TRANSFINITE NORMAL AND COMPOSITION SERIES OF MODULES.

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ABSTRACT. Normal and composition series of modules enumerated by ordinal numbers are studied. The Jordan-Hölder theorem for them is discussed.

1. INTRODUCTION AND SOME PRELIMINARIES.

Transfinite normal and composition series of groups were studied in [1]. They generalize classical normal and composition series which are finitely long. In this paper we reproduce the results of [1] in the case of left modules over associative rings and associative algebras.

Definition 1.1. Let A be an associative ring and let V be an additive Abelian group. The Abelian group V is called a left module over the ring A (or a left A-module) if some homomorphism of rings

$$\varphi \colon A \to \mathrm{End}(V)$$

is given and fixed. Here $\mathrm{End}(V)$ is the ring of endomorphisms of the additive Abelian group V.

Definition 1.2. Let A be an associative algebra over some field $\mathbb{K}$ and let V be a linear vector space over the same field $\mathbb{K}$. The linear vector space V is called a left module over the algebra A (left A-module) if some homomorphism of algebras

$$\varphi \colon A \to \mathrm{End}(V)$$

is given and fixed. Here $\mathrm{End}(V)$ is the algebra of endomorphisms of the linear vector space V.

In both cases each element $a \in A$ produces an operator $\varphi(a)$ that can act upon any element $v \in V$. The result of applying $\varphi(a)$ to v is denoted as

$$u = \varphi(a)(v). \tag{1.1}$$

In some cases the formula (1.1) is written as follows:

$$u = \varphi(a)v. \tag{1.2}$$

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The endomorphism $\varphi(a) \in \mathrm{End}(V)$ in this formula is presented as a left multiplier for the element $v \in V$. In some cases the symbol φ is also omitted:

$$u = a\,v. \tag{1.3}$$

The formula (1.3) is a purely algebraic version of the formulas (1.1) and (1.2). The action of $\varphi(a)$ upon v here is written as a multiplication of v on the left by the element $a \in A$. This is the reason for which V is called a left A-module. Using (1.3), the definitions 1.1 and 1.2 can reformulated as follows.

Definition 1.3. A left module over an associative ring A is an additive Abelian group V equipped with an auxiliary operation of left multiplication by elements of the ring A such that the following identities are fulfilled:

1) $a\,(v_1 + v_2) = a\,v_1 + a\,v_2$ for all $a \in A$ and for all $v_1, v_2 \in V$;
2) $(a_1 + a_2)\,v = a_1\,v + a_2\,v$ for all $a_1, a_2 \in A$ and for all $v \in V$;
3) $(a_1\,a_2)\,v = a_1\,(a_2\,v)$ for all $a_1, a_2 \in A$ and for all $v \in V$.

Definition 1.4. A left module over an associative $\mathbb{K}$-algebra A is a linear vector space V over the field $\mathbb{K}$ equipped with an auxiliary operation of left multiplication by elements of the algebra A such that the following identities are fulfilled:

1) $a\,(v_1 + v_2) = a\,v_1 + a\,v_2$ for all $a \in A$ and for all $v_1, v_2 \in V$;
2) $(a_1 + a_2)\,v = a_1\,v + a_2\,v$ for all $a_1, a_2 \in A$ and for all $v \in V$;
3) $(a_1\,a_2)\,v = a_1\,(a_2\,v)$ for all $a_1, a_2 \in A$ and for all $v \in V$;
4) $(k\,a)\,v = a\,(k\,v) = k\,(a\,v)$ for all $k \in \mathbb{K}$, $a \in A$, and $v \in V$.

Right A-modules are similar to left ones. Here elements $v \in V$ are multiplied by elements $a \in A$ on the right. However, each right A-module can be treated as a left $A^\bullet$-module, where $A^\bullet$ is the opposite ring (opposite algebra) for A (see Chapter 5 in [2]). For this reason below we consider left modules only.

2. Submodules and factormodules.

Definition 2.1. Let V be a left A-module, where A is a ring. A subset $W \subseteq V$ is called a submodule of V if it is closed with respect to the inversion, with respect to the addition, and with respect to the multiplication by elements of the ring A:

1) $v \in W$ implies $-v \in W$;
2) $v_1 \in W$ and $v_2 \in W$ imply $(v_1 + v_2) \in W$;
3) $v \in W$ and $a \in A$ imply $(a\,v) \in W$.

In other words, a submodule W is a subgroup of the additive group V invariant with respect to the endomorphisms $\varphi(a)$ for all $a \in A$.

Definition 2.2. Let V be a left A-module, where A is some $\mathbb{K}$-algebra. A subset $W \subseteq V$ is called a submodule of V if it is closed with respect to the addition and with respect to the multiplication by elements of the field $\mathbb{K}$ and the algebra A:

1) $v \in W$ and $k \in \mathbb{K}$ imply $(k\,v) \in W$;
2) $v_1 \in W$ and $v_2 \in W$ imply $(v_1 + v_2) \in W$;
3) $v \in W$ and $a \in A$ imply $(a\,v) \in W$.

In other words, a submodule W is a subspace of the vector space V invariant with respect to the endomorphisms $\varphi(a)$ for all $a \in A$.

In both cases a submodule $W \subseteq V$ inherits the structure of a left A-module from V. Each submodule W is associated with the corresponding factorset V/W. The factorset V/W is composed by cosets

$$\mathrm{Cl}_W(v) = \{u \in V : u = v + w \text{ for some } w \in W\}. \tag{2.1}$$

The element v in (2.1) is a representative of the coset $\mathrm{Cl}_W(v)$. Any element $v \in \mathrm{Cl}_W(v)$ can be chosen as its representative. In both cases (where A is a ring or where A is an algebra) the factorset V/W inherits the structure of a left A-module from V. For this reason it is called a factormodule. Algebraic operations with cosets are given by the formulas

1) $\mathrm{Cl}_W(v_1) + \mathrm{Cl}_W(v_2) = \mathrm{Cl}_W(v_1 + v_2)$ for any $v_1, v_2 \in V$;
2) $a\,\mathrm{Cl}_W(v) = \mathrm{Cl}_W(a\,v)$ for any $a \in A$ and for any $v \in V$.

If A is a $\mathbb{K}$-algebra, then

3) $k\,\mathrm{Cl}_W(v) = \mathrm{Cl}_W(k\,v)$ for any $k \in \mathbb{K}$ and for any $v \in V$.

3. Transfinite normal and composition series.

Definition 3.1. Let V be a left A-module. A transfinite sequence of its submodules

$$\{0\} = V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n = V \tag{3.1}$$

is called a transfinite normal series for V if $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$ for any limit ordinal $\alpha \leqslant n$.

The term "normal series" in the above definition comes from the group theory where each normal series $\{1\} = G_1 \subsetneq G_2 \subsetneq \ldots \subsetneq G_n = G$ of a group G should be composed by its subgroups such that G_i is a normal subgroup of G_{i+1}. Otherwise we would not be able to build the factorgroup G_{i+1}/G_i. In the case of modules each submodule V_i with $i < n$ produces the factormodule V_{i+1}/V_i. So the term "normal series" here is vestigial, it is used for ear comfort only.

Definition 3.2. A module V is called hypertranssimple if it has no normal series (neither finite nor transfinite) other than trivial one $\{0\} = V_1 \subsetneq V_2 = V$.

Definition 3.3. A transfinite normal series (3.1) of a module V is called a transfinite composition series of V if for each ordinal number $i < n$ the factormodule V_{i+1}/V_i is hypertranssimple.

The concept of hypertranssimplicity is nontrivial in the case of groups. In the case of modules each nontrivial submodule W of V produces the nontrivial normal series $\{0\} = V_1 \subsetneq V_2 \subsetneq V_3 = V$, where $V_2 = W$. For this reason the concept of hypertranssimplicity of modules reduces to the standard concept of simplicity. The definitions 3.2 and 3.3 then are reformulated as follows.

Definition 3.4. A module V is called simple if it has no submodules other than trivial ones $V_1 = \{0\}$ and $V_2 = V$.

Definition 3.5. A transfinite normal series (3.1) of a module V is called a transfinite composition series of V if for each ordinal number $i < n$ the corresponding factormodule V_{i+1}/V_i is simple.

4. Intersections and sums of submodules.

The intersection of two or more submodules of a given module V is again a submodule of V. As for unions, the union of submodules in general case is not a submodule. Unions of submodules are used to define their sums.

Definition 4.1. Let U_i with $i \in I$ be submodules of some module V. The submodule U generated by the union of submodules U_i is called their sum:

$$U = \sum_{i \in I} U_i = \left\langle \bigcup_{i \in I} U_i \right\rangle. \tag{4.1}$$

If the number of submodules in (4.1) is finite, one can use the following notations:

$$U = U_1 + \ldots + U_n = \langle U_1 \cup \ldots \cup U_n \rangle. \tag{4.2}$$

In both cases (4.1) and (4.2) each element $u \in U$ is presented as a finite sum

$$u = u_{i_1} + \ldots + u_{i_s}, \quad \text{where} \quad u_{i_r} \in U_{i_r} \quad \text{and} \quad i_r \in I \quad \text{for all} \quad r = 1, \ldots, s. \tag{4.3}$$

Definition 4.2. The sum of submodules (4.1) is called a direct sum if for each element $u \in U$ its presentation (4.3) is unique.

Lemma 4.1 (Zassenhaus). *Let $\tilde U$ and $\tilde W$ be submodules of some left A-module and let U and W be submodules of $\tilde U$ and $\tilde W$ respectively. Then*

$$(U + (\tilde U \cap \tilde W))/(U + (\tilde U \cap W)) \cong (W + (\tilde W \cap \tilde U))/(W + (\tilde W \cap U)). \tag{4.4}$$

The lemma 4.1 is also known as the butterfly lemma. Typically the butterfly lemma is formulated for groups (see §3 of Chapter I in [3]). However, its proof can be easily adapted for the case of modules.

Proof. Let's denote $M = U + (\tilde U \cap W)$ and $N = W + (\tilde W \cap U)$. Elements of the factormodule in the left hand side of the formula (4.4) are cosets of the form

$$\mathrm{Cl}_M(u + a), \quad \text{where} \quad u \in U \quad \text{and} \quad a \in \tilde U \cap \tilde W.$$

Note that $U \subseteq M$. Therefore $\mathrm{Cl}_M(u+a) = \mathrm{Cl}_M(a)$ which means that each element of the factormodule $(U + (\tilde U \cap \tilde W))/M$ is represented by some element $a \in \tilde U \cap \tilde W$. Thus we have a surjective homomorphism of modules

$$\varphi \colon \tilde U \cap \tilde W \longrightarrow (U + (\tilde U \cap \tilde W))/M. \tag{4.5}$$

Repeating the above arguments for the factormodule in the right hand side of the formula (4.4), we get another surjective homomorphism of modules

$$\psi \colon \tilde U \cap \tilde W \longrightarrow (W + (\tilde W \cap \tilde U))/N. \tag{4.6}$$

The rest is to prove that $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\psi$. Assume that $a \in \mathrm{Ker}\,\varphi$. In this case $a \in \tilde U \cap \tilde W$ and $a \in M$. The inclusion $a \in M$ means that $a = u + w$, where $u \in U$ and $w \in \tilde U \cap W$. Since $\tilde U \cap W \subseteq \tilde U \cap \tilde W$, from $u = a - w$ we derive $u \in \tilde U \cap \tilde W$. On

the other hand $u \in U$. Hence $u \in U \cap (\tilde{U} \cap \tilde{W})$, which means $u \in U \cap \tilde{W}$. Thus we have proved that each element $a \in \operatorname{Ker} \varphi$ is presented as a sum

$$a = u + w, \ \text{ where } \ u \in \tilde{W} \cap U \ \text{ and } \ w \in \tilde{U} \cap W. \tag{4.7}$$

Conversely, it is easy to see that the presentation (4.7) leads to $a \in \tilde{W} \cap \tilde{U}$ and $a \in M$, i.e. $a \in \operatorname{Ker} \varphi$. For this reason we have $\operatorname{Ker} \varphi = (\tilde{W} \cap U) + (\tilde{U} \cap W)$. The equality $\operatorname{Ker} \psi = (\tilde{W} \cap U) + (\tilde{U} \cap W)$ is proved similarly. Now the formula (4.4) is immediate from $\operatorname{Ker} \varphi = \operatorname{Ker} \psi$ due to the surjectivity of the homomorphisms (4.5) and (4.6). The butterfly lemma 4.1 is proved. $\square$

5. The Jordan-Hölder theorem.

Definition 5.1. A transfinite normal series $\{0\} = \tilde{V}_1 \subsetneq \tilde{V}_2 \subsetneq \ldots \subsetneq \tilde{V}_p = G$ is called a refinement for a transfinite normal series $\{0\} = V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n = G$ if each submodule V_i coincides with some submodule $\tilde{V}_j$.

Definition 5.2. Two transfinite normal series $\{0\} = V_1 \subsetneq V_2 \subsetneq \ldots \subsetneq V_n = V$ and $\{0\} = W_1 \subsetneq W_2 \subsetneq \ldots \subsetneq W_m = V$ of a module V are called isomorphic if there is a one-to-one mapping that associates each ordinal number $i < n$ with some ordinal number $j < m$ in such a way that $V_{i+1}/V_i \cong W_{j+1}/W_j$.

Theorem 5.1. *Arbitrary two transfinite normal series of a left A-module V have isomorphic refinements.*

Lemma 5.1. *If $\{1\} = G_1 \subsetneq \ldots \subsetneq G_n = G$ is a transfinite composition series of a group G, then it has no refinements different from itself.*

Theorem 5.2 (Jordan-Hölder). *Any two transfinite composition series of a left A-module V are isomorphic.*

The theorem 5.2 is immediate from the theorem 5.1 and the lemma 5.1. As for the theorem 5.1 and the lemma 5.1, their proof is quite similar to the proof of the theorem 3.1 and the lemma 3.10 in [1]. The algebraic part of this proof is based on the Zassenhaus butterfly lemma. Its version for modules is given above (see lemma 4.1). The other part of the proof deals with indexing sets and ordinal numbers, not with algebraic structures. For this reason it does not differ in the case of groups and in the case of modules.

6. External direct sums.

Sums and direct sums introduced in the definitions 4.1 and 4.2 are internal ones. They are formed by submodules of a given module. External direct sums are formed by separate modules which are not necessarily enclosed in a given module.

Definition 6.1. Let V_i be left A-modules enumerated by elements $i \in I$ of some indexing set I. Finite formal sums of the form

$$v = v_{i_1} + \ldots + v_{i_s}, \ \text{ where } \ v_{i_r} \in V_{i_r} \ \text{ and } \ i_r \in I \ \text{ for all } \ r = 1, \ldots, s, \tag{6.1}$$

constitute a left A-module V which is called the direct sum of the modules V_i.

Once the external direct sum V is constructed, we find that it comprises submodules U_i isomorphic to the initial modules V_i. Indeed, we can set $s = 1$ in (6.1).

References

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