

TRANSFINITE NORMAL AND COMPOSITION SERIES OF GROUPS.

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ABSTRACT. Normal and composition series of groups enumerated by ordinal numbers are studied. The Jordan-Hölder theorem for them is proved.

1. INTRODUCTION.

Finitely long normal and composition series of groups are well-known in group theory. They are given by the following two definitions.

Definition 1.1. Let G be a group. A sequence of its subgroups

$$\{1\} = G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n = G \quad (1.1)$$

is called a normal series of subgroups for G if the subgroup G_i is a normal subgroup in G_{i+1} for all $i = 1, \dots, n-1$.

Definition 1.2. A normal series (1.1) is called a composition series for G if each factorgroup G_{i+1}/G_i is simple.

According to the well-known Jordan-Hölder theorem (see § 3 of Chapter I in [1]), the composition factors G_{i+1}/G_i in a composition series of any group G are unique up to some permutation of indices.

Theorem 1.1 (Jordan-Hölder). *For any two composition series*

$$\{1\} = G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n = G,$$

$$\{1\} = H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_m = G$$

of a group G their lengths are equal to each other, i. e. $n = m$, and there is some permutation σ of the numbers $1, \dots, n-1$ such that

$$G_{i+1}/G_i = H_{\sigma(i)+1}/H_{\sigma(i)}.$$

In this paper we consider normal and composition series of the form (1.1) where n is some ordinal (transfinite) number. We prove the Jordan-Hölder theorem for such transfinite composition series.

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2. BASIC DEFINITIONS.

Let's consider a transfinite series of subgroups of the form (1.1) for some group G . Then n is some ordinal number (see Appendix 5 in [2]). Other indices in the series (1.1) are ordinal numbers less than n . Let's recall that any ordinal number α is either a limit ordinal or a non-limit ordinal:

- 1) if α is a limit ordinal, then for any $\beta < \alpha$ its successor $\beta + 1$ is also less than α ;
- 2) if α is a non-limit ordinal, then $\alpha = \beta + 1$ for some unique $\beta < \alpha$.

Definition 2.1. Let G be a group. A transfinite sequence of subgroups

$$\{1\} = G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n = G \quad (2.1)$$

is called a transfinite normal series of subgroups for the group G if the following two conditions are fulfilled:

- 1) the subgroup G_i is a normal subgroup in G_{i+1} for any ordinal $i < n$;
- 2) $G_\alpha = \bigcup_{\beta < \alpha} G_\beta$ for any limit ordinal $\alpha \leq n$.

Definition 2.2. A group G is called hypertranssimple if it has no normal series (nether finite nor transfinite) other than trivial one $\{1\} = G_1 \subsetneq G_2 = G$.

Definition 2.3. A transfinite normal series (2.1) of a group G is called a transfinite composition series of G if for each ordinal $i < n$ the factorgroup G_{i+1}/G_i is hypertranssimple.

3. THE JORDAN-HÖLDER THEOREM.

Definition 3.1. Let G and H be subgroups of some group. Then $G \cdot H$ is the subgroup generated by elements of both G and H , i. e. $G \cdot H = \langle G \cup H \rangle$.

The subgroup $G \cdot H$ is composed by products of the form $(g_1 h_1) \cdot \dots \cdot (g_n h_n)$ for some integer n , where $g_i \in G$ and $h_i \in H$ for all $i = 1, \dots, n$. If G or H is a normal subgroup in a group enclosing both G and H , then

$$(g_1 h_1) \cdot \dots \cdot (g_n h_n) = g h$$

for some $g \in G$ and some $h \in H$. In this case $G \cdot H$ is composed by products $g h$, where $g \in G$ and $h \in H$.

Note that $G \cdot H = H \cdot G$ even if G and H are not subgroups of an Abelian group. Indeed, $G \cup H = H \cup G$. For this reason $G \cdot H = \langle G \cup H \rangle = \langle H \cup G \rangle = H \cdot G$.

Lemma 3.1 (Zassenhaus). *Let \tilde{G} and \tilde{H} be subgroups of some group and let G and H be normal subgroups of \tilde{G} and \tilde{H} respectively. Then $G \cdot (\tilde{G} \cap H)$ is a normal subgroup in $G \cdot (\tilde{G} \cap \tilde{H})$ and $H \cdot (\tilde{H} \cap G)$ is a normal subgroup in $H \cdot (\tilde{H} \cap \tilde{G})$. The corresponding factorgroups are isomorphic, i. e.*

$$(G \cdot (\tilde{G} \cap \tilde{H})) / (G \cdot (\tilde{G} \cap H)) \cong (H \cdot (\tilde{H} \cap \tilde{G})) / (H \cdot (\tilde{H} \cap G)).$$

The lemma 3.1 is also known as the butterfly lemma. Its proof can be found in §3 of Chapter I in [1].

Definition 3.2. A transfinite normal series $\{1\} = \tilde{G}_1 \subsetneq \tilde{G}_2 \subsetneq \dots \subsetneq \tilde{G}_p = G$ is called a refinement for a transfinite normal series $\{1\} = G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n = G$ if each subgroup G_i coincides with some subgroup \tilde{G}_j .

Definition 3.3. Two transfinite normal series $\{1\} = G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n = G$ and $\{1\} = H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_m = G$ of a group G are called isomorphic if there is a one-to-one mapping that associates each ordinal number $i < n$ with some ordinal number $j < m$ in such a way that $G_{i+1}/G_i \cong H_{j+1}/H_j$.

Theorem 3.1. *Arbitrary two transfinite normal series of a group G have isomorphic refinements.*

Let $\{1\} = G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n = G$ and $\{1\} = H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_m = G$ be two transfinite normal series of some group G . The subgroups G_i in the first series are indexed by ordinal numbers $i \leq n$, the subgroups H_j in the second series are indexed by ordinal numbers $j \leq m$. Let's consider the following subgroups of G indexed by two indices i and j being ordinal numbers:

$$\begin{aligned} G_{ij} &= G_i \cdot (G_{i+1} \cap H_j), \text{ where } i < n \text{ and } j \leq m \\ H_{ji} &= H_j \cdot (H_{j+1} \cap G_i), \text{ where } i \leq n \text{ and } j < m. \end{aligned} \tag{3.1}$$

Applying the Zassenhaus butterfly lemma 3.1 to the subgroups (3.1), we find that G_{ij} is a normal subgroup in G_{i+1} , H_{ji} is a normal subgroup in H_{j+1} , and

$$G_{i+1}/G_{ij} \cong H_{j+1}/H_{ji}. \tag{3.2}$$

The isomorphism (3.2) is a base for proving the theorem 3.1. However, it is not a proof since the subgroups G_{ij} and H_{ji} do not form transfinite normal series yet. In order to complete our proof we need some auxiliary lemmas.

Definition 3.4. An ordered set I is called totally ordered or linearly ordered if any two elements i_1 and i_2 of I are comparable, i. e. $i_1 \neq i_2$ means $i_1 < i_2$ or $i_2 < i_1$.

Definition 3.5. A linearly ordered set I is called well ordered if every non-empty subset $A \subseteq I$ has a minimal element $a_{\min} \in A$. It is clear that such a minimal element a_{\min} in A is unique.

Lemma 3.2. *Let I and J be two well ordered sets. If we denote through $I \times J$ the set of pairs (i, j) , where $i \in I$ and $j \in J$, and if we equip $I \times J$ with the lexicographic ordering, then $I \times J$ is also a well ordered set.*

Proof. The lexicographic ordering in $I \times J$ means that $(i_1, j_1) < (i_2, j_2)$ if $i_1 < i_2$ or if $i_1 = i_2$ and $j_1 < j_2$. It is easy to see that any two pairs (i_1, j_1) and (i_2, j_2) are comparable in this lexicographic ordering. Indeed, if $i_1 \neq i_2$, since I is linearly ordered, we have $i_1 < i_2$ or $i_2 < i_1$. The inequality $i_1 < i_2$ implies $(i_1, j_1) < (i_2, j_2)$, the converse inequality $i_2 < i_1$ implies $(i_2, j_2) < (i_1, j_1)$. In both of these cases the pairs (i_1, j_1) and (i_2, j_2) are comparable.

If $i_1 = i_2$ and $(i_1, j_1) \neq (i_2, j_2)$, then, since J is linearly ordered, we have $j_1 < j_2$ or $j_2 < j_1$. In the first case the equality $i_1 = i_2$ and the inequality $j_1 < j_2$ lead to $(i_1, j_1) < (i_2, j_2)$. Otherwise, from $i_2 = i_1$ and from $j_2 < j_1$ we derive $(i_2, j_2) < (i_1, j_1)$. Again, the pairs (i_1, j_1) and (i_2, j_2) appear to be comparable.

Thus, the set $I \times J$ with the lexicographic ordering is linearly ordered. Let $A \subseteq I \times J$ be some non-empty subset of $I \times J$. Let's denote through A_I the projection of A onto the first component of the direct product $I \times J$:

$$A_I = \{i \in I: \exists j \in J \text{ such that } (i, j) \in A\}.$$

The subset $A_I \subset I$ is not empty. Since I is well ordered, there is a unique minimal element $i_{\min} \in A_I$. Now let's denote through A_J the following subset of J :

$$A_J = \{j \in J: (i_{\min}, j) \in A\}.$$

The subset $A_J \subseteq J$ is also not empty. Since J is well ordered, the subset A_J has a unique minimal element $j_{\min} \in A_J$. Then (i_{\min}, j_{\min}) is a unique minimal element of the subset $A \subseteq I \times J$. The lemma 3.2 is proved. \square

Definition 3.6. Let $\overset{R}{\sim}$ be an equivalence relation in a linearly ordered set I . The equivalence relation $\overset{R}{\sim}$ is said to be concordant with the linear ordering in I if $i_1 < i_2 < i_3$ and $i_1 \overset{R}{\sim} i_3$ imply $i_1 \overset{R}{\sim} i_2$ and $i_2 \overset{R}{\sim} i_3$.

Let I be a linearly ordered set and let $\overset{R}{\sim}$ be an equivalence relation in I concordant with its linear ordering. Then the factorset I/R can be equipped with the factorordering. It is introduced as follows: for any two distinct equivalence classes $\text{Cl}_R(i_1) \neq \text{Cl}_R(i_2)$ we declare $\text{Cl}_R(i_1) < \text{Cl}_R(i_2)$ if $i_1 < i_2$. This definition is self-consistent, i. e. the inequality $\text{Cl}_R(i_1) < \text{Cl}_R(i_2)$ does not depend on our choice of the representatives i_i and i_2 within the equivalence classes $\text{Cl}_R(i_1)$ and $\text{Cl}_R(i_2)$. Indeed, assume that $\text{Cl}_R(i_3) = \text{Cl}_R(i_1)$ and $i_3 \neq i_1$. Since I is linearly ordered, we have two options: $i_3 < i_2$ or $i_2 < i_3$. If $i_2 < i_3$ we would have $i_1 < i_2 < i_3$ and $i_1 \overset{R}{\sim} i_3$. Applying the concordance condition (see Definition 3.6), we would derive $i_1 \overset{R}{\sim} i_2$, which contradicts $\text{Cl}_R(i_1) \neq \text{Cl}_R(i_2)$. Thus, the option $i_2 < i_3$ is excluded and we have $i_3 < i_2$, which means $\text{Cl}_R(i_3) < \text{Cl}_R(i_2)$.

Similarly, if we assume that $\text{Cl}_R(i_4) = \text{Cl}_R(i_2)$ and $i_4 \neq i_2$, then, applying the concordance condition, we derive $i_1 < i_4$, which means $\text{Cl}_R(i_1) < \text{Cl}_R(i_4)$. Thus, the definition of the factorordering in the factorset I/R is self-consistent.

Lemma 3.3. *If I is a well ordered set and if $\overset{R}{\sim}$ is an equivalence relation concordant with the ordering in I , then the factorset I/R equipped with the factorordering is a well ordered set too.*

Proof. The well ordered set I is linearly ordered. It is easy to see that the factorordering in I/R is a linear ordering too. Let A be a non-empty subset of the factorset I/R . Let's denote through \tilde{A} its preimage in I

$$\tilde{A} = \{i \in I: \text{Cl}_R(i) \in A\}.$$

The subset $\tilde{A} \subset I$ is non-empty. Hence, since I is well ordered, there is a unique minimal element \tilde{a}_{\min} in \tilde{A} . Let's denote $a = \text{Cl}_R(\tilde{a}_{\min})$. It is clear that a is a minimal element of A . The proof is over. \square

Let's return back to the subgroups (3.1). In order to describe these subgroups let's introduce the following sets of ordinal numbers:

$$\begin{aligned} I &= \{\alpha \in \text{Ord}: \alpha \leq n\}, & I' &= \{\alpha \in \text{Ord}: \alpha < n\}, \\ J &= \{\beta \in \text{Ord}: \beta \leq m\}, & J' &= \{\beta \in \text{Ord}: \beta < m\}. \end{aligned}$$

The subgroups G_{ij} in (3.1) are indexed by the set $I' \times J$ ordered lexicographically. Similarly, the subgroups H_{ji} are indexed by the set $J' \times I$ ordered lexicographically.

There are two types of elements in the ordered set $I' \times J$ — regular elements of the form (i, j) , where $i < n$ and $j < m$, and irregular elements of the form (i, m) , where $i < n$. Similarly, the ordered set $J' \times I$ has regular elements of the form (j, i) , where $i < n$ and $j < m$, and irregular elements (j, n) , where $j < m$. Note that regular elements of the set $I' \times J$ are in a one-to-one association with regular elements of the set $J' \times I$. Indeed, we have

$$(i, j) \begin{array}{c} \xrightarrow{\theta} \\ \xleftarrow{\theta} \end{array} (j, i). \quad (3.3)$$

The formula (3.3) means that the association θ is composed by two bijective mappings inverse to each other. Both of them are denoted through θ .

Let $\alpha = (i, j)$ be a regular element of the well ordered set $I' \times J$ and let $\beta = (j, i)$ be its associated element in $J' \times I$. Then

$$\beta = \theta(\alpha), \quad \alpha = \theta(\beta).$$

The next element to α is $\alpha + 1 = (i, j + 1)$. Similarly, the next element to β is $\beta + 1 = (j, i + 1)$. Applying the formula (3.2), we derive

$$G_{\alpha+1}/G_{\alpha} \cong H_{\beta+1}/H_{\beta}$$

for each regular element $\alpha \in I' \times J$ and its associated regular element $\beta \in J' \times I$.

Now let $\alpha = (i, m)$ be an irregular element of the well ordered set $I' \times J$. Then, applying the first formula (3.1), we get $G_{\alpha} = G_{im} = G_{i+1}$. The next element to α in this case is $\alpha + 1 = (i + 1, 1)$. Applying the first formula (3.1) again, we get $G_{\alpha+1} = G_{i+11} = G_{i+1}$. Thus we have

$$G_{\alpha+1} = G_{\alpha} \quad (3.4)$$

for each irregular element $\alpha \in I' \times J$. Similarly, the second formula (3.1) yields

$$H_{\beta+1} = H_{\beta} \quad (3.5)$$

for each irregular element $\beta \in J' \times I$.

If n is a non-limit ordinal, then $n = (n - 1) + 1$. In this case the well ordered set $I' \times J$ has the maximal element $\alpha_{\max} = (n - 1, m)$ which is irregular. If n is a limit ordinal, then the well ordered set $I' \times J$ has no maximal elements. In this case we extend this set by adjoining an auxiliary element $\alpha_{\max} = (n, 1)$ to it:

$$IJ = \begin{cases} I' \times J & \text{if } n \text{ is a non-limit ordinal;} \\ (I' \times J) \cup \{(n, 1)\} & \text{if } n \text{ is a limit ordinal.} \end{cases} \quad (3.6)$$

We extend the ordering of $I' \times J$ to IJ by declaring the auxiliary element $(n, 1)$ to be greater than all elements of $I' \times J$. It is easy to see that the extended set (3.6) is a well ordered set possessing the maximal element α_{\max} . If n is a non-limit ordinal, the first formula (3.1) yields

$$G_{\alpha_{\max}} = G_{n-1m} = G. \quad (3.7)$$

If n is a limit ordinal, we extend the first formula (3.1) by setting

$$G_{\alpha_{\max}} = G_{n-1} = G. \quad (3.8)$$

The second well ordered set $J' \times I$ can also be incomplete. In this case we extend it by adjoining an auxiliary element to this set:

$$JI = \begin{cases} J' \times I & \text{if } m \text{ is a non-limit ordinal;} \\ (J' \times I) \cup \{(m, 1)\} & \text{if } m \text{ is a limit ordinal.} \end{cases} \quad (3.9)$$

The ordering of the extended set (3.9) extends the ordering of $J' \times I$ so that the auxiliary element $\beta_{\max} = (m, 1)$ is the maximal element of JI . We extend the second formula (3.1) to this element by setting

$$H_{\beta_{\max}} = H_{m-1} = G. \quad (3.10)$$

If m is a non-limit ordinal the maximal element of JI is given by the formula

$$H_{\beta_{\max}} = H_{m-1} = G. \quad (3.11)$$

Like the extended set IJ in (3.6), the extended set JI in (3.9) is a well ordered set possessing its maximal element β_{\max} .

Due to (3.4) and (3.5) irregular elements of the well ordered sets IJ and JI can be identified (glued) with the regular elements next to them. As for the regular elements, some of them can also be identified with other regular elements. In order to perform such identifications we introduce some special equivalence relations in the well ordered sets IJ and JI .

Definition 3.7. Two elements $\alpha_1 = (i_1, j_1)$ and $\alpha_2 = (i_2, j_2)$ of the set IJ are declared to be equivalent $\alpha_1 \stackrel{R}{\sim} \alpha_2$ if $G_{\alpha_1} = G_{\alpha_2}$, i. e. if $G_{i_1 j_1} = G_{i_2 j_2}$.

Definition 3.8. Two elements $\beta_1 = (j_1, i_1)$ and $\beta_2 = (j_2, i_2)$ of the set JI are declared to be equivalent $\beta_1 \stackrel{R}{\sim} \beta_2$ if $G_{\beta_1} = G_{\beta_2}$, i. e. if $G_{j_1 i_1} = G_{j_2 i_2}$.

Lemma 3.4. *The equivalence relations introduced by the definitions 3.7 and 3.8 are concordant with the orderings in IJ and JI in the sense of the definition 3.6.*

Proof. Using the formulas (3.1) and its extensions (3.8) and (3.10), it is easy to prove that for two elements α_1 and α_2 of the ordered set IJ the inequality $\alpha_1 < \alpha_2$ implies $G_{\alpha_1} \subseteq G_{\alpha_2}$. Relying on the definition 3.6, assume that $\alpha_1 < \alpha_2 < \alpha_3$ and $\alpha_1 \stackrel{R}{\sim} \alpha_3$. Then we have the following relationships:

$$G_{\alpha_1} \subseteq G_{\alpha_2} \subseteq G_{\alpha_3} \qquad G_{\alpha_1} = G_{\alpha_3}. \quad (3.12)$$

From (3.12) we immediately derive $G_{\alpha_1} = G_{\alpha_2}$ and $G_{\alpha_2} = G_{\alpha_3}$ which means that $\alpha_1 \stackrel{R}{\sim} \alpha_2$ and $\alpha_2 \stackrel{R}{\sim} \alpha_3$. Thus the proposition of the lemma 3.4 is proved for the equivalence relation $\stackrel{R}{\sim}$ in IJ . In the case of the second ordered set JI the proof is similar since in this case the inequality $\beta_1 < \beta_2$ implies $H_{\beta_1} \subseteq H_{\beta_2}$. \square

Factorizing the well ordered sets IJ and JI with respect to the equivalence relations introduced in the definitions 3.7 and 3.8, we get two factorsets

$$IJR = (IJ)/R, \qquad JIR = (JI)/R. \quad (3.13)$$

According to the lemmas 3.3 and 3.4, the factorsets (3.13) are well ordered. It is known that each well ordered set is isomorphic to some initial segment in the class of ordinal numbers (see Appendix 3 in [2]). The well ordered sets IJ and JJ have the maximal elements α_{\max} and β_{\max} . Their equivalence classes are maximal elements in the factorsets IJR and JIR respectively. For this reason there are two ordinal numbers p and q such that

$$\begin{aligned} IJR &\cong \{1, \dots, p\} = \{\alpha \in \text{Ord} : \alpha \leq p\}, \\ JIR &\cong \{1, \dots, q\} = \{\beta \in \text{Ord} : \beta \leq q\}. \end{aligned} \quad (3.14)$$

The ordinal numbers p and q in (3.14) correspond to the equivalence classes of the maximal elements α_{\max} and β_{\max} respectively.

According to the definitions 3.7 and 3.8, the factorsets (3.13) can be used for indexing the subgroups G_{ij} and H_{ji} . Applying (3.14) and taking into account (3.7), (3.8), (3.10), and (3.11), we get two transfinite sequences of subgroups of G :

$$\begin{aligned} 1 &= G_1 \subsetneq \dots \subsetneq G_p = G, \\ 1 &= H_1 \subsetneq \dots \subsetneq H_q = G. \end{aligned} \quad (3.15)$$

The next step is to prove that the series (3.15) are transfinite normal series in the sense of the definition 2.1. Let's consider the first series (3.15). Assume that r be some ordinal less than p . Then $r+1 \leq p$ is the ordinal next to r . The ordinals r and $r+1$ are associated with two immediately adjacent classes A and $A+1$

$$A = \text{Cl}_R(a), \quad A+1 = \text{Cl}_R(b) \quad (3.16)$$

in the factorset IJR . The upper class $A+1$ in (3.16) is a non-empty subset of the well ordered set IJ (as well as the lower class A). It has some unique minimal element b_{\min} . Without loss of generality we can assume that $b = b_{\min}$.

Lemma 3.5. *For any two immediately adjacent classes A and $A+1$ of the factorset IJR the upper class $A+1$ has the minimal element b_{\min} , while the lower class A has the maximal element a_{\max} such that $b_{\min} = a_{\max} + 1$.*

Proof. In order to prove the lemma 3.5 it is sufficient to prove that b_{\min} is a non-limit element of the well ordered set IJ . The proof is by contradiction. Assume that b_{\min} is a limit element of the set IJ . There are two options for b_{\min} :

- 1) $b_{\min} = (i, 1)$, where $i \leq n$ is a limit ordinal;
- 2) $b_{\min} = (i, j)$, where $i < n$ is an arbitrary ordinal and $j \leq m$ is a limit ordinal.

In the first case, applying the first formula (3.1) or the formula (3.8), we derive

$$G_{b_{\min}} = G_{i1} = G_i. \quad (3.17)$$

Since i is a limit ordinal, applying the item 2 of the definition 2.1, we get

$$G_i = \bigcup_{\alpha < i} G_\alpha = \bigcup_{\alpha < i} G_{\alpha+1} = \bigcup_{\alpha < i} G_{\alpha m}. \quad (3.18)$$

Combining the formula (3.17) with the formula (3.18), we derive

$$G_{b_{\min}} = \bigcup_{\alpha < i} G_{\alpha m}. \quad (3.19)$$

Note that $\alpha < i$ implies the inequality $(\alpha, m) < (i, 1)$, where $(i, 1) = b_{\min}$. Since b_{\min} is the minimal element of its class, we have $\text{Cl}_R(\alpha, m) < \text{Cl}_R(b_{\min}) = A + 1$. This inequality can be rewritten as follows:

$$\text{Cl}_R(\alpha, m) \leq A \text{ for all } \alpha < i. \quad (3.20)$$

The class A is not empty. Therefore there is at least one ordinal number $\alpha < i$ such that $\text{Cl}_R(\alpha, m) = A$. This equality and the above inequality (3.20) lead to the following relationships for subgroups:

$$\begin{aligned} G_{\alpha m} &\subseteq G_A \text{ for all } \alpha < i; \\ G_{\alpha m} &= G_A \text{ for some } \alpha < i. \end{aligned} \quad (3.21)$$

Applying (3.21) to (3.19), we derive $G_{b_{\min}} = G_A$. But, on the other hand, we have $G_{b_{\min}} = G_{A+1} \neq G_A$ since $A \neq A + 1$. Thus, the first of the above two options for b_{\min} leads to a contradiction.

Let's proceed to the second option. In this case $b_{\min} = (i, j)$, where $i < n$ is some arbitrary ordinal and $j \leq m$ is some limit ordinal. The item 2 of the definition 2.1 applied to the series $\{1\} = H_1 \subsetneq \dots \subsetneq H_m = G$ says that

$$H_j = \bigcup_{\beta < j} H_\beta. \quad (3.22)$$

Combining (3.22) with the first formula (3.1), we derive the formula

$$G_{b_{\min}} = G_{ij} = \bigcup_{\beta < j} G_{i\beta}. \quad (3.23)$$

This formula (3.23) is similar to (3.19). Using the arguments quite similar to the above ones, we get the following relationships:

$$\begin{aligned} G_{i\beta} &\subseteq G_A \text{ for all } \beta < j; \\ G_{i\beta} &= G_A \text{ for some } \beta < j. \end{aligned} \quad (3.24)$$

Applying (3.24) to (3.23), we derive $G_{b_{\min}} = G_A$. But, on the other hand, we have $G_{b_{\min}} = G_{A+1} \neq G_A$. As we see, the second of the above two options for b_{\min} also leads to a contradiction. Due to these two contradictions we conclude that b_{\min} is a non-limit element of the well ordered set IJ . Then $b_{\min} = a + 1$ for some unique element $a \in IJ$. It is clear that $a = a_{\max}$ is the maximal element in the lower class A . The lemma is proved. \square

Lemma 3.6. *For any two immediately adjacent classes B and $B+1$ of the factorset JIR the upper class $B + 1$ has the minimal element b_{\min} , while the lower class B has the maximal element a_{\max} such that $b_{\min} = a_{\max} + 1$.*

The lemma 3.6 is an analog of the previous lemma 3.5. For this reason it does not require a separate proof.

A remark. Not each subset in a well ordered set has a maximal element, but, according to the lemma 3.5, each equivalence class in IJ has. According to the lemma 3.6, the same is true for equivalence classes in JI .

Lemma 3.7. *Let A be a limit class of the factorset IJR . Then*

$$G_A = \bigcup_{B < A} G_B. \quad (3.25)$$

Proof. Let a_{\min} be the minimal element of the class A . Then a_{\min} is a limit element of the well ordered set IJ . There are the following options for a_{\min} :

- 1) $a_{\min} = (i, 1)$, where $i \leq n$ is a limit ordinal;
- 2) $a_{\min} = (i, j)$, where $i < n$ is an arbitrary ordinal and $j \leq m$ is a limit ordinal.

In the first case we have the following formula for the subgroup G_A :

$$G_A = G_{a_{\min}} = \bigcup_{\alpha < i} G_{\alpha}. \quad (3.26)$$

The arguments are the same as in deriving the formulas (3.17) and (3.18). Note that each class $B < A$ in this case is represented by some element $b = (\alpha, j)$, where $\alpha < i$ and $1 \leq j \leq m$. The inequalities $1 \leq j \leq m$ yield

$$G_{\alpha} = G_{\alpha 1} \subseteq G_{\alpha j} \subseteq G_{\alpha m} = G_{\alpha+1}. \quad (3.27)$$

Since i is a limit ordinal, combining (3.26) and (3.27), we derive

$$G_A = \bigcup_{\substack{\alpha < i \\ j \leq m}} G_{\alpha j} = \bigcup_{B < A} G_B.$$

Thus, the formula (3.25) is proved for the first case where $a_{\min} = (i, 1)$ and $i \leq n$ is some limit ordinal.

Let's proceed to the second case $a_{\min} = (i, j)$, where $i < n$ is some arbitrary ordinal and where $j \leq m$ is some limit ordinal. In this case we have

$$G_{a_{\min}} = G_{ij} = \bigcup_{\beta < j} G_{i\beta}. \quad (3.28)$$

The formula (3.28) is identical to (3.23). Each class $B < A$ is represented by some element $b = (\alpha, \beta)$ such that $b < a_{\min}$. Due to the lexicographic ordering in IJ there are two types of such elements $b = (\alpha, \beta)$ — those with $\alpha < i$ and those with $\alpha = i$ and $\beta < j$. For the elements of the first type we have

$$G_{\alpha\beta} \subseteq G_{\alpha m} = G_{\alpha+11} \subseteq G_{i1}. \quad (3.29)$$

Due to the inclusions (3.29) we can derive the following formula:

$$\bigcup_{B < A} G_B = \left(\bigcup_{\substack{\alpha < i \\ \beta \leq m}} G_{\alpha\beta} \right) \cup \left(\bigcup_{\beta < j} G_{i\beta} \right) = \bigcup_{\beta < j} G_{i\beta}. \quad (3.30)$$

Comparing (3.30) with (3.28), we see that the formula (3.25) is proved for the second case. Thus the lemma 3.7 is completely proved. \square

Lemma 3.8. *Let B be a limit class of the factorset JIR . Then*

$$G_B = \bigcup_{A < B} G_A. \quad (3.31)$$

The lemma 3.8 is an analog of the lemma 3.7, while the formula (3.31) is an analog of the formula (3.25).

Now let's return back to the series (3.15). Recall that they were built by the subgroups (3.1), (3.8) and (3.10) in such a way that they obey the item 1 of the definition 2.1. On the other hand, the lemmas 3.7 and 3.8 prove that they obey the item 2 of the definition 2.1 as well. Thus the series (3.15) are transfinite normal series of subgroups for the group G in the sense of the definition 2.1.

Note that $G_{i_1} = G_i$ and $H_{j_1} = H_j$. For this reason all subgroups of the initial series $\{1\} = G_1 \subsetneq \dots \subsetneq G_n = G$ and $\{1\} = H_1 \subsetneq \dots \subsetneq H_m = G$ are among the subgroups of the series (3.15), i. e. the transfinite normal series (3.15) are refinements of the initial series $\{1\} = G_1 \subsetneq \dots \subsetneq G_n = G$ and $\{1\} = H_1 \subsetneq \dots \subsetneq H_m = G$ in the sense of the definition 3.2.

Lemma 3.9. *The transfinite normal series (3.15) of the group G are isomorphic to each other in the sense of the definition 3.3.*

Proof. The proof is based on the lemmas 3.5 and 3.6. Due to (3.14) each factor-group G_{A+1}/G_A of the first normal series (3.15) is associated with some pair of immediately adjacent equivalence classes A and $A + 1$ being the elements of the factorset $IJR = (IJ)/R$. According to the lemma 3.5, the lower of these two classes has the maximal element $a_{\max} = (i, j)$. Then

$$G_{i_{j+1}}/G_{ij} = G_{A+1}/G_A \neq \{1\}. \quad (3.32)$$

Applying the formula (3.2) to (3.32), we get

$$G_{A+1}/G_A \cong H_{j_{i+1}}/H_{ji} \neq \{1\}. \quad (3.33)$$

Let's denote through B the equivalence class of the element $b = (j, i)$, i. e. let $B = \text{Cl}_R(j, i)$. Note that $i < n$ and $j < m$ in (3.33). Hence $b + 1 = (j, i + 1)$ and

$$H_{b+1}/H_b = H_{j_{i+1}}/H_{ji} \neq \{1\}. \quad (3.34)$$

The formula (3.34) means that $b = b_{\max}$ is the maximal element of its class B and

$$G_{A+1}/G_A \cong H_{B+1}/H_B \neq \{1\}. \quad (3.35)$$

Thus, we have proved that the upper mapping θ in (3.3) goes through the factorization procedures in IJ and JI and associates each class A having its successor $A + 1$ in IJR with some unique class B having its successor $B + 1$ in JIR . Applying the above arguments to a class B of JIR , we find that same is true for the lower mapping θ in (3.3). Like the initial mappings (3.3), the factorized mappings θ are bijective and inverse to each other. The formula (3.35) shows that these factorized mappings constitute an isomorphism for the transfinite normal series (3.15). The lemma 3.9 is proved. \square

The lemma 3.9 proved just above completes the proof of the theorem 3.1.

Lemma 3.10. *If $\{1\} = G_1 \subsetneq \dots \subsetneq G_n = G$ is a transfinite composition series of a group G , then it has no refinements different from itself.*

Proof. The proof is by contradiction. Assume that the transfinite composition series $\{1\} = G_1 \subsetneq \dots \subsetneq G_n = G$ of the group G has some nontrivial refinement $\{1\} = \tilde{G}_1 \subsetneq \dots \subsetneq \tilde{G}_p = G$ different from $\{1\} = G_1 \subsetneq \dots \subsetneq G_n = G$. According to the definition 3.3, each G_i of the first series coincides with some G_j in the second series. In other words we have an injective mapping

$$\sigma: \{i \in \text{Ord}: i \leq n\} \longrightarrow \{j \in \text{Ord}: j \leq p\}$$

such that $\tilde{G}_{\sigma(i)} = G_i$ for each $i \leq n$. Since $G_1 = \tilde{G}_1$ and $G_n = \tilde{G}_p$, we have

$$\sigma(1) = 1, \quad \sigma(n) = p \neq n. \quad (3.36)$$

The set $I = \{i \in \text{Ord}: i \leq n\}$ subdivides into two subsets

$$I_1 = \{i \in I: \sigma(i) = i\}, \quad I_2 = \{i \in I: \sigma(i) \neq i\}. \quad (3.37)$$

Due to (3.36) both subsets (3.37) are not empty. Let's denote through $i_2 = i_{\min}$ the minimal element of the subset I_2 .

The index i_2 is a non-limit ordinal. Indeed, otherwise we would have

$$G_{i_2} = \bigcup_{\alpha < i_2} G_\alpha, \quad \tilde{G}_{i_2} = \bigcup_{\alpha < i_2} \tilde{G}_\alpha = \bigcup_{\alpha < i_2} G_\alpha,$$

which yields $\tilde{G}_{i_2} = G_{i_2}$ and $\sigma(i_2) = i_2$. Since i_2 is a non-limit ordinal, the first subset I_1 in (3.37) has the maximal element $i_1 = i_{\max}$ such that $i_2 = i_1 + 1$. In other words, i_1 and i_2 are two neighboring ordinals such that

$$\sigma(i_1) = i_1, \quad \sigma(i_2) > i_2. \quad (3.38)$$

Due to the relationships (3.38) we have the following segment in the refined normal series $\{1\} = \tilde{G}_1 \subsetneq \dots \subsetneq \tilde{G}_p = G$ of the group G :

$$G_{i_1} = \tilde{G}_{i_1} \subsetneq \tilde{G}_{i_1+1} \subsetneq \dots \subsetneq \tilde{G}_{i_1+q} = G_{i_2} \quad (3.39)$$

Here $q > 1$ is some ordinal number. Note that $i_2 = i_1 + 1$. Hence G_{i_1} is a normal subgroup of G_{i_2} . Then G_{i_1} is a normal subgroup of each group in the sequence (3.39). Passing to factorgroups in the sequence (3.39), we get

$$\{1\} = \tilde{G}_{i_1}/G_{i_1} \subsetneq \dots \subsetneq \tilde{G}_{i_1+q}/G_{i_1} = G_{i_1+1}/G_{i_1}. \quad (3.40)$$

It is easy to show that (3.40) is a normal series of the factorgroup G_{i_1+1}/G_{i_1} . But, since $\{1\} = G_1 \subsetneq \dots \subsetneq G_n = G$ is a transfinite composition series, its factorgroup G_{i_1+1}/G_{i_1} is hypertranssimple (see Definition 2.3). Applying the definition 2.2, we find that the normal series (3.40) should be trivial, i. e. it should look like

$$\{1\} = \tilde{G}_{i_1}/G_{i_1} \subsetneq \tilde{G}_{i_1+1}/G_{i_1} = G_{i_1+1}/G_{i_1}.$$

This means that $q = 1$ and $\tilde{G}_{i_1+1} = G_{i_1+1}$. Then $\tilde{G}_{i_2} = G_{i_2}$ and $\sigma(i_2) = i_2$ which contradicts (3.38). The contradiction obtained completes the proof. \square

Theorem 3.2 (Jordan-Hölder). *Any two transfinite composition series of a group G are isomorphic.*

Proving the theorem 3.2 is the main goal of this paper. Now its proof is immediate from the theorem 3.1 and the lemma 3.10.

A remark. Note that if we have two transfinite composition series

$$\{1\} = G_1 \subsetneq G_2 \subsetneq \dots \subsetneq G_n = G,$$

$$\{1\} = H_1 \subsetneq H_2 \subsetneq \dots \subsetneq H_m = G,$$

their isomorphism (see Definition 3.3) does not mean $n = m$. It means only that n and m are two ordinal numbers of the same cardinality, i. e. $|n| = |m|$.

4. DEDICATORY.

This paper is dedicated to my aunt Halila Yusupova who was born before the First World War and lived her long life in works and permanent cares.

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