A NOTE ON KHABIBULLIN'S CONJECTURE
FOR INTEGRAL INEQUALITIES.

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Abstract. An integral transformation relating two inequalities in Khabibullin's conjecture is found. Another proof of this conjecture for some special values of its numeric parameters is suggested.

1. Introduction.

Conjecture 1.1 (Khabibullin). *Let* $\alpha > 1/2$ *and let* $q = q(t)$ *be a positive continuous function on the half-line* $[0, +\infty)$. *Then the inequality*

$$\int\limits_0^1 \left(\int\limits_x^1 (1-y)^{n-1}\, \frac{dy}{y} \right) q(tx)\, dx \leqslant t^{\alpha-1} \tag{1.1}$$

fulfilled for all $0 \leqslant t < +\infty$ *implies the inequality*

$$\int\limits_0^{+\infty} q(t)\, \ln\left(1 + \frac{1}{t^{2\alpha}}\right) dt \leqslant \pi\,\alpha \prod_{k=1}^{n-1} \left(1 + \frac{\alpha}{k}\right). \tag{1.2}$$

The conjecture 1.1 arose in [1] (see also [2]), though in some different form. The statement of this conjecture in the above form is given in [3].

Note that the conjecture 1.1 is formulated for $\alpha > 1/2$. Actually it could be formulated for all positive α in the following way.

Conjecture 1.2 (Khabibullin). *Let* $\alpha > 0$ *and let* $q = q(t)$ *be a positive continuous function on the half-line* $[0, +\infty)$. *Then the inequality*

$$\int\limits_0^1 \left(\int\limits_x^1 (1-y)^{n-1}\, \frac{dy}{y} \right) q(tx)\, dx \leqslant t^{\alpha-1} \tag{1.3}$$

fulfilled for all $0 \leqslant t < +\infty$ *implies the inequality*

$$\int\limits_0^{+\infty} q(t)\, \ln\left(1 + \frac{1}{t^{2\alpha}}\right) dt \leqslant \pi\,\alpha \prod_{k=1}^{n-1} \left(1 + \frac{\alpha}{k}\right). \tag{1.4}$$

The matter is that in [4] the conjecture 1.2 is already proved to be valid for $0 < \alpha \leqslant 1/2$. Despite this fact, in the present paper we shall use the conjecture 1.1 in its more general form 1.2. The main goal of this paper is to give another treatment of the inequalities (1.1), (1.2), (1.3), (1.4) and to give another proof of the conjecture 1.2 for the case $0 < \alpha \leqslant 1/2$. Some other particular values of the parameters n and α are also considered.

2. Relation to Euler's Beta function.

The product in the right hand side of the inequalities (1.2) and (1.4) is related to Euler's Beta function. We have the following relationship:

$$\alpha \prod_{k=1}^{n-1} \left(1 + \frac{\alpha}{k}\right) = \frac{1}{\mathrm{B}(\alpha, n)}. \tag{2.1}$$

Let's recall that Euler's Beta function $\mathrm{B}(\alpha, \beta)$ is defined through the integral

$$\mathrm{B}(\alpha, \beta) = \int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, dx. \tag{2.2}$$

The Beta function (2.2) is expressed through Euler's Gamma function

$$\mathrm{B}(\alpha, \beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha + \beta)}. \tag{2.3}$$

The value of the Gamma function $\Gamma(\beta)$ in (2.3) is expressed through the factorial for integer values of the argument $\beta = n$:

$$\Gamma(n) = (n-1)! \ \text{ for all } \ n = 1, 2, 3 \dots. \tag{2.4}$$

Applying (2.4) to (2.3), we recall the following property of the Gamma function:

$$\Gamma(\alpha) = (\alpha - 1)\,\Gamma(\alpha - 1). \tag{2.5}$$

Now from (2.3), (2.4), and (2.5) we derive the formula

$$\mathrm{B}(\alpha, n) = \frac{(n-1)!}{(\alpha + n - 1)\,(\alpha + n - 2) \cdot \ldots \cdot (\alpha + 1)\,\alpha}. \tag{2.6}$$

The formula (2.6) can be transformed as follows:

$$\mathrm{B}(\alpha, n) = \frac{1}{\alpha} \prod_{k=1}^{n-1} \frac{k}{k + \alpha} = \frac{1}{\alpha} \prod_{k=1}^{n-1} \frac{1}{1 + \dfrac{\alpha}{k}}. \tag{2.7}$$

Now it is easy to see that the formula (2.7) is equivalent to the relationship (2.1). Thus, the formula (2.1) is proved.

Note that the formula (2.3) as well as the formulas equivalent to (2.4) and (2.5) can be found in Chapter XVII of the book [5]. The proofs of these formulas are also available there.

3. THE STUDY OF THE KERNEL.

Let's denote through $A_n(x)$ the following integral:

$$A_n(x) = \int\limits_x^1 (1 - y)^n \, \frac{dy}{y}. \tag{3.1}$$

The integral (3.1) is used as a kernel in the integral inequality (1.3). In terms of our notation $A_n(x)$ the inequality (1.3) looks like

$$\int\limits_0^1 A_{n-1}(x) \, q(tx) \, dx \leqslant t^{\alpha-1}. \tag{3.2}$$

The integral (3.1) can be calculated explicitly. We use the relationship

$$A_n(x) = A_{n-1}(x) - \frac{(1-x)^n}{n} \quad \text{for} \ \ n \geqslant 1. \tag{3.3}$$

The recurrent relationship is derived immediately from (3.1). Indeed, we have

$$A_n(x) = \int\limits_x^1 (1 - y) \, (1 - y)^{n-1} \, \frac{dy}{y} = \int\limits_x^1 (1 - y)^{n-1} \, \frac{dy}{y} -$$

$$- \int\limits_x^1 (1 - y)^{n-1} \, dy = A_{n-1}(x) + \frac{(1-y)^n}{n} \bigg|_x^1 = A_{n-1}(x) - \frac{(1-x)^n}{n}.$$

Applying the relationship (3.3) recursively, we derive

$$A_n(x) = A_0(x) - \sum_{m=1}^n \frac{(1-x)^m}{m}. \tag{3.4}$$

Note that the term $A_0(x)$ in (3.3) is calculated explicitly. Indeed, we have

$$A_0(x) = \int\limits_x^1 \frac{dy}{y} = \ln y \bigg|_x^1 = -\ln x. \tag{3.5}$$

Lemma 3.1. *The kernel $A_n(x)$ is given by the explicit formula*

$$A_n(x) = -\ln x - \sum_{m=1}^n \frac{(1-x)^m}{m}. \tag{3.6}$$

The proof of the lemma 3.1 is immediate from (3.4) and (3.5).

Lemma 3.2. *The kernel $A_n(x)$ is a continuous function on the interval $(0, 1]$ with a logarithmic singularity at the the point $x = 0$. It vanishes at the point $x = 1$.*

The proof of the lemma 3.2 is immediate from the formula (3.6).

Note that the logarithmic function has the following Tailor expansion:

$$- \ln x = - \ln(1 - (1 - x)) = \sum_{m=1}^{\infty} \frac{(1 - x)^m}{m}. \tag{3.7}$$

Combining (3.7) with the formula (3.6), we get

$$A_n(x) = \sum_{m=n+1}^{\infty} \frac{(1 - x)^m}{m}. \tag{3.8}$$

Lemma 3.3. *The kernel $A_n(x)$ is a positive decreasing function on the interval $(0, 1)$ vanishing at the point $x = 1$. The kernel function $A_n(x)$ and its derivatives $A_n'(x)$, $A_n''(x)$, $\ldots$, $A_n^{(n)}(x)$ up to the n-th order do vanish at the point $x = 1$.*
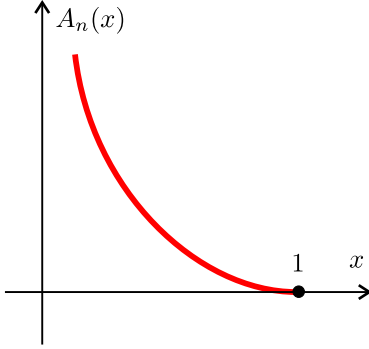
Proof. The power series (3.8) converges on the interval $(0, 1]$. Each term of this series is a positive decreasing function on the interval $(0, 1]$. Hence the kernel $A_n(x)$ is a positive decreasing function on the interval $(0, 1]$.

The equality $A_n(1) = 0$ and the equalities $A_n^{(k)}(1) = 0$ for $k = 1, \ldots, n$ are derived from the formula (3.8). $\square$

Fig 3.1

Relying on the lemmas 3.2 and 3.3, one can plot the graph of the kernel function $A_n(x)$. This graph is shown above on Fig. 3.1.

Since all terms of the power series (3.8) are positive on the interval $(0, 1)$, we derive the following inequalities for the kernel functions on this interval:

$$A_0(x) > A_1(x) > A_2(x) > \ldots > A_n(x) > \ldots . \tag{3.9}$$

These kernel functions (3.9) vanish simultaneously at the point $x = 1$.

4. THE BASIC EXAMPLE.

Let's consider the function $q(x) = C\, x^{\alpha-1}$, where $\alpha > 1/2$ and $C = \text{const} > 0$. If $\alpha < 1$, this function is not continuous at $x = 0$. Nevertheless, the integral

$$\int_0^1 A_{n-1}(x)\, q(tx)\, dx = C\, t^{\alpha-1} \int_0^1 A_{n-1}(x)\, x^{\alpha-1}\, dx \tag{4.1}$$

converges at the point $x = 0$ for any $\alpha > 0$. Relying on (4.1), we denote

$$I_{n\,\alpha} = \int_0^1 A_{n-1}(x)\, x^{\alpha-1}\, dx. \tag{4.2}$$

The constant $I_{n\,\alpha}$ can be calculated explicitly. Substituting (3.6) into (4.2), we get

$$I_{n\,\alpha} = -\int_0^1 \ln(x)\, x^{\alpha-1}\, dx - \sum_{m=1}^{n-1} \int_0^1 \frac{x^{\alpha-1}\,(1-x)^m}{m}\, dx. \qquad (4.3)$$

For $\alpha > 0$ the first integral from (4.3) is calculated as follows:

$$\int_0^1 \ln(x)\, x^{\alpha-1}\, dx = \ln(x)\, \frac{x^\alpha}{\alpha}\Big|_0^1 - \int_0^1 \frac{x^{\alpha-1}}{\alpha}\, dx = -\frac{x^\alpha}{\alpha^2}\Big|_0^1 = -\frac{1}{\alpha^2}. \qquad (4.4)$$

Due to (2.2) the second integral from (4.3) is expressed through the Beta function:

$$\int_0^1 x^{\alpha-1}\,(1-x)^m\, dx = B(\alpha, m+1). \qquad (4.5)$$

Combining (4.4) and (4.5), we derive

$$I_{n\,\alpha} = \frac{1}{\alpha^2} - \sum_{m=1}^{n-1} \frac{B(\alpha, m+1)}{m}. \qquad (4.6)$$

Note that there is another way for calculating the constant $I_{n\,\alpha}$. From the formula (4.2) we immediately derive the following expression for $I_{n\,\alpha}$:

$$I_{n\,\alpha} = A_{n-1}(x)\, \frac{x^\alpha}{\alpha}\Big|_0^1 - \int_0^1 \frac{A'_{n-1}(x)\, x^\alpha}{\alpha}\, dx. \qquad (4.7)$$

Since $\alpha > 0$ and $A_{n-1}(1) = 0$ for each integer $n \geqslant 1$, the formula (4.7) yields

$$I_{n\,\alpha} = -\int_0^1 \frac{A'_{n-1}(x)\, x^\alpha}{\alpha}\, dx = \int_0^1 \frac{x^{\alpha-1}\,(1-x)^{n-1}}{\alpha}\, dx = \frac{B(\alpha, n)}{\alpha}. \qquad (4.8)$$

Comparing (4.8) and (4.6), for each integer $n \geqslant 1$ we derive the identity

$$\frac{B(\alpha, n)}{\alpha} = \frac{1}{\alpha^2} - \sum_{m=1}^{n-1} \frac{B(\alpha, m+1)}{m}. \qquad (4.9)$$

For $n = 1$ the identity (4.9) simplifies. In this case it is written as follows:

$$B(\alpha, 1) = \frac{1}{\alpha}. \qquad (4.10)$$

The equality (4.10) is valid for $\alpha > 0$. It is easily derived immediately from the formula (2.2) defining Euler's Beta function.

For $\alpha > 0$ and for each integer $n \geqslant 2$, applying the formula (2.7) to (4.9), we transform the identity (4.9) to the following one:

$$\frac{1}{\alpha} \prod_{k=1}^{n-1} \frac{k}{k+\alpha} = \frac{1}{\alpha} - \sum_{m=1}^{n-1} \left(\prod_{k=1}^{m} \frac{k}{m\,(k+\alpha)} \right). \tag{4.11}$$

For $n = 2$ the identity (4.11) is verified by means of direct calculations. Then for each integer $n > 2$ it is proved by induction on n.

Now, returning back to the function $q(x) = C\,x^{\alpha-1}$, we choose $C = 1/I_{n\,\alpha}$. Under this special choice of the constant C the function

$$q(x) = \frac{x^{\alpha-1}}{I_{n\,\alpha}} = \frac{\alpha\,x^{\alpha-1}}{B(\alpha, n)} \tag{4.12}$$

turns the inequality (3.2) into the equality

$$\int_0^1 A_{n-1}(x)\,q(tx)\,dx = t^{\alpha-1}. \tag{4.13}$$

Let's substitute the function (4.12) into the integral in the left hand side of (1.4):

$$\int_0^{+\infty} q(t)\,\ln\left(1 + \frac{1}{t^{2\alpha}}\right) dt = \frac{\alpha}{B(\alpha, n)} \int_0^{+\infty} t^{\alpha-1}\,\ln\left(1 + \frac{1}{t^{2\alpha}}\right) dt. \tag{4.14}$$

The integral in the right hand side of (4.14) is calculated explicitly as an indefinite integral. Indeed, by differentiation one can verify that

$$\int^x t^{\alpha-1}\,\ln\left(1 + \frac{1}{t^{2\alpha}}\right) dt = \frac{t^\alpha}{\alpha}\,\ln\left(1 + \frac{1}{t^{2\alpha}}\right) - \frac{2}{\alpha}\,\arctan(t^{-\alpha}). \tag{4.15}$$

Assuming that $\alpha > 0$ and applying the formula (4.15), we calculate the integral

$$\int_0^{+\infty} t^{\alpha-1}\,\ln\left(1 + \frac{1}{t^{2\alpha}}\right) dt = \frac{\pi}{\alpha}. \tag{4.16}$$

Now from (4.14) and (4.16) for the function (4.12) we derive the equality

$$\int_0^{+\infty} q(t)\,\ln\left(1 + \frac{1}{t^{2\alpha}}\right) dt = \frac{\pi}{B(\alpha, n)}. \tag{4.17}$$

Combining (4.17) with (2.1), we find that the inequality (1.4) turns to the equality for the function (4.12). Indeed, we have

$$\int_0^{+\infty} q(t)\,\ln\left(1 + \frac{1}{t^{2\alpha}}\right) dt = \pi\,\alpha \prod_{k=1}^{n-1} \left(1 + \frac{\alpha}{k}\right). \tag{4.18}$$

Lemma 4.1. *The special choice of the function $q(x)$ given by the formula* (4.12) *turn both inequalities* (1.3) *and* (1.4) *into equalities.*

The lemma 4.1 is immediate from (4.13) and (4.18). This lemma could be a motivation for the conjecture 1.2.

5. SOME INTEGRAL RELATIONSHIPS.

Let $\varphi(t)$ be a smooth function defined on the interval $(0, +\infty)$ and such that it satisfies the following asymptotic condition for some $\varepsilon > 0$:

$$\varphi(t) = O(t^{-\varepsilon}) \quad \text{as} \quad t \to +\infty. \tag{5.1}$$

In addition to (5.1), assume that its derivatives satisfy the conditions

$$\varphi^{(s)}(t) = \frac{d^s\varphi}{dt^s} = O(t^{-s-\varepsilon}) \quad \text{as} \quad t \to +\infty \quad \text{for all} \quad s = 1, \dots, n+2. \tag{5.2}$$

By means of the function $\varphi(t)$ we define the function

$$\Phi_n(t) = -\frac{d}{dt}\left(\frac{(-t)^{n+1}}{n!}\,\varphi^{(n+1)}(t)\right), \quad \text{where} \quad \varphi^{(n+1)}(t) = \frac{d^{n+1}\varphi}{dt^{n+1}}. \tag{5.3}$$

Lemma 5.1. *The function* (5.3) *satisfies the integral relationship*

$$\int\limits_y^{+\infty} \Phi_n(t)\, A_n(y/t)\, dt = \varphi(y) \quad \text{for} \quad n \geqslant 0, \tag{5.4}$$

provided the conditions (5.1) *and* (5.2) *are fulfilled.*

The kernel function $A_n(x)$ has the logarithmic singularity at the point $x = 0$ (see Lemma 3.2). Indeed, from (3.6) we derive $A_n(x) \sim -\ln(x)$ as $x \to 0$. Then the function $A_n(y/t)$ has the logarithmic singularity at the infinity:

$$A_n(y/t) \sim \ln t \quad \text{as} \quad t \to +\infty. \tag{5.5}$$

On the other hand, from (5.1) and (5.2) for the function (5.3) we derive

$$\Phi_n(t) = O(t^{-1-\varepsilon}) \quad \text{as} \quad t \to +\infty. \tag{5.6}$$

Combining (5.5) with (5.6), we find that the integral (5.4) converges at infinity. Since the product $\Phi_n(t)\, A_n(y/t)$ has no singularities at finite points $t \in [y, +\infty)$, the integral (5.4) converges in whole.

Proof of the lemma 5.1. The proof is pure calculations. Upon substituting (5.3) into the integral (5.4) we can integrate by parts:

$$\int\limits_y^{+\infty} \Phi_n(t)\, A_n(y/t)\, dt = -\int\limits_y^{+\infty} \frac{d}{dt}\left(\frac{(-t)^{n+1}}{n!}\,\varphi^{(n+1)}(t)\right) A_n(y/t)\, dt =$$

$$= -\frac{(-t)^{n+1}}{n!}\,\varphi^{(n+1)}(t)\,A_n(y/t)\Big|_y^\infty - \int\limits_y^{+\infty}\frac{(-t)^{n+1}}{n!}\,\varphi^{(n+1)}(t)\,A'_n(y/t)\,\frac{y\,dt}{t^2}.$$

From (5.2) we derive $(-t)^{n+1}\,\varphi^{(n+1)}(t) = O(t^{-\varepsilon})$ as $t \to +\infty$. Therefore the product $(-t)^{n+1}\,\varphi^{(n+1)}(t)$ suppresses the logarithmic singularity of the function $A_n(y/t)$ at infinity. As for the lower limit $t = y$, the function $A_n(y/t)$ vanishes at this point. Indeed, we have $A_n(y/y) = A_n(1) = 0$ (see Lemma 3.2 or Lemma 3.3). As a result the integral (5.4) reduces to the following one:

$$\int\limits_y^{+\infty}\Phi_n(t)\,A_n(y/t)\,dt = -\int\limits_y^{+\infty}\frac{(-t)^{n-1}}{n!}\,\varphi^{(n+1)}(t)\,A'_n(y/t)\,y\,dt. \tag{5.7}$$

The first derivative $A'_n(y/t)$ in (5.7) can be calculated explicitly. Indeed, differentiating the formula (3.6), we derive the following expression for $A'_n(x)$:

$$A'_n(x) = -\frac{1}{x} + \sum_{m=1}^{n}(1-x)^{m-1}. \tag{5.8}$$

The sum in (5.8) is the sum of a geometric progression. It is calculated explicitly:

$$A'_n(x) = -\frac{1}{x} + \frac{(1-x)^n - 1}{(1-x) - 1} = \frac{-(1-x)^n}{x}. \tag{5.9}$$

Substituting $x = y/t$ into the formula (5.9), we get

$$A'_n(y/t) = \frac{-(1-y/t)^n}{y/t} = \frac{-(t-y)^n}{t^{n-1}\,y}. \tag{5.10}$$

The next step is to substitute (5.10) into (5.7). As a result we derive

$$\int\limits_y^{+\infty}\Phi_n(t)\,A_n(y/t)\,dt = -\int\limits_y^{+\infty}\frac{(y-t)^n}{n!}\,\varphi^{(n+1)}(t)\,dt. \tag{5.11}$$

The integral in the right hand side of the formula can be calculated by means of integrating by parts. Indeed, we easily derive the formula

$$\int\limits_y^{+\infty}\Phi_n(t)\,A_n(y/t)\,dt = -\frac{(y-t)^n}{n!}\,\varphi^{(n)}(t)\Big|_y^\infty - \int\limits_y^{+\infty}\frac{(y-t)^{n-1}}{(n-1)!}\,\varphi^{(n)}(t)\,dt.$$

Note that $(y-t)^n\,\varphi^{(n)}(t) = O(t^{-\varepsilon})$ as $t \to +\infty$ due to (5.2). Moreover, $(y-t)^n = 0$ if $n > 0$ and $t = y$. Therefore the above formula reduces to

$$\int\limits_y^{+\infty}\Phi_n(t)\,A_n(y/t)\,dt = -\int\limits_y^{+\infty}\frac{(y-t)^{n-1}}{(n-1)!}\,\varphi^{(n)}(t)\,dt. \tag{5.12}$$

Comparing (5.11) and (5.12), we see that the above calculations let us to pass from n to $n-1$ in the right had side of the formula (5.11). Performing these calculations repeatedly, we derive the following formula:

$$\int\limits_y^{+\infty} \Phi_n(t)\, A_n(y/t)\, dt = -\int\limits_y^{+\infty} \varphi'(t)\, dt. \tag{5.13}$$

Due to (5.1) the integral in the right hand side of (5.13) is transformed to

$$-\int\limits_y^{+\infty} \varphi'(t)\, dt = -\varphi(t)\Big|_y^{\infty} = \varphi(y). \tag{5.14}$$

Combining the formulas (5.13) and (5.14) we derive the required formula (5.4). Thus, the lemma 5.1 is proved for $n \geqslant 2$.

If $n=1$ the formula (5.12) coincides with (5.13). Similarly, if $n=0$ the formula (5.11) coincides with (5.13). Therefore, the formula (5.4) is proved for all $n \geqslant 0$. The proof of the lemma 5.1 is over. $\square$

Now, in addition to (5.1) and (5.2), assume that the function $\varphi(t)$ satisfies the following auxiliary condition at the point $t=0$:

$$\lim_{t\to+0} \left(t^{s+\omega}\, \varphi^{(s)} \right) = 0 \ \text{ for any } \ \omega > 0 \ \text{ and for all } \ s = 0,\, \ldots,\, n+2. \tag{5.15}$$

Applying (5.15) to (5.3), we find that $\Phi_n(t)$ satisfies the condition

$$\lim_{t\to+0} \left(t^{1+\omega}\, \Phi_n(t) \right) = 0 \ \text{ for any } \ \omega > 0. \tag{5.16}$$

Lemma 5.2. *For any $0 < \alpha < \varepsilon$ the function (5.3) satisfies the integral relationship*

$$\int\limits_0^{+\infty} \Phi_n(t)\, t^\alpha\, dt = -\alpha \prod_{k=1}^n \left(1 + \frac{\alpha}{k} \right) \int\limits_0^{+\infty} t^\alpha\, \varphi'(t)\, dt \quad \textit{for} \quad n \geqslant 0, \tag{5.17}$$

provided the conditions (5.1), (5.2), and (5.15) are fulfilled.

Assume that the conditions (5.1), (5.2), and (5.15) are fulfilled. Then the function $\Phi_n(t)$ satisfies the condition (5.6). If $\alpha < \varepsilon$, the condition (5.6) means that the integral in the left hand side of the equality (5.17) converges at infinity. The integral in the right hand side of the equality (5.17) also converges at infinity due to the condition (5.2) and the inequality $\alpha < \varepsilon$.

The condition (5.15) leads to the condition (5.16) for the function $\Phi_n(t)$. Hence from $\alpha > 0$ we derive that the integral in the left hand side of the equality (5.17) converges at the point $t = 0$. Similarly the integral in the right hand side of the equality (5.17) converges at $t = 0$ due to (5.15) and the inequality $\alpha > 0$.

Proof of the lemma 5.2. The proof is pure calculations. Upon substituting the

formula (5.3) into the left hand side of (5.17) we can integrate by parts:

$$\int\limits_0^{+\infty} \Phi_n(t)\, t^\alpha\, dt = -\int\limits_0^{+\infty} \frac{d}{dt}\left(\frac{(-t)^{n+1}}{n!}\, \varphi^{(n+1)}(t)\right) t^\alpha\, dt =$$

$$= -\frac{(-1)^{n+1}}{n!}\, t^{\alpha+n+1}\, \varphi^{(n+1)}(t)\Big|_0^\infty - \alpha\, \frac{(-1)^n}{n!}\int\limits_0^{+\infty} t^{\alpha+n}\, \varphi^{(n+1)}(t)\, dt.$$

Note that $t^{\alpha+n+1}\, \varphi^{(n+1)}(t) \to 0$ as $t \to 0$ due to (5.15) since $\alpha > 0$. Similarly $t^{\alpha+n+1}\, \varphi^{(n+1)}(t) \to 0$ as $t \to +\infty$ due to (5.2) since $\alpha < \varepsilon$. As a result we get

$$\int\limits_0^{+\infty} \Phi_n(t)\, t^\alpha\, dt = -\alpha\, \frac{(-1)^n}{n!}\int\limits_0^{+\infty} t^{\alpha+n}\, \varphi^{(n+1)}(t)\, dt. \tag{5.18}$$

In order to transform (5.18) we continue integrating by parts and we shall decrease by 1 the order of derivatives in each step. In the first step we get

$$\int\limits_0^{+\infty} \Phi_n(t)\, t^\alpha\, dt = -\alpha\, \frac{(-1)^n}{n!}\, t^{\alpha+n}\, \varphi^n(t)\Big|_0^\infty - \alpha\, \frac{(-1)^{n-1}}{n!}\, (\alpha+n)\int\limits_0^{+\infty} t^{\alpha+n-1}\, \varphi^{(n)}(t)\, dt.$$

The non-integral term in the above formula vanishes due to (5.2) and (5.15) since $\alpha > 0$ and $\alpha < \varepsilon$. Hence the formula (5.18) transforms to

$$\int\limits_0^{+\infty} \Phi_n(t)\, t^\alpha\, dt = -\alpha\, \frac{(-1)^{n-1}}{(n-1)!}\left(1+\frac{\alpha}{n}\right)\int\limits_0^{+\infty} t^{\alpha+n-1}\, \varphi^{(n)}(t)\, dt. \tag{5.19}$$

Comparing the right hand sides of (5.18) and (5.19), we see that they differ by passing from n to $n-1$ and in (5.19) we have the additional factor $(1 - \alpha/n)$. Performing the above procedure repeatedly we shall reduce n to 0 and gain more additional factors. They form the following product

$$\left(1+\frac{\alpha}{n}\right)\left(1+\frac{\alpha}{n-1}\right) \cdot \ldots \cdot \left(1+\frac{\alpha}{1}\right) = \prod_{k=1}^n \left(1+\frac{\alpha}{k}\right). \tag{5.20}$$

Due to (5.20) the formula (5.18) reduces to (5.17). Thus, we have proved the lemma 5.2 for $n \geqslant 2$.

If $n = 0$, the formula (5.18) is equivalent to (5.17). Similarly, if $n = 1$ the formula (5.19) is equivalent to (5.17). The lemma 5.2 is proved. $\square$

6. APPLICATION TO KHABIBULLIN'S CONJECTURE.

The integral inequalities (1.3) and (1.4) in Khabibullin's conjecture 1.2 are related to some special choice of the function $\varphi(t)$ in (5.4) and (5.17). Let's recall that the inequality (1.3) now is written as (3.2) in terms of the kernel function

$A_{n-1}(x)$. Upon changing the variable of integration x for $y = tx$ in (3.2) this inequality transforms to the following one:

$$\int\limits_0^t A_{n-1}(y/t)\, q(y)\, \frac{dy}{t} \leqslant t^{\alpha-1}. \tag{6.1}$$

Since $t \geqslant 0$ in (6.1), this inequality can be written as

$$\int\limits_0^t A_{n-1}(y/t)\, q(y)\, dy \leqslant t^{\alpha}. \tag{6.2}$$

Assume that $\Phi_{n-1}(t)$ is positive on the interval $(0, +\infty)$ for some choice of $\varphi(t)$ in (5.3). Then we can multiply both sides of (6.2) by $\Phi_{n-1}(t)$ and integrate over t from 0 to $+\infty$. As a result we derive the following inequality:

$$\int\limits_0^{+\infty} \Phi_{n-1}(t) \left(\int\limits_0^t A_{n-1}(y/t)\, q(y)\, dy \right) dt \leqslant \int\limits_0^{+\infty} \Phi_{n-1}(t)\, t^{\alpha}\, dt. \tag{6.3}$$

Upon changing the order of integration in the left hand side of (6.3) we get

$$\int\limits_0^{+\infty} \left(\int\limits_y^{+\infty} \Phi_{n-1}(t)\, A_{n-1}(y/t)\, dt \right) q(y)\, dy \leqslant \int\limits_0^{+\infty} \Phi_{n-1}(t)\, t^{\alpha}\, dt. \tag{6.4}$$

Now we can apply (5.4) and (5.17) to (6.4). This yields

$$\int\limits_0^{+\infty} \varphi(y)\, q(y)\, dy \leqslant -\alpha \prod_{k=1}^{n-1} \left(1 + \frac{\alpha}{k} \right) \int\limits_0^{+\infty} t^{\alpha}\, \varphi'(t)\, dt \tag{6.5}$$

The left hand side of (6.5) is an integral depending on $q(t)$, while its right hand side is a number. Comparing (6.5) and (1.4), we see that our proper choice is

$$\varphi(t) = \ln(1 + t^{-2\,\alpha}), \quad \text{where } \alpha > 0. \tag{6.6}$$

It is easy to verify that the function (6.6) satisfies the condition (5.15). Moreover, it satisfies the conditions (5.1) and (5.2) for $\varepsilon = 2\,\alpha$.

The integral in the right hand side of the inequality (6.5) for the function (6.6) can be calculated explicitly. Indeed, we have

$$\int\limits_0^{+\infty} t^{\alpha}\, \varphi'(t)\, dt = \int\limits_0^{+\infty} t^{\alpha}\, \frac{-2\,\alpha\, t^{-2\,\alpha-1}}{1 + t^{-2\,\alpha}}\, dt = -2 \int\limits_0^{+\infty} \frac{\alpha\, t^{\alpha-1}}{1 + t^{2\,\alpha}}\, dt =$$

$$= -2 \int\limits_0^{+\infty} \frac{(t^{\alpha})'\, dt}{1 + (t^{\alpha})^2} = -2 \int\limits_0^{+\infty} \frac{dz}{1 + z^2} = -2\, \arctan(z) \Big|_0^{\infty} = -\pi. \tag{6.7}$$

Applying (6.6) and (6.7) to (6.5), we get the inequality coinciding with the required inequality (1.4) in Khabibullin's conjecture.

7. The analysis of the transition function.

The result of the previous section shows that Khabibullin's conjecture 1.2 is valid, provided $\Phi_{n-1}(t) \geqslant 0$ for all $t \in (0, +\infty)$. Unfortunately, the transition function $\Phi_{n-1}(t)$ produced by the function (6.6) is not always positive.

Note that the function (6.6) depends on the parameter $\alpha > 0$ coinciding with the parameter α in Khabibullin's conjecture 1.2. Therefore the transition function $\Phi_{n-1}(t)$ depends on two parameters n and α, i. e. $\Phi_{n-1}(t) = \Phi_{n-1}(\alpha, t)$. Our goal in this section is to search some values of n and α for which

$$\Phi_{n-1}(\alpha, t) \geqslant 0 \quad \text{for all} \quad t > 0. \tag{7.1}$$

Let's begin with $n = 1$. Then $\Phi_{n-1}(\alpha, t) = \Phi_0(\alpha, t)$. From (5.3) and (6.6) for this case we derive the following expression for $\Phi_0(\alpha, t)$:

$$\begin{aligned}
\Phi_0(\alpha, t) &= (t \, \varphi'(t))' = \frac{d}{dt}\left(\frac{-2\,\alpha\,t^{-2\,\alpha}}{1 + t^{-2\,\alpha}}\right) = \\
&= \frac{d}{dt}\left(\frac{-2\,\alpha}{1 + t^{2\,\alpha}}\right) = \frac{4\,\alpha^2}{t} \cdot \frac{t^{2\,\alpha}}{(1 + t^{2\,\alpha})^2}.
\end{aligned} \tag{7.2}$$

It is easy to see that the function (7.2) is positive for all $t > 0$, i.e. the condition (7.1) is fulfilled. As a result we get the following theorem.

Theorem 7.1. *If $n = 1$, Khabibullin's conjecture 1.2 is valid for all $\alpha > 0$.*

Let's proceed to the case $n = 2$. In this case $\Phi_{n-1}(\alpha, t) = \Phi_1(\alpha, t)$. Substituting the function (6.6) into the formula (5.3), we derive

$$\Phi_1(\alpha, t) = -\left(t^2\,\varphi''(t)\right)' = \frac{d}{dt}\left(t^2\,\frac{d}{dt}\left(\frac{2\,\alpha\,t^{-2\,\alpha-1}}{1 + t^{-2\,\alpha}}\right)\right). \tag{7.3}$$

Upon expanding (7.3) and simplifying the obtained expression, we get

$$\Phi_1(\alpha, t) = \frac{4\,\alpha^2}{t} \cdot \frac{(2\,\alpha + 1)\,t^{4\,\alpha} + (1 - 2\,\alpha)\,t^{2\,\alpha}}{(1 + t^{2\,\alpha})^3}. \tag{7.4}$$

The formula (7.4) can be written as

$$\Phi_1(\alpha, t) = \frac{4\,\alpha^2}{t} \cdot \frac{t^{2\,\alpha}\,P_1(\alpha, t^{2\,\alpha})}{(1 + t^{2\,\alpha})^3}, \tag{7.5}$$

where $P_1 = P_1(\alpha, z)$ is a polynomial of the variable $z = t^{2\,\alpha}$:

$$P_1(\alpha, z) = (2\,\alpha + 1)\,z + (1 - 2\,\alpha). \tag{7.6}$$

The polynomial $P_1(z)$ in (7.6) is positive for all $z > 0$ if and only if $0 < \alpha \leqslant 1/2$. Hence we have the following theorem.

Theorem 7.2. *If $n = 2$, Khabibullin's conjecture 1.2 is valid for all $0 < \alpha \leqslant 1/2$.*

The next case is $n = 3$. In this case $\Phi_{n-1}(\alpha, t) = \Phi_2(\alpha, t)$. Substituting the function (6.6) into the formula (5.3), we derive

$$\Phi_2(\alpha, t) = \frac{\left(t^3 \, \varphi'''(t)\right)'}{2} = -\frac{d}{dt}\left(t^3 \, \frac{d^2}{dt^2}\left(\frac{\alpha \, t^{-2\alpha-1}}{1 + t^{-2\alpha}}\right)\right). \tag{7.7}$$

It is preferable to write the formula (7.7) as follows:

$$\Phi_2(\alpha, t) = -\frac{d}{dt}\left(t^3 \, \frac{d^2}{dt^2}\left(\frac{\alpha \, t^{-1}}{1 + t^{2\alpha}}\right)\right). \tag{7.8}$$

By mens of direct calculations we transform the formula (7.8) to

$$\Phi_2(\alpha, t) = \frac{4\,\alpha^2}{t} \cdot \frac{t^{2\alpha} \, P_2(\alpha, t^{2\alpha})}{(1 + t^{2\alpha})^4}, \tag{7.9}$$

where $P_2 = P_2(\alpha, z)$ is the following quadratic polynomial of the variable $z = t^{2\alpha}$:

$$P_2(\alpha, z) = (\alpha + 1)\,(1 + 2\,\alpha)\,z^2 - 2\,(2\,\alpha - 1)\,(1 + 2\,\alpha)\,z + (2\,\alpha - 1)(\alpha - 1). \tag{7.10}$$

Lemma 7.1. *Let $P(z) = A\,z^2 + B\,z + C$ be a general quadratic polynomial with real coefficients A, B, and C. Then $P(z) \geqslant 0$ for all $z > 0$ if and only if one of the following three conditions is fulfilled:*
1) *$A > 0$, $B < 0$, $B^2 - 4\,A\,C \leqslant 0$;*
2) *$A > 0$, $B \geqslant 0$, $C \geqslant 0$;*
3) *$A = 0$, $B \geqslant 0$, $C \geqslant 0$.*

Proof. If $A \neq 0$, then $P(z) \to -\infty$ as $z \to +\infty$ for $A < 0$ and $P(z) \to +\infty$ as $z \to +\infty$ for $A > 0$. Therefore $A \geqslant 0$ is a necessary condition for $P(z) \geqslant 0$ for all $z > 0$. If $A > 0$, the graph of the function $P(z)$ is a parabola (see Fig. 7.1 and Fig. 7.2). If $A = 0$, it is a straight line (see Fig. 7.3). In the last case $P(z) \geqslant 0$ for
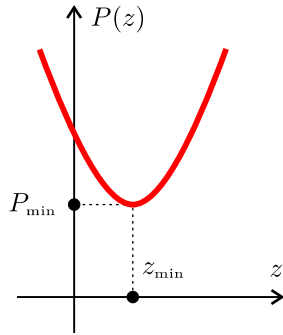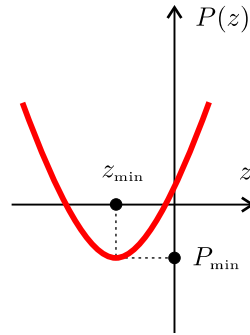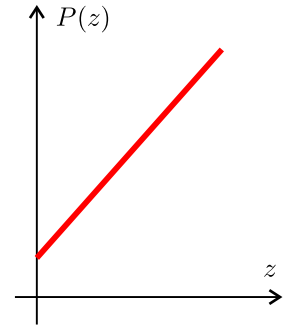

Fig 7.1   Fig 7.2   Fig 7.3

all $z > 0$ if and only if the following two inequalities are fulfilled:

$$C \geqslant 0, \qquad\qquad\qquad B \geqslant 0. \tag{7.11}$$

In the case of a parabolic graph, i.e. if $A > 0$, the function $P(z)$ decreases for $z < z_{\min}$ and $P(z)$ increases for $z > z_{\min}$. Hence there are two subcases where $P(z) \geqslant 0$ for all $z > 0$. The first subcase is given by the inequalities (see Fig. 7.1):

$$z_{\min} > 0, \qquad\qquad\qquad P_{\min} \geqslant 0. \qquad\qquad (7.12)$$

The second subcase corresponds to Fig. 7.2. It is given by the inequalities

$$z_{\min} \leqslant 0, \qquad\qquad\qquad P(0) \geqslant 0. \qquad\qquad (7.13)$$

The elementary calculus yields

$$z_{\min} = -\frac{B}{2\,A}, \qquad\qquad P_{\min} = C - \frac{B^2}{4\,A}, \qquad\qquad P(0) = C. \qquad (7.14)$$

Applying (7.14) to (7.12) and taking into account that $A > 0$, we obtain

$$B < 0, \qquad\qquad\qquad B^2 - 4\,A\,C \leqslant 0. \qquad\qquad (7.15)$$

Similarly, applying (7.14) to (7.13) and taking into account that $A > 0$, we get

$$B \geqslant 0, \qquad\qquad\qquad C \geqslant 0. \qquad\qquad (7.16)$$

Now in order to complete the proof of the lemma 7.1 it is sufficient to complement the inequalities (7.15) and (7.16) with the inequality $A > 0$ and to complement the inequalities (7.11) with the equality $A = 0$. $\square$

having proved the lemma 7.1, we apply it to the polynomial (7.10). In this case $A = (\alpha + 1)\,(1 + 2\,\alpha)$. Since $\alpha > 0$ in the conjecture 1.2, we have $A > 0$, i.e. the third option of the lemma 7.1 can be dropped from our further considerations. Since $B = -2\,(2\,\alpha - 1)\,(1 + 2\,\alpha)$ and $C = (2\,\alpha - 1)(\alpha - 1)$, the second option of the lemma 7.1 combined with $\alpha > 0$, leads to the following system of inequalities:

$$\begin{cases} (1 - 2\,\alpha)\,(1 + 2\,\alpha) \geqslant 0, \\ (2\,\alpha - 1)(\alpha - 1) \geqslant 0, \\ \alpha > 0. \end{cases} \qquad\qquad (7.17)$$

The inequalities (7.17) resolve to $0 < \alpha \leqslant 1/2$.

Let's proceed to the first option of the lemma 7.1. By means of direct calculations we get $B^2 - 4\,A\,C = 12\,\alpha^2\,(2\,\alpha - 1)(1 + 2\,\alpha)$. Therefore the first option of the lemma 7.1 combined with the inequality $\alpha > 0$ yields

$$\begin{cases} (1 - 2\,\alpha)\,(1 + 2\,\alpha) < 0, \\ \alpha^2\,(2\,\alpha - 1)(1 + 2\,\alpha) \leqslant 0, \\ \alpha > 0. \end{cases} \qquad\qquad (7.18)$$

It is easy to find that the inequalities (7.18) are mutually contradictory. They cannot be satisfied simultaneously. As a result we conclude that the values of the polynomial (7.10) are non-negative for all $z > 0$ if and only if $0 < \alpha \leqslant 1/2$.

Applying this fact to the transition function (7.9), we get the following theorem.

Theorem 7.3. *If* $n = 3$, *Khabibullin's conjecture 1.2 is valid for all* $0 < \alpha \leqslant 1/2$.

8. RECURRENT FORMULAS FOR THE TRANSITION FUNCTIONS.

The transition function $\Phi_n(\alpha, t)$ introduced in Section 5 is given by the explicit formula (5.3). However, for our purposes we need to have the recurrent formula expressing $\Phi_n(\alpha, t)$ through the function $\Phi_{n-1}(\alpha, t)$. Here is this formula

$$\Phi_n = -\frac{t^n}{n} \frac{d}{dt} \left(\frac{\Phi_{n-1}}{t^{n-1}} \right). \tag{8.1}$$

The formula (8.1) is easily proved by means of direct calculations with the use of the initial formula (5.3).

The transition functions $\Phi_n(\alpha, t)$ associated with Khabibullin's conjecture 1.2 correspond to the special choice (6.6) of the function $\varphi(t)$ in (5.3). The formula (7.2), (7.5), and (7.9) were derived exactly for this special choice of $\varphi(t)$. Comparing these three formulas with each other, we write

$$\Phi_n(\alpha, t) = \frac{4\,\alpha^2}{t} \cdot \frac{t^{2\,\alpha}\,P_n(\alpha, z)}{(1 + t^{2\,\alpha})^{n+2}}, \tag{8.2}$$

where $z = t^{2\,\alpha}$ and $P_n(\alpha, z)$ is a polynomial of the degree n with respect to the variable z. The formula (8.2) is proved by induction on n. Using (8.1) one can derive the following recurrent formula for the polynomials $P_n(z)$ in (8.2):

$$P_n(\alpha, z) = \left((2\,\alpha + 1)z + \left(1 - \frac{2\,\alpha}{n} \right) \right) P_{n-1}(\alpha, z) -$$
$$- \frac{2\,\alpha\,z\,(z+1)}{n} P'_{n-1}(\alpha, z). \tag{8.3}$$

The formula (7.2) yields the base for applying the recurrent formula (8.3):

$$P_0(\alpha, z) = 1. \tag{8.4}$$

Combining the formulas (8.3) and (8.4), one can calculate the polynomial $P_n(\alpha, z)$ explicitly for each particular n.

Theorem 8.1. *If* $0 < \alpha \leqslant 1/2$, *then* $P_n(\alpha, z) \geqslant 0$ *for all* $z > 0$.

Theorem 8.2. *If* $0 < \alpha \leqslant 1/2$, *then* $\Phi_n(\alpha, t) \geqslant 0$ *for all* $t > 0$.

Due to (8.2) the theorems 8.1 and 8.2 are equivalent to each other. However, the theorem 8.2 is easier to prove.

Proof of the theorem 8.2. Applying the recurrent formula (8.1) we immediately derive the following expression for $\Phi_n(\alpha, t)$:

$$\Phi_n(\alpha, t) = \frac{(-1)^n}{n!} \frac{d^n \Phi_0(\alpha, t)}{dt^n}. \tag{8.5}$$

The function $\Phi_0(\alpha, t)$ is given by the formula (7.2). We write it as

$$\Phi_0(\alpha, t) = \frac{4\,\alpha^2}{(1 + t^{2\,\alpha})^2\,t^{1-2\,\alpha}}. \tag{8.6}$$

The function (8.6) belongs to the class of functions $KK(\beta)$ with $\beta = 2\,\alpha$. By definition this class of functions consists of all linear combinations of functions

$$\psi(t) = \frac{1}{t^{\gamma}\,(1 + t^{\beta})^k}, \quad \gamma \geqslant 0, \quad k \geqslant 0, \tag{8.7}$$

with non-negative coefficients. The class of functions $KK(\beta)$ is similar to the class of functions $K(\beta)$ defined in [4].

Obviously, each function $f \in KK(\beta)$ is non-negative, i. e. $f(t) \geqslant 0$ for all $t > 0$. For $0 < \beta \leqslant 1$ the class of functions $KK(\beta)$ is closed with respect to the operator

$$D = -\frac{d}{dt}. \tag{8.8}$$

Indeed, applying the operator (8.8) to the function (8.7), we get

$$D\psi(t) = \gamma \cdot \frac{1}{t^{\gamma+1}\,(1 + t^{\beta})^k} + (k\,\beta) \cdot \frac{1}{t^{\gamma+1-\beta}\,(1 + t^{\beta})^{k+1}} \tag{8.9}$$

As we see, for $0 < \beta \leqslant 1$ the right hand side of (8.9) is a linear combination of two functions of the form (8.7) with two non-negative coefficients γ and $k\,\beta$.

Now let's return back to the formulas (8.5) and (8.6). In terms of the operator (8.8) the formula (8.5) is written as follows:

$$\Phi_n(\alpha, t) = \frac{D^n \Phi_0(\alpha, t)}{n!}. \tag{8.10}$$

Since $\beta = 2\,\alpha$ and $0 < \alpha \leqslant 1/2$ is equivalent to $0 < \beta \leqslant 1$, from $\Phi_0(\alpha, t) \in KK(\beta)$ and (8.10) we derive $\Phi_n(\alpha, t) \in KK(\beta)$. Hence $\Phi_n(\alpha, t) \geqslant 0$ for all $t > 0$. This means that the theorem 8.2 is proved. $\square$

From the theorem 8.2 we immediately derive the following theorem.

Theorem 8.3. *Khabibullin's conjecture 1.2 is valid for all $0 < \alpha \leqslant 1/2$ and for all integer $n > 0$.*

9. CONCLUSIONS.

The theorem 8.3 is not a new result. It is known from [4]. However, using the transition function $\Phi_{n-1}(\alpha, t)$, we define the integral transformation

$$\psi(t) \mapsto \int\limits_0^{+\infty} \Phi_{n-1}(t)\,\psi(t)\,dt. \tag{9.1}$$

Due to the lemma 5.1 the transformation (9.1) can be worth in studying Khabibullin's conjecture 1.2 for its parameters n and α in ranges beyond those covered by the theorem 8.3.

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