A COUNTEREXAMPLE TO KHABIBULLIN'S CONJECTURE FOR INTEGRAL INEQUALITIES.

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Abstract. Khabibullin's conjecture for integral inequalities has two numeric parameters n and α in its statement, n being a positive integer and α being a positive real number. This conjecture is already proved in the case where $n>0$ and $0<\alpha\leqslant 1/2$. However, for $\alpha>1/2$ it is not always valid. In this paper a counterexample is constructed for $n=2$ and $\alpha=2$. Then Khabibullin's conjecture is reformulated in a way suitable for all $\alpha>0$.

1. Introduction.

Conjecture 1.1 (Khabibullin). *Let $\alpha>0$ be a positive number and let $q=q(t)$ be a continuous function such that $q(t)\geqslant 0$ for all $t>0$. Then the inequality*

$$\int\limits_0^1\left(\,\int\limits_x^1(1-y)^{n-1}\,\frac{dy}{y}\right)q(tx)\,dx\leqslant t^{\alpha-1}\tag{1.1}$$

fulfilled for all $0\leqslant t<+\infty$ implies the inequality

$$\int\limits_0^{+\infty}q(t)\,\ln\!\left(1+\frac{1}{t^{2\alpha}}\right)dt\leqslant\pi\,\alpha\prod_{k=1}^{n-1}\left(1+\frac{\alpha}{k}\right).\tag{1.2}$$

This conjecture was initially formulated in [1] and [2], though in some different form. In [3] it was reformulated in a form very close to the above statement. In [4] it was proved that the conjecture 1.1 is valid for $0<\alpha\leqslant 1/2$. Another proof of this result is given in [5].

Unfortunately, beyond the scope of $0<\alpha\leqslant 1/2$ the conjecture 1.1 is not always valid. The main goal of this paper is to construct a counterexample to the conjecture 1.1 for $n=2$ and $\alpha=2$ and reformulate this conjecture in a way suitable for all integer $n>0$ and for all $\alpha>0$.

2. The kernel function and the transition function.

The kernel function $A_n(x)$ and the transition function $\varPhi_n(t)$ associated with the conjecture 1.1 were introduced in [5]. The kernel function is defined by the formula

$$A_n(x)=\int\limits_x^1(1-y)^n\,\frac{dy}{y}.\tag{2.1}$$

2000 *Mathematics Subject Classification.* 26D10, 26D15, 39B62, 47A63.

In terms of the kernel function (2.1) the inequality (1.1) is written as

$$\int\limits_0^1 A_{n-1}(x)\,q(t\,x)\,dx \leqslant t^{\alpha-1}, \ \text{ where } \ t \geqslant 0. \tag{2.2}$$

Omitting the inessential value $t = 0$ and changing the variable x for the variable $y = t\,x$ in the integral (2.2), we transform it to

$$\int\limits_0^t A_{n-1}(y/t)\,q(y)\,dy \leqslant t^{\alpha}, \ \text{ where } \ t > 0. \tag{2.3}$$

The concept of the transition function $\varPhi_n(t)$ is more complicated. It was introduced in [5] by means of the following formula:

$$\varPhi_n(t) = -\frac{d}{dt}\left(\frac{(-t)^{n+1}}{n!}\,\frac{d^{\,n+1}\varphi}{dt^{n+1}}\right). \tag{2.4}$$

Here $\varphi = \varphi(t)$ is some smooth function of one variable t. The basic property of the transition function (2.4) is its interaction with the kernel function (2.1):

$$\int\limits_y^{+\infty} \varPhi_n(t)\,A_n(y/t)\,dt = \varphi(y) \ \text{ for } \ n \geqslant 0 \tag{2.5}$$

(see Lemma 5.1 in [5]). In order to apply the formula (2.5) to Khabibullin's conjecture 1.1 we should specify our choice of $\varphi(t)$ in (2.4):

$$\varphi(t) = \ln(1 + t^{-2\,\alpha}), \ \text{ where } \ \alpha > 0. \tag{2.6}$$

The function (2.6) depends on α. This dependence is inherited by the transition function (2.4). Therefore we write $\varPhi_n = \varPhi_n(\alpha, t)$. Then (2.5) is written as

$$\int\limits_y^{+\infty} \varPhi_{n-1}(\alpha, t)\,A_{n-1}(y/t)\,dt = \ln(1 + y^{-2\,\alpha}) \ \text{ for } \ n \geqslant 1. \tag{2.7}$$

Note that the function (2.6) enters the integrand in the left hand side of the inequality (1.2). For this reason the the formula (2.7) is a bridge binding two inequalities (1.1) and (1.2) in Khabibullin's conjecture.

3. Application to Khabibullin's conjecture.

Let's recall that the first inequality of Khabibullin's conjecture 1.1 is written as (2.3). Let's multiply both sides of (2.3) by $\varPhi_{n-1}(\alpha, t)$:

$$\int\limits_0^t \varPhi_{n-1}(\alpha, t)\,A_{n-1}(y/t)\,q(y)\,dy \leqslant \varPhi_{n-1}(\alpha, t)\ t^{\alpha}. \tag{3.1}$$

The inequality (3.1) would follow from (2.3) provided $\varPhi_{n-1}(\alpha,t) \geqslant 0$. But actually, positive values of $\varPhi_{n-1}(\alpha,t)$ alternate with negative ones. For this reason we subdivide the half-line $t > 0$ into two subsets M_+ and M_-:

$$\begin{aligned} M_+ = M_+(n,\alpha) &= \{t \in \mathbb{R}\colon\ t > 0 \ \text{ and } \ \varPhi_{n-1}(\alpha,t) \geqslant 0\}, \\ M_- = M_-(n,\alpha) &= \{t \in \mathbb{R}\colon\ t > 0 \ \text{ and } \ \varPhi_{n-1}(\alpha,t) < 0\}. \end{aligned} \tag{3.2}$$

The inequality (3.1) is valid for $t \in M_+(n,\alpha)$. If $t \in M_-(n,\alpha)$, the inequality (3.1) is not valid. In this case we recall that the kernel function $A_n(x)$ is positive, i. e. $A_n(x) > 0$ for $0 < x < 1$ (see [5]). The function $q(y)$ is non-negative according to the conjecture 1.1. As for the function $\varPhi_{n-1}(\alpha,t)$, it is negative for $t \in M_-(n,\alpha)$ (see (3.2)). As a result we get the following inequality for $t \in M_-(n,\alpha)$:

$$\int\limits_0^t \varPhi_{n-1}(\alpha,t)\, A_{n-1}(y/t)\, q(y)\, dy \leqslant 0. \tag{3.3}$$

Combining the inequalities (3.1) and (3.3), we write

$$\int\limits_0^t \varPhi_{n-1}(\alpha,t)\, A_{n-1}(y/t)\, q(y)\, dy \leqslant \begin{cases} \varPhi_{n-1}(\alpha,t)\ t^\alpha & \text{for } \ t \in M_+, \\ 0 & \text{for } \ t \in M_-. \end{cases} \tag{3.4}$$

Now let's integrate (3.4) over t from 0 to infinity:

$$\int\limits_0^{+\infty} \left(\int\limits_0^t \varPhi_{n-1}(\alpha,t)\, A_{n-1}(y/t)\, q(y)\, dy \right) dt \ \leqslant \int\limits_{t\in M_+} \varPhi_{n-1}(\alpha,t)\ t^\alpha\, dt. \tag{3.5}$$

Upon changing the order of integration in the left hand side of (3.5) we get

$$\int\limits_0^{+\infty} \left(\int\limits_y^{+\infty} \varPhi_{n-1}(\alpha,t)\, A_{n-1}(y/t)\, dt \right) q(y)\, dy \ \leqslant \int\limits_{t\in M_+} \varPhi_{n-1}(\alpha,t)\ t^\alpha\, dt. \tag{3.6}$$

Applying (2.7) to (3.6), we derive the following inequality:

$$\int\limits_0^{+\infty} \ln(1 + y^{-2\,\alpha})\, q(y)\, dy \ \leqslant \int\limits_{t\in M_+} \varPhi_{n-1}(\alpha,t)\ t^\alpha\, dt. \tag{3.7}$$

The integral in the left hand side of (3.7) coincides with that of the second inequality (1.2) in Khabibullin's conjecture 1.1. There is the formula

$$\varPhi_n(\alpha,t) = \frac{4\,\alpha^2}{t} \cdot \frac{t^{2\,\alpha}\, P_n(\alpha,z)}{(1 + t^{2\,\alpha})^{n+2}}. \tag{3.8}$$

It was derived in [5]. Here $z = t^{2\,\alpha}$ and $P_n(\alpha,z)$ is a polynomial of the degree n

with respect to the variable z. From the formula (3.8) we derive

$$\varPhi_n(\alpha,t)=\begin{cases} O(t^{2\,\alpha-1}) & \text{as}\ \ t\longrightarrow +0,\\ O(t^{-2\,\alpha-1}) & \text{as}\ \ t\longrightarrow +\infty.\end{cases} \tag{3.9}$$

Due to (3.9) the integral in the right hand side of (3.7) is finite. In's value is a positive number depending on n and α. Let's denote it through $C(n,\alpha)$:

$$C(n,\alpha)\ =\int\limits_{t\in M_+}\varPhi_{n-1}(\alpha,t)\ t^{\alpha}\,dt<\infty. \tag{3.10}$$

Apart from (3.10), we consider two other integrals

$$\int\limits_{t\in M_-}\varPhi_{n-1}(\alpha,t)\ t^{\alpha}\,dt,\qquad\qquad \int\limits_{0}^{+\infty}\varPhi_{n-1}(\alpha,t)\ t^{\alpha}\,dt. \tag{3.11}$$

Due to (3.9) both integrals (3.11) are finite. According to (3.2), the first of them is non-positive. These integrals are related to (3.10) as follows:

$$C(n,\alpha)\ +\int\limits_{t\in M_-}\varPhi_{n-1}(\alpha,t)\ t^{\alpha}\,dt\ =\int\limits_{0}^{+\infty}\varPhi_{n-1}(\alpha,t)\ t^{\alpha}\,dt. \tag{3.12}$$

In [5] the second integral (3.11) was calculated explicitly. As appears, this integral coincides with the number in the right hand side of the second inequality (1.2) in Khabibullin's conjecture 1.1, i. e. we have

$$\int\limits_{0}^{+\infty}\varPhi_{n-1}(\alpha,t)\ t^{\alpha}\,dt=\pi\,\alpha\prod_{k=1}^{n-1}\Bigl(1+\frac{\alpha}{k}\Bigr). \tag{3.13}$$

Since the first integral (3.11) is non-positive, from (3.12) and (3.13) we derive

$$0<\pi\,\alpha\prod_{k=1}^{n-1}\Bigl(1+\frac{\alpha}{k}\Bigr)\leqslant C(n,\alpha)<\infty \tag{3.14}$$

The formula (3.10) for $C(n,\alpha)$ is not so simple as compared to (3.13). Nevertheless, $C(n,\alpha)$ should be considered as a quite certain quantity that can be effectively computed numerically for each particular numeric value of n and α. Replacing the right hand side of the inequality (1.2) by $C(n,\alpha)$, we cam formulate Khabibullin's conjecture 1.1 not as a conjecture, but as a proved result.

Theorem 3.1. *Let $\alpha>0$ be a positive number and let $q=q(t)$ be a continuous function such that $q(t)\geqslant 0$ for all $t>0$. Then the inequality*

$$\int\limits_{0}^{1}\Biggl(\,\int\limits_{x}^{1}(1-y)^{n-1}\,\frac{dy}{y}\Biggr)\,q(tx)\,dx\leqslant t^{\alpha-1}$$

fulfilled for all $0 < t < +\infty$ *implies the inequality*

$$\int\limits_0^{+\infty} q(t)\,\ln\Bigl(1+\frac{1}{t^{\,2\alpha}}\Bigr)\,dt \leqslant C(n,\alpha). \tag{3.15}$$

4. A counterexample to Khabibullin's conjecture.

In this section we show that Khabibullin's conjecture in its form 1.1 is not valid for $\alpha > 1/2$. Since the first inequality (1.1) in Khabibullin's conjecture for $t > 0$ is already transformed to (2.3), we introduce the function

$$g(t) = \int\limits_0^t A_{n-1}(y/t)\,q(y)\,dy. \tag{4.1}$$

The formula (4.1) is called the direct conversion formula. It was studied in [6], where the inverse conversion formula expressing $q(t)$ back through $g(t)$ was derived:

$$q(t) = \frac{d^n}{dt^n}\biggl(\frac{t^n\,g'(t)}{(n-1)!}\biggr). \tag{4.2}$$

As it was shown in [6], under the assumptions of Khabibullin's conjecture 1.1 the function (4.1) is an $(n+1)$ times differentiable function satisfying the inequality

$$0 \leqslant g(t) \leqslant t^{\alpha} \quad \text{for all} \quad t > 0. \tag{4.3}$$

Before constructing a counterexample to the conjecture 1.1 we specify $n = 2$ and $\alpha = 2$. Then the transition function (3.8) turns to

$$\varPhi_{n-1}(\alpha,t) = \frac{16\,t^3\,P_1(\alpha,z)}{(1+t^4)^3}, \tag{4.4}$$

where $z = t^4$ and $P_1(\alpha,z)$ is the following polynomial:

$$P_1(\alpha,z) = (2\,\alpha+1)\,z + (1-2\,\alpha) = 5\,z - 3 \tag{4.5}$$

(see the formula (7.6) in [5]). Substituting (4.5) into (4.4), we derive

$$\varPhi_1(2,t) = \frac{16\,t^3\,(5\,t^4-3)}{(1+t^4)^3}. \tag{4.6}$$

The basic feature of the function (4.6) is that it is not always positive. According to (3.2), we have two non-empty subsets M_- and M_+ of the half-line $t > 0$:

$$M_- = \{t\in\mathbb{R}\colon\ 0 < t < t_0\}, \qquad\qquad M_+ = \{t\in\mathbb{R}\colon\ t \geqslant t_0\}, \tag{4.7}$$

where $t_0 = \sqrt[4]{3/5}$ is the root of the polynomial $5\,t^4 - 3$ in the numerator of the fraction in (4.6). Relying on (4.7), we construct the function $g = g(t)$ defining it by two different formulas in M_- and in M_+. For this purpose we define the polynomial

$$h(t)=\frac{(t-t_0)^4}{t_0^4}. \tag{4.8}$$

Since $\alpha=2$, we set $g(t)=t^\alpha=t^2$ for $t\in M_+$. For $t\in M_-$ we define $g(t)$ by a spline polynomial composed with the use of the polynomial (4.8):

$$g(t)=\begin{cases} t^2\,(1-\varepsilon h(t)) & \text{for } \ 0<t<t_0,\\ \qquad t^2 & \text{for } \ t\geqslant t_0.\end{cases} \tag{4.9}$$

Let's recall that $n=2$ and $n+1=3$. Hence (4.9) should be a three times differentiable function. This leads to the following conditions at the point $t=t_0$:

$$\begin{aligned} &\lim_{t\to t_0} g(t)=t_0^2, && \lim_{t\to t_0} g'(t)=2\,t_0,\\ &\lim_{t\to t_0} g''(t)=2, && \lim_{t\to t_0} g'''(t)=0. \end{aligned} \tag{4.10}$$

It is easy to verify that the function (4.9) obeys all of the gluing conditions (4.10).

Note that $h=h(t)$ in (4.8) is a monotonic function on the interval $0\leqslant t\leqslant t_0$ such that $h(0)=1$ and $h(t_0)=0$. Therefore, we have the following lemma.

Lemma 4.1. *The function (4.9) satisfies the inequalities (4.3) for $\alpha=2$ if and only if its parameter ε satisfies the inequalities $0\leqslant\varepsilon\leqslant 1$.*

Having constructed the function $g(t)$ by means of the formula (4.9), we define the function $q=q(t)$ by applying the inverse conversion formula (4.2). It yields

$$q(t)=\begin{cases} 12\,t\,(1-\varepsilon\,r(t)) & \text{for } \ 0<t<t_0,\\ \qquad 12\,t & \text{for } \ t\geqslant t_0,\end{cases} \tag{4.11}$$

where $r(t)$ is a polynomial of the degree 4. In order to simplify the formula for the polynomial $r(t)$ we write this polynomial as

$$r(t)=R(\tau),\ \ \text{where } \ \tau=\frac{t_0-t}{t_0}. \tag{4.12}$$

Then the polynomial $R(\tau)$ in (4.12) is given by the following formula:

$$R(\tau)=21\,\tau^4-34\,\tau^3+16\,\tau^2-2\,\tau. \tag{4.13}$$

As we see, the polynomial (4.13) factorizes into the product of two polynomials:

$$R(\tau)=(21\,\tau^3-34\,\tau^2+16\,\tau-2)\,\tau. \tag{4.14}$$

Let's denote through $R_3(\tau)$ the first multiplicand in (4.14). Then

$$R_3(\tau)=21\,\tau^3-34\,\tau^2+16\,\tau-2. \tag{4.15}$$

The function (4.15) is a cubic polynomial of the variable τ. The graph of this polynomial on the segment $0 \leqslant \tau \leqslant 1$ is shown in Fig. 4.1. At the ending points of this segment we have

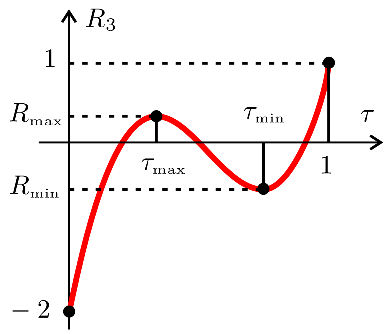

Fig 4.1

$$R_3(0) = -2, \qquad R_3(1) = 1. \tag{4.16}$$

The values (4.16) are easily verified by means of direct calculations. Apart from (4.16), there are two local extrema of the function $R_3(\tau)$ within this segment:

$$\begin{aligned} \tau_{\max} &= \frac{34 - 2\sqrt{37}}{63} \approx 0.34, \\ \tau_{\min} &= \frac{34 + 2\sqrt{37}}{63} \approx 0.73. \end{aligned} \tag{4.17}$$

Substituting the quantities (4.17) into the formula (4.15), we easily find the values of the polynomial $R_3(\tau)$ at its local extrema $\tau_{\max}$ and $\tau_{\min}$:

$$\begin{aligned} R_{\max} &= \frac{394 + 592\sqrt{37}}{11907} \approx 0.33, \\ R_{\min} &= \frac{394 - 592\sqrt{37}}{11907} \approx -0.26. \end{aligned} \tag{4.18}$$

From (4.16) and (4.18) we immediately derive the inequality

$$R_3(\tau) \leqslant 1 \quad \text{for all} \quad 0 \leqslant \tau \leqslant 1. \tag{4.19}$$

Since $\tau \geqslant 0$ in (4.19), we can multiply both sides of (4.19) by τ. This yields $R_3(\tau)\,\tau \leqslant \tau$. Then, combining this inequality with $\tau \leqslant 1$ and taking into account the formulas (4.14) and (4.15), we obtain

$$R(\tau) \leqslant 1 \quad \text{for all} \quad 0 \leqslant \tau \leqslant 1. \tag{4.20}$$

Applying (4.12), we transform (4.20) to the inequality

$$r(t) \leqslant 1 \quad \text{for all} \quad 0 \leqslant t \leqslant t_0, \tag{4.21}$$

while (4.16) combined with (4.15), (4.14) and (4.12) yields

$$r(0) = 1, \qquad\qquad r(t_0) = 0. \tag{4.22}$$

On the base of (4.11), (4.21) and (4.22) we can formulate the following lemma similar to the lemma 4.1.

Lemma 4.2. *The function* (4.11) *satisfies the inequalities $q(t) \geqslant 0$ for all $t > 0$ if and only if its parameter ε satisfies the inequalities $0 \leqslant \varepsilon \leqslant 1$.*

Theorem 4.1. *For each particular value of the parameter ε satisfying the inequalities $0 < \varepsilon \leqslant 1$ the function* (4.11) *is a counterexample to Khabibullin's conjecture 1.1 for $n = 2$ and $\alpha = 2$.*

The theorem 4.1 follows from the lemma 4.2 and the above considerations in § 3 and in § 4 preceding it. Indeed, the function (4.11) is derived from the function (4.9) by means of the inverse conversion formula (4.2). Therefore, substituting (4.11) into the direct conversion formula (4.1), we get the function (4.9).

The formula (4.1) is related to the inequality (1.1) in Khabibullin's conjecture 1.1 through the formulas (2.3), (2.2), and (2.1). Therefore, substituting the function (4.11) into the left hand side of the inequality (1.1) for $n = 2$, we get

$$\int\limits_0^1 \left(\,\int\limits_x^1 (1-y)^{n-1}\,\frac{dy}{y}\right) q(tx)\,dx = \frac{g(t)}{t}, \tag{4.23}$$

where $g(t)$ is given by the formula (4.9). Now let's recall that $\alpha = 2$ and apply the lemma 4.1 to (4.23). This yields the following result.

Lemma 4.3. *For each particular value of the parameter ε such that $0 \leqslant \varepsilon \leqslant 1$ the function (4.11) satisfies the inequality (1.1) for $n = 2$ and $\alpha = 2$.*

The next step is to calculate the integral in the left hand side of the inequality (1.2) for the function (4.11). For this purpose we use the formula

$$\int\limits_0^{+\infty} q(t)\,\ln\!\left(1+\frac{1}{t^{\,2\alpha}}\right)dt \;=\; \int\limits_0^{+\infty} \varPhi_{n-1}(\alpha,t)\,g(t)\,dt. \tag{4.24}$$

The formula (4.24) is derived from (4.1) with the use of the formula (2.7). Let's recall that in our case $\alpha = 2$ and $g(t)$ is given by the formula (4.9). Then the formula (4.24) is written in the following form:

$$\int\limits_0^{+\infty} q(t)\,\ln\!\left(1+\frac{1}{t^{\,2\alpha}}\right)dt \;=\; \int\limits_0^{+\infty} \varPhi_{n-1}(\alpha,t)\;t^{\alpha}\,dt \,-\, \varepsilon\!\int\limits_0^{t_0} \varPhi_{n-1}(\alpha,t)\,t^2\,h(t)\,dt. \tag{4.25}$$

Now we apply the formula (3.13) to (4.25) and write this formula as

$$\int\limits_0^{+\infty} q(t)\,\ln\!\left(1+\frac{1}{t^{\,2\alpha}}\right)dt = \pi\,\alpha\prod_{k=1}^{n-1}\left(1+\frac{\alpha}{k}\right) + \,\delta I. \tag{4.26}$$

The term δI in the right hand side of (4.26) is given by the formula

$$\delta I = -\,\varepsilon\!\int\limits_0^{t_0} \varPhi_{n-1}(\alpha,t)\;t^2\,h(t)\,dt. \tag{4.27}$$

In our case the parameter ε is positive due to the condition $0 < \varepsilon \leqslant 1$ in the theorem 4.1. The function $h(t)$ is given by the formula (4.8). It is positive for $0 < t < t_0$. As for the function $\varPhi_{n-1}(\alpha,t)$, since $n = 2$ and $\alpha = 2$, in our case $\varPhi_{n-1}(\alpha,t)$ is given by the formula (4.6). It is negative for $0 < t < t_0$. As a result we derive the following inequality for the integral (4.27):

$$\delta I > 0 \tag{4.28}$$

The inequality (4.28) completes the proof of the theorem 4.1.

The integral (4.27) can be computed numerically. The product in the right hand side of (4.27) can also be computed numerically. As a result we get

$$\int\limits_0^{+\infty} q(t)\,\ln\Bigl(1+\frac{1}{t^{2\alpha}}\Bigr)\,dt\approx 18.84955592+0.01299443\,\varepsilon. \tag{4.29}$$

On the other hand there is the estimate (3.15) for the integral (4.29). The constant $C(n,\alpha)$ in (3.15) is given by the integral (3.10). It can be calculated numerically. For $n=2$ and $\alpha=2$ we have $C(n,\alpha)\approx 19.65507202$. Then (3.15) yields

$$\int\limits_0^{+\infty} q(t)\,\ln\Bigl(1+\frac{1}{t^{2\alpha}}\Bigr)\,dt\leqslant 19.65507203. \tag{4.30}$$

Note that $0<\varepsilon\leqslant 1$ in the theorem 4.1. As we see, even for $\varepsilon=1$ there is a substantial gap between the integral (4.29) and the estimate (4.30) for it.

5. Conclusions.

The theorem 4.1 shows that Khabibullin's conjecture is not valid for $\alpha>1/2$ in its present form 1.1. However, due to the theorem 3.1 and the inequalities (3.14) it can be replaced by the following theorem which is certainly valid for all $\alpha>0$.

Theorem 5.1. *For each positive integer $n>0$ and for each positive $\alpha>0$ there is a positive constant $C[\mathrm{Kh}](n,\alpha)$ that yields the best (non-improvable) estimate*

$$\int\limits_0^{+\infty} q(t)\,\ln\Bigl(1+\frac{1}{t^{2\alpha}}\Bigr)\,dt\leqslant C[\mathrm{Kh}](n,\alpha) \tag{5.1}$$

in the class of all non-negative continuous functions $q(t)\geqslant 0$ on the half-line $t>0$ satisfying the integral inequality

$$\int\limits_0^1\Biggl(\,\int\limits_x^1(1-y)^{n-1}\,\frac{dy}{y}\Biggr)\,q(tx)\,dx\leqslant t^{\alpha-1}\quad\textit{for all}\ \ t>0. \tag{5.2}$$

In (5.2) we omit the inessential value $t=0$ as compared to the initial inequality (1.1) in Khabibullin's conjecture 1.1.

Note that the constants $C[\mathrm{Kh}](n,\alpha)$ in (5.1) are finite. They satisfy the inequality $C[\mathrm{Kh}](n,\alpha)\leqslant C(n,\alpha)<\infty$, where the constants $C(n,\alpha)$ are given by the formula (3.10). I think the constants $C[\mathrm{Kh}](n,\alpha)$ in (5.1) should be called the Khabibullin constants, appreciating the efforts of Prof. B. N. Khabibullin in formulating the conjecture 1.1 and advertising this conjecture for many years.

Note that the theorem 5.1 claims the existence of the constants $C[\mathrm{Kh}](n,\alpha)$, but it gives neither a formula nor an algorithm for calculating these constants. Nowadays the problem of finding the exact values of the Khabibullin constants $C[\mathrm{Kh}](n,\alpha)$ in (5.1) is yet an unsolved problem.

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