DIRECT AND INVERSE CONVERSION FORMULAS ASSOCIATED WITH KHABIBULLIN'S CONJECTURE FOR INTEGRAL INEQUALITIES.

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Abstract. Khabibullin's conjecture deals with two linear integral inequalities for some non-negative continuous function $q(t)$. The integral in the first of these two inequalities converts $q(t)$ into another function of one variable $g(t)$. This integral yields the direct conversion formula. An inverse conversion formula means a formula expressing $q(t)$ back through $g(t)$. Such an inverse conversion formula is derived.

1. Introduction.

In this paper the following statement of Khabibullin's conjecture is used.

Conjecture 1.1 (Khabibullin). *Let $\alpha > 0$ and let $q = q(t)$ be a continuous function such that $q(t) \geqslant 0$ for all $t > 0$. Then the inequality*

$$\int\limits_0^1 \left(\,\int\limits_x^1 (1-y)^{n-1}\,\frac{dy}{y}\right) q(tx)\,dx \leqslant t^{\alpha-1} \tag{1.1}$$

fulfilled for all $0 \leqslant t < +\infty$ implies the inequality

$$\int\limits_0^{+\infty} q(t)\,\ln\Bigl(1+\frac{1}{t^{2\alpha}}\Bigr)\,dt \leqslant \pi\,\alpha\prod_{k=1}^{n-1}\Bigl(1+\frac{\alpha}{k}\Bigr).$$

Initially, the conjecture 1.1 was formulated in [1] and [2], though in some different form. Later in [3] it was reformulated in a form very close to the above statement. In [4] the conjecture 1.1 was proved to be valid for $0 < \alpha \leqslant 1/2$. Another proof of this result was given in [5].

The approach of the paper [5] is based on the kernel function $A_n(x)$. The kernel function $A_n(x)$ is defined by the inner integral in the formula (1.1):

$$A_n(x) = \int\limits_x^1 (1-y)^n\,\frac{dy}{y}. \tag{1.2}$$

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In terms of the kernel function (1.2) the inequality (1.1) is written as

$$\int\limits_0^1 A_{n-1}(x)\,q(t\,x)\,dx \leqslant t^{\alpha-1},\ \text{ where }\ t\geqslant 0. \tag{1.3}$$

By changing the variable x for the variable $y = t\,x$ in the integral (1.3) we get

$$\int\limits_0^t A_{n-1}(y/t)\,q(y)\,dy \leqslant t^{\alpha},\ \text{ where }\ t>0. \tag{1.4}$$

The value $t = 0$ is an exception when transforming (1.3) into (1.4). We omit this exceptional value from our further considerations.

Looking at the left hand side of the inequality (1.4), we define the following integral transformation that converts a function $q = q(t)$ defined on the half-line $t > 0$ into another function $g = g(t)$ defined on the same half-line $t > 0$:

$$g(t) = \int\limits_0^t A_n(y/t)\,q(y)\,dy. \tag{1.5}$$

The formula (1.5) is called the direct conversion formula. The main goal of this paper is to derive an inverse conversion formula that converts $g(t)$ back to $q(t)$.

2. Properties of the kernel function.

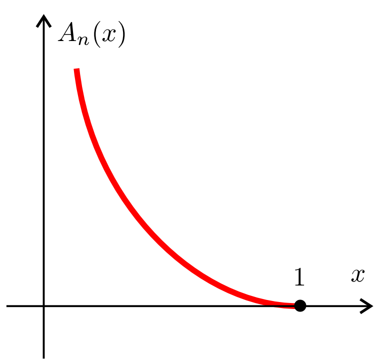

Fig 2.1

The kernel function $A_n(x)$ used in the direct conversion formula (1.5). Its properties were studied in [5]. This is a decreasing smooth function on the segment $(0,\,1]$ vanishing at the point $x = 1$ and having the logarithmic singularity

$$A_n(x) \sim -\ln x \tag{2.1}$$

at the point $x = 0$. Its graph is shown on Fig. 2.1. There are two explicit formulas for $A_n(x)$. Here is the first of them

$$A_n(x) = \sum_{m=n+1}^{\infty} \frac{(1-x)^m}{m}. \tag{2.2}$$

From (2.2) we immediately derive the following vanishing conditions:

$$\left.\frac{d^k A_n(x)}{dx^k}\right|_{x=1} = 0\ \text{ for }\ k = 0,\,1,\,\ldots\,,\,n. \tag{2.3}$$

The sum in the second explicit formula for the kernel function $A_n(x)$ is finite

$$A_n(x) = -\ln x - \sum_{m=1}^{n} \frac{(1-x)^m}{m}. \tag{2.4}$$

The formula (2.1) is immediate from (2.4).

Note that the function $A_n(x)$ enters the formula (1.5) in the form of $A_n(y/t)$ with the composite argument $x = y/t$. Substituting $x = y/t$ into (2.4), we get

$$A_n(y/t) = -\ln y + \ln t - \sum_{m=1}^{n} \frac{(1 - y/t)^m}{m}. \tag{2.5}$$

The function (2.5) can be treated as a function of two variables y and t. The partial derivatives of $A_n(y/t)$ with respect to y and t can be calculated explicitly:

$$\frac{\partial A_n(y/t)}{\partial y} = -\frac{(t - y)^n}{t^n\, y}, \qquad \frac{\partial A_n(y/t)}{\partial t} = \frac{(t - y)^n}{t^{n+1}}. \tag{2.6}$$

The formulas (2.6) are easily derived from the following formula:

$$\frac{dA_n(x)}{dx} = \frac{-(1 - x)^n}{x}. \tag{2.7}$$

As for the formula (2.7), it is derived from (2.4) by means of direct calculations. The reader can find more details in [5].

3. The first derivative of the function $g(t)$.

The integral in the direct conversion formula (1.5) is assumed to be finite for at least one value of $t = t_0 > 0$ as a convergent improper Riemann integral or as a Lebesque integral. Under this assumption we have the following lemma.

Lemma 3.1. *Let $q = q(t)$ be a non-negative continuous function on the open half-line $t > 0$, i. e. $q(t) \geqslant 0$ for all $t > 0$. If the integral (1.5) is finite for some $t_0 > 0$ then for all $t > 0$ the following integrals are finite:*

$$\int_0^t q(y)\, dy < \infty, \qquad \int_0^t |\ln y|\, q(y)\, dy < \infty. \tag{3.1}$$

Proof. Since $q = q(t)$ is a continuous function, in order to prove the inequalities (3.1) for all $t > 0$ it is sufficient to prove them for some particular $t = y_0 > 0$. Since

$$\int_0^{t_0} A_n(y/t_0)\, q(y)\, dy < \infty, \tag{3.2}$$

we choose $t = t_0$ and from (2.4) we derive $A_n(y/t_0) \to +\infty$ as $y \to +0$. Hence, there is some $y_0 > 0$ such that $A_n(y/t_0) > 1$ for all $0 < y \leqslant y_0$. Multiplying by $q(y)$ and taking into account that $q(y) \geqslant 0$, from $A_n(y/t_0) > 1$ we derive

$$q(y) \leqslant A_n(y/t_0)\, q(y) \quad \text{for} \quad 0 < y \leqslant y_0. \tag{3.3}$$

Integrating the inequality (3.3) we get

$$\int_0^{y_0} q(y)\, dy \leqslant \int_0^{y_0} A_n(y/t_0)\, q(y)\, dy. \tag{3.4}$$

Note that the integration interval in (3.4) differs from that of (3.2). However, since both $A_n(y/t_0)$ and $q(y)$ are continuous functions, extending or shrinking the integration interval does not affect the finiteness of the integral (3.2). Therefore, combining (3.2) and (3.4), we get the inequality

$$\int\limits_0^{y_0} q(y)\,dy < \infty.$$

Thus, the first inequality (3.1) of the lemma 3.1 is proved.

In order to prove the second inequality we use the formula (2.4) again. Applying this formula to the function $A_n(y/t_0)$, we obtain

$$\lim_{y\to+0}\frac{A_n(y/t_0)}{|\ln y|}=1. \tag{3.5}$$

The equality (3.5) means that there is some $y_0>0$ such that

$$|\ln y|<2\,A_n(y/t_0)\ \ \text{for all}\ \ 0<y\leqslant y_0. \tag{3.6}$$

Multiplying (3.6) by $q(y)$ and taking into account that $q(y)\geqslant 0$, we get

$$|\ln y|\,q(y)\leqslant 2\,A_n(y/t_0)\,q(y)\ \ \text{for all}\ \ 0<y\leqslant y_0. \tag{3.7}$$

Now, integrating the inequality (3.7), we obtain

$$\int\limits_0^{y_0}|\ln y|\,q(y)\,dy\leqslant 2\int\limits_0^{y_0}A_n(y/t_0)\,q(y)\,dy. \tag{3.8}$$

Combining (3.2) and (3.8), we derive

$$\int\limits_0^{y_0}|\ln y|\,q(y)\,dy<\infty. \tag{3.9}$$

The second inequality (3.1) of the lemma 3.1 is also proved. As we already said above, the difference in upper limits of the integrals (3.1) and (3.9) does not matter for finiteness of the integral (3.9). $\square$

Let's return back to the direct conversion formula (1.5) and, choosing some constant b, let's subdivide the integral (1.5) into two integrals:

$$I_1(t)=\int\limits_0^{b}A_n(y/t)\,q(y)\,dy,\qquad\qquad I_2(t)=\int\limits_b^{t}A_n(y/t)\,q(y)\,dy. \tag{3.10}$$

Writing (3.10), we assume that $0<b<t$. Let c be another constant such that $0<b<t<c$. The first integral (3.10) is an improper integral with the singularity

at its lower limit $y = 0$. The second integral (3.10) is a proper integral. In addition to (3.10), let's consider the following two integrals:

$$I_3(t) = \int\limits_0^b \frac{\partial A_n(y/t)}{\partial t}\, q(y)\, dy, \qquad\qquad I_4(t) = \int\limits_b^t \frac{\partial A_n(y/t)}{\partial t}\, q(y)\, dy. \tag{3.11}$$

Due to (2.5) and (2.6) the functions

$$A_n(y/t)\, q(y), \qquad\qquad \frac{\partial A_n(y/t)}{\partial t}\, q(y) \tag{3.12}$$

both are functions of two variables y and t which are continuous within the closed rectangle $R_2 = \{(y,t) \in \mathbb{R}^2 :\; b \leqslant y \leqslant c,\; b \leqslant t \leqslant c\}$. Therefore we can apply the theorem 4' from § 53 of Chapter VI in [6] to the integrals $I_2(t)$ and $I_4(t)$. This theorem says that $I_2(t)$ is a differentiable function such that

$$\frac{dI_2(t)}{dt} = I_4(t) + A_n(t/t)\, q(t). \tag{3.13}$$

Note that $t/t = 1$ and, according to (2.3), $A_n(1) = 0$. Therefore (3.13) reduces to

$$\frac{dI_2(t)}{dt} = I_4(t). \tag{3.14}$$

Now let's proceed to the integrals $I_1(t)$ and $I_3(t)$ in (3.10) and (3.11). The functions (3.12) both are continuous functions of two variables within the semi-open rectangle $R_1 = \{(y,t) \in \mathbb{R}^2 :\; 0 < y \leqslant b,\; b \leqslant t \leqslant c\}$. Using (2.5) and (2.6), one can easily prove that there are two constants C_1 and C_2 such that

$$|A_n(y/t)\, q(y)| \leqslant C_1\, (|\ln y| + 1)\, q(y), \qquad\qquad \left|\frac{\partial A_n(y/t)}{\partial t}\, q(y)\right| \leqslant C_2\, q(y) \tag{3.15}$$

for all (y,t) within the semi-open rectangle R_1. Due to (3.15) and (3.1) we can apply the theorem 1 from § 54 of Chapter VI in [6] to the improper integrals $I_1(t)$ and $I_3(t)$ in (3.10) and (3.11). This theorem says that both of these two improper integrals converge uniformly in t over the interval $b \leqslant t \leqslant c$. Due to the uniform convergence, we can apply the theorem 8 from § 54 of Chapter VI in [6] to $I_1(t)$ and $I_3(t)$. This theorem says that $I_1(t)$ is a differentiable function and

$$\frac{dI_1(t)}{dt} = I_3(t). \tag{3.16}$$

Since $g(t) = I_1(t) + I_2(t)$, now we can combine the formulas (3.14) and (3.16) and derive the following formula for the first derivative of $g(t)$:

$$g'(t) = \int\limits_0^t \frac{\partial A_n(y/t)}{\partial t}\, dy. \tag{3.17}$$

More precisely this result is formulated in the following theorem.

Theorem 3.1. *If $q = q(t)$ is a non-negative continuous function on the half-line $t > 0$ and if the integral (1.5) is finite for at least one value $t = t_0 > 0$, then $g(t)$ is a differentiable function and its derivative is given by the formula (3.17).*

4. Higher order derivatives of the function $g(t)$.

Let's apply the second formula (2.6) to (3.17). Then the formula (3.17) for $g'(t)$ is brought to the following more explicit form:

$$t^{n+1}\,g'(t) = \int\limits_0^t (t-y)^n\,q(y)\,dy. \tag{4.1}$$

Let's denote $\tilde{g}(t) = t^{n+1}\,g'(t)$ and write (4.1) as

$$\tilde{g}(t) = \int\limits_0^t (t-y)^n\,q(y)\,dy. \tag{4.2}$$

The formula is quite similar to (1.5). It is even simpler than (1.5) since the function $(t-y)^n$ has no logarithmic singularity, though this fact does not matter for us. Applying the same arguments as in proving the theorem 3.1 above, we can prove the following theorem for $\tilde{g}(t)$.

Theorem 4.1. *If $q = q(t)$ is a non-negative continuous function on the half-line $t > 0$ and if the integral (1.5) is finite for at least one value $t = t_0 > 0$, then $\tilde{g}(t)$ in (4.2) is a differentiable function and its derivative is given by the formula*

$$\tilde{g}'(t) = \int\limits_0^t n\,(t-y)^{n-1}\,q(y)\,dy. \tag{4.3}$$

The next theorem yields the second derivative of the function $\tilde{g}(t)$.

Theorem 4.2. *If $q = q(t)$ is a non-negative continuous function on the half-line $t > 0$ and if the integral (1.5) is finite for at least one value $t = t_0 > 0$, then the function $\tilde{g}(t)$ in (4.2) is a twice differentiable function and its second order derivative is given by the formula*

$$\tilde{g}''(t) = \int\limits_0^t n\,(n-1)\,(t-y)^{n-2}\,q(y)\,dy. \tag{4.4}$$

Acting repeatedly, one can prove the series of theorems saying that $\tilde{g}(t)$ is an n times differentiable function and derive the formula generalizing (4.3) and (4.4):

$$\frac{d^k\tilde{g}(t)}{dt^k} = \frac{n!}{(n-k)!}\int\limits_0^t (t-y)^{n-k}\,q(y)\,dy, \qquad k = 0,\,\ldots\,,\,n. \tag{4.5}$$

For $k = n$ the formula (4.5) reduces to the following one:

$$\frac{d^n \tilde{g}(t)}{dt^n} = n! \int\limits_0^t q(y)\, dy. \tag{4.6}$$

Due to (4.6) the function $g(t)$ in (1.5) is an $(n+1)$ times differentiable function.

5. An inverse conversion formula.

An inverse conversion formula is almost immediate from (4.6). Indeed, due to (3.1) the right hand side of the formula (4.6) is a differentiable function. Differentiating (4.6) and substituting $\tilde{g}(t) = t^{n+1}\, g'(t)$, we get

$$q(t) = \frac{d^{n+1}}{dt^{n+1}}\left(\frac{t^{n+1}\, g'(t)}{n!}\right). \tag{5.1}$$

This is a required inverse conversion formula. Tne following ultimate theorem is associated with the formula (5.1).

Theorem 5.1. *If $q = q(t)$ is a non-negative continuous function on the half-line $t > 0$ and if the integral (1.5) is finite for at least one value $t = t_0 > 0$, then the function $g(t)$ in (1.5) is an $(n+2)$ times differentiable function such that $q(t)$ is expressed through its derivatives according to the formula (5.1).*

The theorem 5.1 does not claim the formula (5.1) to be the only way for expressing $q(t)$ through $g(t)$. But hopefully, the formula (5.1) could be a useful tool for studying Khabibullin's conjecture 1.1.

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