski matrix in this frame:

$$g_{ij} = g^{ij} = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right\|. \quad (2.1)$$

In physical literature the matrix (2.1) is often denoted by η_{ij} . However, this is not a good tradition. I prefer to use the symbol g for the components of the metric tensor irrespective to the choice of an orthonormal or a non-orthonormal frame.

Definition 2.4. An orthonormal spatial frame $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ is called *positively polarized* if its first vector Υ_0 is a time-like vector directed to the future in the sense of the polarization in M .

A positively polarized right orthonormal frame $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ is a typical choice when dealing with spinors. Note, however, that in general case of a non-flat space-time M such a frame is not holonomic, i. e. its vector fields do not commute:

$$[\Upsilon_i, \Upsilon_j] = \sum_{k=0}^3 c_{ij}^k \Upsilon_k. \quad (2.2)$$

The coefficients c_{ij}^k in (2.2) are called the *commutation coefficients* of the frame $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$. This frame is called *holonomic* if all of its commutation coefficients are identically zero. Otherwise, it is called a *non-holonomic* frame.

It is known that the metric \mathbf{g} induces the 4-form ω in M which is called the *volume form* or the *volume tensor*. This differential form is used for integration over M . In the coordinate form the volume tensor ω is given by the formula

$$\omega_{ijklm} = \pm \sqrt{-\det(g_{ij})} \varepsilon_{ijklm}, \quad (2.3)$$

where ε is the Levi-Civita symbol:

$$\varepsilon_{ijklm} = \varepsilon^{ijklm} = \begin{cases} 1 & \text{if } (ijklm) \text{ is an even permutation} \\ & \text{of the numbers } 0, 1, 2, 3; \\ -1 & \text{if } (ijklm) \text{ is an odd permutation} \\ & \text{of the numbers } 0, 1, 2, 3; \\ 0 & \text{in all other cases.} \end{cases} \quad (2.4)$$

Typically ω is treated as a pseudotensor. However, we assume M to be an orientable manifold with a fixed orientation. In this case we can fix the choice of sign in (2.3) by setting plus for right frames and setting minus for left frames. Therefore, we treat ω as a tensor.

The dual volume tensor is denoted by the same symbol ω . Its components are produced from ω_{ijklm} by means of the standard index raising procedure:

$$\omega^{ijklm} = \sum_{p=0}^3 \sum_{q=0}^3 \sum_{r=0}^3 \sum_{s=0}^3 \omega_{pqrs} g^{pi} g^{qj} g^{rk} g^{sm}. \quad (2.5)$$

Applying the formula (2.3) to (2.5), we derive the formula

$$\omega^{ijklm} = \mp \sqrt{-\det(g^{ij})} \varepsilon^{ijklm}, \quad (2.6)$$

where ε again is the Levi-Civita symbol (2.4). In the case of a right orthonormal frame the formulas (2.3) and (2.6) are reduced to

$$\omega_{ijklm} = \varepsilon_{ijklm}, \quad \omega^{ijklm} = -\varepsilon^{ijklm}. \quad (2.7)$$

Definition 2.5. A spinor frame in the bundle of Dirac spinors DM is a quadruple $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ of smooth sections of this bundle over some open domain of M linearly independent at each point of this domain.

Definition 2.6. A spinor frame $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ is called an *orthonormal frame* the spinor metric \mathbf{d} is presented by the following matrices in this frame:

$$d_{ij} = \left\| \begin{array}{cccc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{array} \right\|, \quad d^{ij} = \left\| \begin{array}{cccc} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{array} \right\|. \quad (2.8)$$

The matrices (2.8) are inverse to each other. They present the spinor metric \mathbf{d} and its dual metric in an orthonormal spinor frame. Irrespective to the choice of

a spinor frame (orthonormal or non-orthonormal) the components of the spinor metric \mathbf{d} are used for lowering spinor indices. The components of the dual spinor metric are used for raising spinor indices.

Definition 2.7. A spinor frame $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ of the bundle DM is called a *chiral frame* if the chirality operator \mathbf{H} given by the following matrix in this frame:

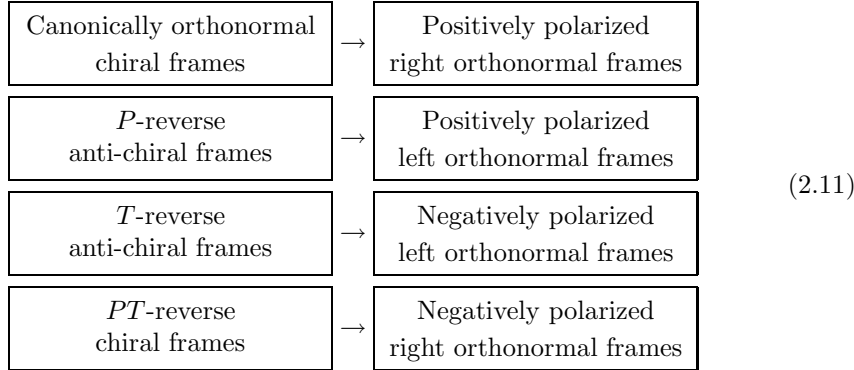
$$H_j^i = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \quad (2.9)$$

Definition 2.8. A spinor frame $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ of the Dirac bundle DM is called a *self-adjoint frame* if the Hermitian metric tensor \mathbf{D} (the Dirac form) is represented by the following matrix in this frame:

$$D_{i\bar{j}} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}. \quad (2.10)$$

Definition 2.9. *Canonically orthonormal chiral frames* in DM are those which are orthonormal, chiral, and self-adjoint simultaneously .

Canonically orthonormal chiral frames in DM do exist. Moreover, each such frame is canonically associated with some positively polarized right orthonormal frame in TM . Apart from canonically orthonormal chiral frames, there are three other special types of frames in DM . All of these frame types and their associated frame types in TM are listed in the following diagram.



More details concerning the diagram (2.11) can be found in [2]. In this paper we shall use canonically orthonormal chiral frames in DM and their associated positively polarized right orthonormal frames in TM only. They are sufficient for our purposes.

The bundle of Dirac spinors DM is a complex vector bundle. Therefore it is equipped with the involution of complex conjugation τ that acts upon spin-tensorial fields and changes their spin tensorial type as follows:

$$(\alpha, \beta | \nu, \gamma | r, s) \xrightleftharpoons[\tau]{\tau} (\nu, \gamma | \alpha, \beta | r, s). \quad (2.12)$$

As we see in (2.12), the involution τ exchanges spinor and conjugate spinor indices. In the coordinate form it acts through complex conjugation upon the components of spin-tensors.

Applying τ to \mathbf{H} and \mathbf{d} we get two other basic fields $\bar{\mathbf{H}} = \tau(\mathbf{H})$ and $\bar{\mathbf{d}} = \tau(\mathbf{d})$. They are called the *conjugate chirality operator* and the *conjugate spinor metric* respectively. In a canonically orthonormal chiral frame $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ the conjugate chirality operator $\bar{\mathbf{H}}$ is given by the matrix

$$\bar{H}_{\bar{j}}^{\bar{i}} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}. \quad (2.13)$$

The conjugate spinor metric in such a frame is given by the matrices

$$\bar{d}_{\bar{i}\bar{j}} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad \bar{d}^{\bar{i}\bar{j}} = \begin{vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{vmatrix}. \quad (2.14)$$

Though the matrix (2.13) coincides with the matrix (2.9) and the matrices (2.14) coincide with the matrices (2.8), $\bar{\mathbf{H}} \neq \mathbf{H}$ and $\bar{\mathbf{d}} \neq \mathbf{d}$ because the spin tensorial types of these fields are different. The coincidence of their matrices occurring in a special frame is destroyed in an arbitrary non-special frame.

The Dirac form \mathbf{D} is invariant with respect to the involution τ , i.e. we have $\tau(\mathbf{D}) = \mathbf{D}$. In the coordinate form this equality is written as follows:

$$D_{i\bar{j}} = \overline{D_{\bar{j}i}} \quad (2.15)$$

The equality (2.15) is easily derived from (2.10). Being derived in a special frame, it remains valid in an arbitrary frame too.

In order to present the Dirac γ -field in the coordinate form we need to fix two frames — some spinor frame $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ and some spatial frame $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$. We do it according to the first line in the diagram (2.11). In other words, we choose some canonically orthonormal chiral frame $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ in DM and take its associated positively polarized right orthonormal frame in TM . Then the Dirac γ -field γ is given by the following four matrices:

$$\begin{aligned} \gamma^0 &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, & \gamma^1 &= \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \\ \gamma^2 &= \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, & \gamma^3 &= \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}. \end{aligned} \quad (2.16)$$

The number of a matrix in (2.16) is determined by the spatial index k of the component γ_b^{ak} . Two spinor indices a and b specify the position of the component γ_b^{ak} within the matrix γ^k .

3. SOME ALGEBRAIC RELATIONSHIPS FOR THE BASIC FIELDS.

First of all let's note that the square of the chirality operator \mathbf{H} is equal to the unit operator $\mathbf{1}$. The same is true for the conjugate chirality operator $\bar{\mathbf{H}}$:

$$\mathbf{H}^2 = \mathbf{1}, \quad \bar{\mathbf{H}}^2 = \mathbf{1}. \quad (3.1)$$

In the coordinate form the identities (3.1) are written as

$$\sum_{c=1}^4 H_c^a H_b^c = \delta_b^a, \quad \sum_{\bar{c}=1}^4 \bar{H}_{\bar{c}}^{\bar{a}} \bar{H}_{\bar{b}}^{\bar{c}} = \delta_{\bar{b}}^{\bar{a}}. \quad (3.2)$$

The formulas (3.2) immediately follow from (2.9) and (2.13).

When some spatial basis $\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ is fixed, the matrices (2.16) can be treated as the components of four operators $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ acting in fibers of the bundle DM . These operators satisfy the following well-known relationships:

$$\{\gamma^p, \gamma^q\} = 2 g^{pq} \mathbf{1}. \quad (3.3)$$

The curly brackets in (3.3) denotes the anticommutator of operators. In the coordinate form the relationships (3.3) are written as

$$\sum_{c=1}^4 \gamma_c^{ap} \gamma_b^{cq} + \sum_{c=1}^4 \gamma_c^{aq} \gamma_b^{cp} = 2 g^{pq} \delta_b^a.$$

These relationships are proved by direct calculations with the use of (2.16). Lowering the upper spatial index of $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ by means of the metric \mathbf{g} , we get other four operators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$:

$$\gamma_k = \sum_{q=0}^3 g_{kq} \gamma^q. \quad (3.4)$$

The operators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ satisfy the following relationships:

$$\{\gamma^p, \gamma_q\} = 2 \delta_q^p \mathbf{1}, \quad \{\gamma_p, \gamma_q\} = 2 g_{pq} \mathbf{1}. \quad (3.5)$$

The relationships (3.5) are easily derived from (3.3) with the use of (3.4). The explicit matrix presentation for the operators (3.4) is derived from (2.16):

$$\begin{aligned} \gamma_0 &= \left\| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right\|, & \gamma_1 &= \left\| \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array} \right\|, \\ \gamma_2 &= \left\| \begin{array}{cccc} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{array} \right\|, & \gamma_3 &= \left\| \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right\|. \end{aligned} \quad (3.6)$$

Like the formulas (2.16), these formulas (3.6) are valid if we choose some canonically orthonormal chiral frame in DM and its associated positively polarized right orthonormal frame in TM .

Note that the chirality operator \mathbf{H} can be expressed through the operators $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ and through the other four operators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$:

$$\begin{aligned}\mathbf{H} &= \frac{i}{24} \sum_{p=0}^3 \sum_{q=0}^3 \sum_{k=0}^3 \sum_{m=0}^3 \omega_{pqkm} \gamma^p \gamma^q \gamma^k \gamma^m, \\ \mathbf{H} &= \frac{i}{24} \sum_{p=0}^3 \sum_{q=0}^3 \sum_{k=0}^3 \sum_{m=0}^3 \omega^{pqkm} \gamma_p \gamma_q \gamma_k \gamma_m.\end{aligned}\tag{3.7}$$

Here $24 = 4!$. The formulas (3.7) are easily proved in special frames by means of the formulas (2.7), (2.16), and (3.6). In this case they are reduced to

$$\mathbf{H} = i \gamma^0 \gamma^1 \gamma^2 \gamma^3, \quad \mathbf{H} = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3.\tag{3.8}$$

By means of the direct calculations we find that

$$i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \left\| \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right\| = -i \gamma_0 \gamma_1 \gamma_2 \gamma_3.\tag{3.9}$$

Comparing (3.9) with (2.9), we prove the formulas (3.8). Note that (3.7) are proper tensorial formulas. Having been proved in special frames, they remain valid in an arbitrary pair of frames.

Note that in physical literature the operator $\gamma^5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3$ is introduced (see § 22 in [1]). This is another bad tradition since γ^5 is not a part of the Dirac γ -field. It is a separate spin-tensorial field $\gamma^5 = -\mathbf{H}$. I prefer to use the chirality operator \mathbf{H} instead of the operator γ^5 .

The chirality operator \mathbf{H} anticommutes with the operators $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ and with the operators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$, i. e. we have the equalities:

$$\{\mathbf{H}, \gamma^k\} = 0, \quad \{\mathbf{H}, \gamma_k\} = 0.\tag{3.10}$$

The equalities (3.10) are derived in a special frame by means of the formulas (2.9), (2.16), and (3.6). Then they are extended to arbitrary frame pairs by linearity.

Taking pairs of γ -operators, we can write the following commutation relationships for the operators $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ and for the operators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$:

$$[\mathbf{H}, \gamma^k \gamma^q] = 0, \quad [\mathbf{H}, \gamma_k \gamma_q] = 0.\tag{3.11}$$

In the case of three operators we again have the anticommutation relationships

$$\{\mathbf{H}, \gamma^p \gamma^k \gamma^q\} = 0, \quad \{\mathbf{H}, \gamma_p \gamma_k \gamma_q\} = 0.\tag{3.12}$$

The formulas (3.11) and (3.12) are easily derived from the formula (3.10).

Apart from (3.11) and (3.12) we need some additional formulas — not for commutators and anticommutators, but for the products of γ -operators and the chirality operator \mathbf{H} . In the case of two γ -operators we have

$$\begin{aligned}\mathbf{H}\gamma^p\gamma^q &= \mathbf{H}g^{pq} - \frac{i}{2}\sum_{r=0}^3\sum_{s=0}^3\gamma_r\gamma_s\omega^{rspq}, \\ \mathbf{H}\gamma_p\gamma_q &= \mathbf{H}g_{pq} - \frac{i}{2}\sum_{r=0}^3\sum_{s=0}^3\gamma^r\gamma^s\omega_{rspq}.\end{aligned}\tag{3.13}$$

In the case of three γ -operators we have a little bit more complicated formulas:

$$\begin{aligned}\gamma^p\gamma^q\gamma^r &= g^{pq}\gamma^r + g^{qr}\gamma^p - g^{pr}\gamma^q + i\sum_{s=0}^3\omega^{pqr s}\mathbf{H}\gamma_s, \\ \gamma_p\gamma_q\gamma_r &= g_{pq}\gamma_r + g_{qr}\gamma_p - g_{pr}\gamma_q + i\sum_{s=0}^3\omega_{pqr s}\mathbf{H}\gamma^s.\end{aligned}\tag{3.14}$$

The formulas (3.13) and (3.14) are proved by choosing some special pair of frames where the operators $\mathbf{H}, \gamma^0, \gamma^1, \gamma^2, \gamma^3$ and $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ are given by the formulas (2.9), (2.16), (3.6), while the metric tensor \mathbf{g} is given by the matrix (2.1). For the sake of completeness let's write the following four formulas:

$$\begin{aligned}\gamma^p\gamma^q &= \mathbf{1}g^{pq} - \frac{i}{2}\sum_{r=0}^3\sum_{s=0}^3\mathbf{H}\gamma_r\gamma_s\omega^{rspq}, \\ \gamma_p\gamma_q &= \mathbf{1}g_{pq} - \frac{i}{2}\sum_{r=0}^3\sum_{s=0}^3\mathbf{H}\gamma^r\gamma^s\omega_{rspq}.\end{aligned}\tag{3.15}$$

$$\begin{aligned}\mathbf{H}\gamma^p\gamma^q\gamma^r &= g^{pq}\mathbf{H}\gamma^r + g^{qr}\mathbf{H}\gamma^p - g^{pr}\mathbf{H}\gamma^q + i\sum_{s=0}^3\omega^{pqr s}\gamma_s, \\ \mathbf{H}\gamma_p\gamma_q\gamma_r &= g_{pq}\mathbf{H}\gamma_r + g_{qr}\mathbf{H}\gamma_p - g_{pr}\mathbf{H}\gamma_q + i\sum_{s=0}^3\omega_{pqr s}\gamma^s.\end{aligned}\tag{3.16}$$

We derive (3.15) and (3.16) multiplying both sides of (3.13) and (3.14) by \mathbf{H} and taking into account the first identity (3.1).

4. TRACE FORMULAS.

Note that the traces of all of the γ -operators $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ are equal to zero. The same is true for the γ -operators with lower spatial index $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ as well as for the products of γ -operators and the chirality operator \mathbf{H} :

$$\mathrm{tr}\gamma^k = \sum_{a=1}^4\gamma_a^{ak} = 0, \quad \mathrm{tr}\gamma_k = \sum_{a=1}^4\gamma_{ak}^a = 0.\tag{4.1}$$

$$\mathrm{tr}(\mathbf{H}\gamma^k) = 0, \quad \mathrm{tr}(\mathbf{H}\gamma_k) = 0.\tag{4.2}$$

The formulas (4.1) and (4.2) are proved by direct calculations with the use of the formulas (2.9), (2.16), and (3.6).

Note that $\text{tr}(\mathbf{A}\mathbf{B}) = \text{tr}(\mathbf{B}\mathbf{A})$. Therefore, for the traces of the double products of γ -operators we have the following formulas:

$$\text{tr}(\gamma^p \gamma^q) = 4 g^{pq}, \quad \text{tr}(\gamma_p \gamma_q) = 4 g_{pq}, \quad \text{tr}(\gamma^p \gamma_q) = 4 \delta_q^p. \quad (4.3)$$

The formulas (4.3) are derived from the anticommutation relationships (3.3) and (3.5). Applying the formulas (4.3) to (3.13), we derive

$$\text{tr}(\mathbf{H} \gamma^p \gamma^q) = 0, \quad \text{tr}(\mathbf{H} \gamma_p \gamma_q) = 0, \quad \text{tr}(\mathbf{H} \gamma^p \gamma_q) = 0. \quad (4.4)$$

In order to calculate the traces of triple products of γ -operators we use the formulas (3.14). These formulas immediately yield

$$\text{tr}(\gamma^p \gamma^q \gamma^r) = 0, \quad \text{tr}(\gamma_p \gamma_q \gamma_r) = 0, \quad (4.5)$$

$$\text{tr}(\mathbf{H} \gamma^p \gamma^q \gamma^r) = 0, \quad \text{tr}(\mathbf{H} \gamma_p \gamma_q \gamma_r) = 0. \quad (4.6)$$

The formulas (4.6) are derived from (3.16). In deriving both (4.5) and (4.6) we use the formulas (4.1) and (4.2).

Now let's proceed to the quadruple products of γ -operators. For the beginning let's lower the index r in the first formula (3.14):

$$\gamma^p \gamma^q \gamma_r = g^{pq} \gamma_r + \delta_r^q \gamma^p - \delta_r^p \gamma^q + i \sum_{n=0}^3 \omega^{pqmn} g_{mr} \mathbf{H} \gamma_n, \quad (4.7)$$

Then we multiply the equality (4.7) on the right by γ_s :

$$\gamma^p \gamma^q \gamma_r \gamma_s = g^{pq} \gamma_r \gamma_s + \delta_r^q \gamma^p \gamma_s - \delta_r^p \gamma^q \gamma_s + i \sum_{n=0}^3 \omega^{pqmn} g_{mr} \mathbf{H} \gamma_n \gamma_s,$$

Passing to the traces of both sides of this equality, we take into account the formulas (4.3) and (4.4). As a result we derive

$$\text{tr}(\gamma^p \gamma^q \gamma_r \gamma_s) = 4 g^{pq} g_{rs} + 4 \delta_r^q \delta_s^p - 4 \delta_r^p \delta_s^q. \quad (4.8)$$

In addition to the formula (4.8), there is a formula for $\text{tr}(\mathbf{H} \gamma^p \gamma^q \gamma_r \gamma_s)$. However, in this paper we do not need it.

5. THE INVERSE CONVERSION PROCEDURE.

Let's return back to the conversion formula (1.1). Omitting the spinor indices a and b , we can write it as an operator equality:

$$\mathbf{F} = u \mathbf{1} + v \mathbf{H} + \sum_{k=0}^3 \gamma^k u_k + \sum_{k=0}^3 \mathbf{H} \gamma^k v_k + \sum_{p=0}^3 \sum_{q=0}^3 \gamma^p \gamma^q w_{pq}. \quad (5.1)$$

Applying the formulas (4.1), (4.2), and (4.3) to (5.1), taking into account that

$$\operatorname{tr} \mathbf{1} = 4, \quad \operatorname{tr} \mathbf{H} = 0, \quad (5.2)$$

and remembering the skew-symmetry of w_{pq} , we derive

$$u = \frac{1}{4} \operatorname{tr} \mathbf{F}. \quad (5.3)$$

The inverse conversion procedure is a series of formulas expressing u , v , u_k , v_k , and w_{pq} through \mathbf{F} . The formula (5.3) is the first formula in such a series. Here is the second formula. It expresses v through \mathbf{F} :

$$v = \frac{1}{4} \operatorname{tr}(\mathbf{H} \mathbf{F}). \quad (5.4)$$

The formula (5.4) is derived from (5.1) with the use of the formulas (5.2), (4.1), (4.2), (4.4), and (3.1).

The third conversion formula should express the components of the covector field \mathbf{u} through \mathbf{F} . We derive it multiplying (5.1) on the left by γ_k :

$$u_k = \frac{1}{4} \operatorname{tr}(\gamma_k \mathbf{F}). \quad (5.5)$$

In deriving (5.5) we use the formulas (5.2), (3.10), (4.1), (4.2), (4.3), (4.4), (4.5), and (3.1). The fourth conversion formula is similar to (5.5):

$$v_k = \frac{1}{4} \operatorname{tr}(\gamma_k \mathbf{H} \mathbf{F}). \quad (5.6)$$

In order to derive the fifth conversion formula we multiply the formula (5.1) on the left by $\gamma_q \gamma_p$. Taking the traces of both sides, then we get

$$w_{pq} = \frac{1}{16} \operatorname{tr}(\gamma_q \gamma_p \mathbf{F}) - \frac{1}{16} \operatorname{tr}(\gamma_p \gamma_q \mathbf{F}). \quad (5.7)$$

In deriving (5.7) we use the formulas (4.6), (4.7) and take into account the skew symmetry of w_{pq} with respect to the indices p and q .

The formulas (5.3), (5.4), (5.5), (5.6), and (5.7) constitute the inverse conversion procedure. They prove that the mapping (5.1), which produces a spin-operator \mathbf{F} from a collection of purely spatial fields u , v , \mathbf{u} , \mathbf{v} , \mathbf{w} , is bijective.

6. SYMMETRIC AND SKEW-SYMMETRIC OPERATORS.

Note that the spinor metric \mathbf{d} given by the matrices (2.8) in orthonormal spinor frames defines a skew-symmetric bilinear form in fibers of the Dirac bundle DM :

$$d(\psi, \phi) = \sum_{a=1}^4 \sum_{b=1}^4 d_{ab} \psi^a \phi^b. \quad (6.1)$$

Definition 6.1. A spin-operator field \mathbf{F} is called a *symmetric operator* if it is

symmetric with respect to the bilinear form (6.1), i. e. if $d(\mathbf{F}\psi, \phi) = d(\psi, \mathbf{F}\phi)$ for any two spinor fields ψ and ϕ .

Definition 6.2. A spin-operator field \mathbf{F} is called a *skew-symmetric operator* if it is symmetric with respect to the bilinear form (6.1), i. e. if $d(\mathbf{F}\psi, \phi) = -d(\psi, \mathbf{F}\phi)$ for any two spinor fields ψ and ϕ .

In the coordinate form the symmetry and skew-symmetry conditions are written as

$$\sum_{c=1}^4 F_a^c d_{cb} = \sum_{c=1}^4 d_{ac} F_b^c, \quad \sum_{c=1}^4 F_a^c d_{cb} = -\sum_{c=1}^4 d_{ac} F_b^c. \quad (6.2)$$

Using (6.2), we easily prove that the unit operator $\mathbf{1}$ and the chirality operator \mathbf{H} are symmetric, while the γ -operators γ^k and γ_k are skew-symmetric. The products $\mathbf{H}\gamma^k$ and $\mathbf{H}\gamma_k$ are symmetric. As for the products $\gamma^p\gamma^q$ and $\gamma_p\gamma_q$, they have both symmetric and skew-symmetric components:

$$\begin{aligned} (\gamma^p\gamma^q)_{\text{sym}} &= \mathbf{1}g^{pq}, & (\gamma^p\gamma^q)_{\text{skew}} &= -\frac{i}{2} \sum_{r=0}^3 \sum_{s=0}^3 \mathbf{H}\gamma_r\gamma_s \omega^{rspq}, \\ (\gamma_p\gamma_q)_{\text{sym}} &= \mathbf{1}g_{pq}, & (\gamma_p\gamma_q)_{\text{skew}} &= -\frac{i}{2} \sum_{r=0}^3 \sum_{s=0}^3 \mathbf{H}\gamma^r\gamma^s \omega_{rspq}. \end{aligned} \quad (6.3)$$

The formula (6.3) is easily derived from (3.15). Note that the symmetric parts of the products $\gamma^p\gamma^q$ and $\gamma_p\gamma_q$ are symmetric with respect to the indices p and q , while their skew-symmetric parts are skew-symmetric with respect to these indices. This is not a general rule, but a pure coincidence in this particular case since the operator symmetrization and the operator alternation for spin-operators are not the same as the symmetrization and alternation for spatial indices.

Theorem 6.1. *The spin-operator \mathbf{F} given by the formula (5.1) is a symmetric operator if and only if $u_k = 0$ and $w_{pq} = 0$.*

Theorem 6.2. *The spin-operator \mathbf{F} given by the formula (5.1) is a skew-symmetric operator if and only if $u = 0$, $v = 0$, and $v_k = 0$.*

These two theorems are easily proved on the base of the above results in this section concerning the operators in the right hand side of the expansion (5.1).

7. HERMITIAN AND ANTI-HERMITIAN OPERATORS.

Let's recall that the bundle of Dirac spinors DM is equipped with the Dirac form \mathbf{D} . Its components are given by the matrix (2.10) in self-adjoint spinor frames. Using \mathbf{D} , we can define a sesquilinear form in fibers of the bundle DM :

$$D(\psi, \phi) = \sum_{a=1}^4 \sum_{b=1}^4 D_{a\bar{b}} \overline{\psi^b} \phi^a. \quad (7.1)$$

The sesquilinear form (7.1) is not positive. Its signature is $(+, +, -, -)$.

Definition 7.1. A spin-operator field \mathbf{F} is called a *Hermitian operator* if it is Hermitian with respect to the sesquilinear form (7.1), i. e. if $D(\mathbf{F}\psi, \phi) = D(\psi, \mathbf{F}\phi)$ for any two spinor fields ψ and ϕ .

Definition 7.2. A spin-operator field \mathbf{F} is called a *anti-Hermitian operator* if it is anti-Hermitian with respect to the sesquilinear form (7.1), i. e. if for any two spinor fields ψ and ϕ we have $D(\mathbf{F}\psi, \phi) = -D(\psi, \mathbf{F}\phi)$.

In the coordinate form the conditions of being Hermitian and anti-Hermitian for a spin-operator field \mathbf{F} are written as follows:

$$\sum_{\bar{c}=1}^4 \overline{F_{\bar{b}}^{\bar{c}}} D_{a\bar{c}} = \sum_{c=1}^4 D_{c\bar{b}} F_a^c, \quad \sum_{\bar{c}=1}^4 \overline{F_{\bar{b}}^{\bar{c}}} D_{a\bar{c}} = - \sum_{c=1}^4 D_{c\bar{b}} F_a^c. \quad (7.2)$$

Using (7.2), we easily prove that the unit operator $\mathbf{1}$ is Hermitian, the chirality operator \mathbf{H} is anti-Hermitian, and the γ -operators γ^k and γ_k are Hermitian operators. The products $\mathbf{H}\gamma^k$ and $\mathbf{H}\gamma_k$ are also Hermitian operators. The double products $\gamma^p \gamma^q$ and $\gamma_p \gamma_q$ have both Hermitian and anti-Hermitian parts:

$$\begin{aligned} (\gamma^p \gamma^q)_{\text{Herm}} &= \mathbf{1} g^{pq}, & (\gamma^p \gamma^q)_{\text{anti}} &= -\frac{i}{2} \sum_{r=0}^3 \sum_{s=0}^3 \mathbf{H} \gamma_r \gamma_s \omega^{rspq}, \\ (\gamma_p \gamma_q)_{\text{Herm}} &= \mathbf{1} g_{pq}, & (\gamma_p \gamma_q)_{\text{anti}} &= -\frac{i}{2} \sum_{r=0}^3 \sum_{s=0}^3 \mathbf{H} \gamma^r \gamma^s \omega_{rspq}. \end{aligned} \quad (7.3)$$

Comparing (7.3) with (6.3), we see that the subdivision of $\gamma^p \gamma^q$ and $\gamma_p \gamma_q$ into Hermitian and anti-Hermitian components does coincide with their subdivision into symmetric and skew-symmetric parts. This is not a general rule again, but a pure coincidence in our particular case.

Theorem 7.1. *The spin-operator \mathbf{F} given by the formula (5.1) is a Hermitian operator if and only if we have*

$$u = \bar{u}, \quad v = -\bar{v}, \quad u_k = \bar{u}_k, \quad v_k = \bar{v}_k, \quad w_{pq} = -\overline{w_{pq}}.$$

Theorem 7.2. *The spin-operator \mathbf{F} given by the formula (5.1) is an anti-Hermitian operator if and only if we have*

$$u = -\bar{u}, \quad v = \bar{v}, \quad u_k = -\bar{u}_k, \quad v_k = -\bar{v}_k, \quad w_{pq} = \overline{w_{pq}}.$$

The proof of these two theorems is obvious. Indeed, multiplying a Hermitian and an anti-Hermitian operators by a real scalar, we again get a Hermitian operator and an anti-Hermitian operator respectively. Multiplying these operators by an imaginary scalar, we convert a Hermitian operator into an anti-Hermitian operator and vice versa. Therefore, in order to get a Hermitian operator \mathbf{F} the coefficients of Hermitian operators in the expansion (5.1) should be reals, while the coefficients of anti-Hermitian operators in (5.1) should be imaginary numbers.

8. COMMUTATOR EQUATIONS.

Assume that some spatial frame $\Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3$ is fixed. Then the Dirac γ -field is subdivided into four γ -operators $\gamma^0, \gamma^1, \gamma^2, \gamma^3$ or equivalently $\gamma_0, \gamma_1, \gamma_2, \gamma_3$. Under this assumption we consider the following commutator equations:

$$[\mathbf{F}, \gamma_m] = \mathbf{V}_m. \quad (8.1)$$

Here \mathbf{F} is an undetermined operator, while \mathbf{V}_m are some given operators that constitute a spin-tensorial field of the type $(1, 1|0, 0|0, 1)$.

For the beginning we consider the special case where $\mathbf{V}_m = 0$. In this case the operator \mathbf{F} in (8.1) should commute with the operators $\gamma_0, \gamma_1, \gamma_2, \gamma_3$:

$$[\mathbf{F}, \gamma_m] = 0. \quad (8.2)$$

Note that the γ -matrices (3.6) and their products complemented with the unit matrix span the space of all 4×4 complex matrices. Hence (8.2) means \mathbf{F} commutes with all operators acting in fibers of the Dirac bundle. Such an operator is scalar, i. e. it coincides with the unit operator up to a scalar factor:

$$\mathbf{F} = u \mathbf{1}. \quad (8.3)$$

The operator (8.3) is the general solution of the equation (8.2).

Theorem 8.1. *In the case where the equations (8.1) have a solution, this solution is unique up to the additive complement of the form (8.3).*

In order to solve the equations (8.1) we substitute the conversion formula (5.1) into (8.1) for \mathbf{F} . As a result we obtain the equation

$$\begin{aligned} v[\mathbf{H}, \gamma_m] + \sum_{k=0}^3 u_k [\gamma^k, \gamma_m] + \sum_{k=0}^3 v_k [\mathbf{H} \gamma^k, \gamma_m] + \\ + \sum_{p=0}^3 \sum_{q=0}^3 w_{pq} [\gamma^p \gamma^q, \gamma_m] = \mathbf{V}_m. \end{aligned} \quad (8.4)$$

In order to transform the first term in (8.4) we use (3.10). This formula yields

$$[\mathbf{H}, \gamma_m] = 2 \mathbf{H} \gamma_m - \{\mathbf{H}, \gamma_m\} = 2 \mathbf{H} \gamma_m = \sum_{k=0}^3 2 g_{mk} \mathbf{H} \gamma^k. \quad (8.5)$$

For the second term in (8.4) we use the skew symmetry of the commutator:

$$\begin{aligned} \sum_{k=0}^3 u_k [\gamma^k, \gamma_m] &= \sum_{p=0}^3 \sum_{q=0}^3 (\gamma^p \gamma^q - \gamma^q \gamma^p) g_{qm} = \\ &= \sum_{p=0}^3 \sum_{q=0}^3 \gamma^p \gamma^q (u_p g_{qm} - u_q g_{pm}). \end{aligned} \quad (8.6)$$

Transforming the third term in (8.4), we use the formulas (3.3) and (3.10):

$$\begin{aligned} \sum_{k=0}^3 v_k [\mathbf{H} \gamma^k, \gamma_m] &= \sum_{p=0}^3 \sum_{q=0}^3 v_p (\mathbf{H} \gamma^p \gamma^q - \gamma^q \mathbf{H} \gamma^p) g_{qm} = \\ &= \sum_{p=0}^3 \sum_{q=0}^3 v_p \mathbf{H} (\gamma^p \gamma^q + \gamma^q \gamma^p) g_{qm} = 2 v_m \mathbf{H}. \end{aligned} \quad (8.7)$$

And finally, in order to transform the fourth term in (8.4) we apply the formula (3.14). As a result for this term we derive

$$\begin{aligned} \sum_{p=0}^3 \sum_{q=0}^3 w_{pq} [\gamma^p \gamma^q, \gamma_m] &= \sum_{p=0}^3 \sum_{q=0}^3 \sum_{r=0}^3 w_{pq} (\gamma^p \gamma^q \gamma^r - \gamma^r \gamma^p \gamma^q) g_{rm} = \\ &= \sum_{p=0}^3 \sum_{q=0}^3 \sum_{r=0}^3 2 w_{pq} (g^{qr} \gamma^p - g^{pr} \gamma^q) g_{rm} = \sum_{k=0}^3 4 w_{km} \gamma^k. \end{aligned} \quad (8.8)$$

Note that the operators \mathbf{V}_m in (8.1) can also be expressed in the form of (5.1). In order to distinguish this expression from the original expression (5.1) for the operator \mathbf{F} we set the tilde sign over the coefficients of it:

$$\mathbf{V}_m = \tilde{u}_m \mathbf{1} + \tilde{v}_m \mathbf{H} + \sum_{k=0}^3 \gamma^k \tilde{u}_{mk} + \sum_{k=0}^3 \mathbf{H} \gamma^k \tilde{v}_{mk} + \sum_{p=0}^3 \sum_{q=0}^3 \gamma^p \gamma^q \tilde{w}_{mpq}. \quad (8.9)$$

Using the formulas (8.5), (8.6), (8.7), (8.8), (8.9), we prove the following theorem.

Theorem 8.2. *The commutator equations (8.1) are solvable if and only if the operators \mathbf{V}_m are presented by the formula (8.9) where $\tilde{u}_m = 0$; \tilde{u}_{mk} is skew symmetric; $\tilde{v}_{mk} = 2v g_{mk}$ for some scalar v ; $w_{mpq} = u_p g_{qm} - u_q g_{pm}$ for the components u_p of some covector.*

The theorems 8.1 and 8.2 are helpful in studying Lie derivatives for Dirac spinors. However, this is the subject for a separate paper.

REFERENCES

1. Berestetsky V. B., Lifshits E. M., Pitaevsky L. P., *Quantum Electrodynamics*, Vol. IV of *Theoretical Physics* by L. D. Landau and E. M. Lifshits, Nauka publishers, Moscow, 1989.
2. Sharipov R. A., *A note on Dirac spinors in a non-flat space-time of general relativity*, e-print [math.DG/0601262](https://arxiv.org/abs/math/0601262) in Electronic Archive <http://arXiv.org>.

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