

## A NOTE ON A PERFECT EULER CUBOID.

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ABSTRACT. The problem of constructing a perfect Euler cuboid is reduced to a single Diophantine equation of the degree 12.

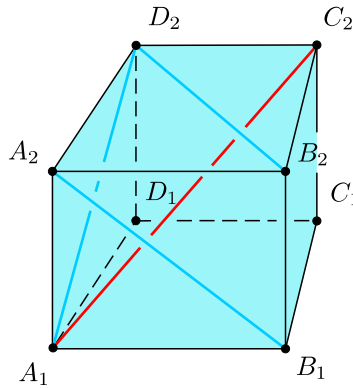
### 1. INTRODUCTION.

An *Euler cuboid*, named after Leonhard Euler, is a rectangular parallelepiped whose edges and face diagonals all have integer lengths. A *perfect cuboid* is an Euler cuboid whose space diagonal is also of an integer length.

In 2005 Lasha Margishvili from the Georgian-American High School in Tbilisi won the Mu Alpha Theta Prize for the project entitled "Diophantine Rectangular Parallelepiped" (see [http://www.mualphatheta.org/Science\\_Fair/...](http://www.mualphatheta.org/Science_Fair/...)). He suggested a proof that a perfect Euler cuboid does not exist. However, by now his proof is not accepted by mathematical community. The problem of finding a perfect Euler cuboid is still considered as an unsolved problem. The history of this problem can be found in [1]. Here are some appropriate references: [2–35].

### 2. PASSING TO RATIONAL NUMBERS.

Let  $A_1B_1C_1D_1A_2B_2C_2D_2$  be a perfect Euler cuboid. Its edges are presented by positive integer numbers. We write this fact as



$$\begin{aligned} |A_1B_1| &= a, \\ |A_1D_1| &= b, \\ |A_1A_2| &= c. \end{aligned} \tag{2.1}$$

Its face diagonals are also presented by positive integers (see Fig. 2.1):

$$\begin{aligned} |A_1D_2| &= \alpha, \\ |A_2B_1| &= \beta, \\ |B_2D_2| &= \gamma. \end{aligned} \tag{2.2}$$

And finally, the spacial diagonal of this cuboid is presented by a positive integer:

$$|A_1C_2| = d. \tag{2.3}$$

Fig. 2.1

From (2.1), (2.2), (2.3) one easily derives a series of Diophantine equations for the integer numbers  $a, b, c, \alpha, \beta, \gamma$ , and  $d$ :

$$\begin{aligned} a^2 + b^2 &= \gamma^2, & b^2 + c^2 &= \alpha^2, \\ c^2 + a^2 &= \beta^2, & a^2 + b^2 + c^2 &= d^2. \end{aligned} \quad (2.4)$$

The main goal of this paper is to reduce the equations (2.4) to a single Diophantine equation for some other integer numbers.

Relying on the last equation (2.4), we introduce the following rational numbers:

$$x_1 = \frac{a}{d}, \quad x_2 = \frac{b}{d}, \quad x_3 = \frac{c}{d}. \quad (2.5)$$

The numbers (2.5) are the components of a three-dimensional unit vector:

$$(x_1)^2 + (x_2)^2 + (x_3)^2 = 1. \quad (2.6)$$

From the first three equations (2.4) one easily derives the equations

$$\begin{aligned} (x_1)^2 + (x_2)^2 &= (d_3)^2, \\ (x_2)^2 + (x_3)^2 &= (d_1)^2, \\ (x_3)^2 + (x_1)^2 &= (d_2)^2, \end{aligned} \quad (2.7)$$

where the rational numbers  $d_1, d_2, d_3$  are given by the following fractions:

$$d_1 = \frac{\alpha}{d}, \quad d_2 = \frac{\beta}{d}, \quad d_3 = \frac{\gamma}{d}. \quad (2.8)$$

The equations (2.6), (2.7), and (2.8) lead to the following theorem.

**Theorem 2.1.** *A perfect Euler cuboid does exist if and only if the equations (2.6) and (2.7) are solvable in positive rational numbers  $x_1, x_2, x_3$  and  $d_1, d_2, d_3$ .*

*Proof.* The direct proposition of the theorem 2.1 is immediate from the formulas (2.4), (2.5), and (2.8). Conversely, assume that  $x_1, x_2, x_3$  and  $d_1, d_2, d_3$  are positive rational numbers obeying the equations (2.6) and (2.7). They are presented by some unique irreducible fractions with positive integer numerators and denominators:

$$x_1 = \frac{\nu_1}{\delta_1}, \quad x_2 = \frac{\nu_2}{\delta_2}, \quad x_3 = \frac{\nu_3}{\delta_3}, \quad d_1 = \frac{\nu_4}{\delta_4}, \quad d_2 = \frac{\nu_5}{\delta_5}, \quad d_3 = \frac{\nu_6}{\delta_6}.$$

Let's denote through  $d$  the least common multiple of their denominators, i. e.

$$d = LCM(\delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6).$$

Then the following products are positive integer numbers:

$$\begin{aligned} a &= x_1 d, & b &= x_2 d, & c &= x_3 d, \\ \alpha &= d_1 d, & \beta &= d_2 d, & \gamma &= d_3 d. \end{aligned} \quad (2.9)$$

Applying (2.6) and (2.7) to (2.9), we derive the equations (2.4) for the integer numbers  $a, b, c, \alpha, \beta, \gamma$ , and  $d$ .  $\square$

### 3. A RATIONAL PARAMETRIZATION.

Combining (2.7) and (2.6), we derive the following equation for  $x_1$  and  $d_1$ :

$$(x_1)^2 + (d_1)^2 = 1. \quad (3.1)$$

Rational solutions of the equation (3.1) are parametrized by a rational number  $u$ :

$$x_1 = \frac{2u}{1+u^2}, \quad d_1 = \frac{1-u^2}{1+u^2}. \quad (3.2)$$

Since both  $x_1$  and  $d_1$  in (3.1) are positive, the parameter  $u$  satisfies the inequalities:

$$0 < u < 1. \quad (3.3)$$

The second equation in (2.7) is  $(x_2)^2 + (x_3)^2 = (d_1)^2$ . This equation can be written in a form quite similar to the equation (3.1):

$$\left(\frac{x_2}{d_1}\right)^2 + \left(\frac{x_3}{d_1}\right)^2 = 1. \quad (3.4)$$

Rational solutions of the equation (3.4) are parametrized by a rational number  $z$ :

$$\frac{x_2}{d_1} = \frac{2z}{1+z^2}, \quad \frac{x_3}{d_1} = \frac{1-z^2}{1+z^2}. \quad (3.5)$$

Combining (3.2) and (3.5), we derive the formulas

$$\begin{aligned} x_2 &= \frac{2z(1-u^2)}{(1+u^2)(1+z^2)}, \\ x_3 &= \frac{(1-u^2)(1-z^2)}{(1+u^2)(1+z^2)}. \end{aligned} \quad (3.6)$$

The parameter  $z$  in (3.5) and (3.6) obeys the inequalities similar to (3.3):

$$0 < z < 1. \quad (3.7)$$

**Theorem 3.1.** *The formulas (3.2) and (3.6) constitute a rational parametrization of the variables  $x_1, x_2, x_3$  and  $d_1$  by means of two parameters  $u$  and  $z$  obeying the inequalities (3.3) and (3.7). The equations*

$$(x_1)^2 + (x_2)^2 + (x_3)^2 = 1, \quad (x_2)^2 + (x_3)^2 = (d_1)^2 \quad (3.8)$$

*are fulfilled identically due to this parametrization.*

The proof of the theorem 3.1 is pure calculations.

## 4. AN EXTENDED PARAMETRIZATION.

Note that the equations (3.8) are two of the four equations (2.6) and (2.7) providing a perfect Euler cuboid. The other two equations are

$$(x_1)^2 + (x_2)^2 = (d_3)^2, \quad (x_3)^2 + (x_1)^2 = (d_2)^2. \quad (4.1)$$

Let's substitute (3.2) and (3.6) into the first equation (4.1). As a result we get

$$(d_3)^2 = \frac{4(u^2 z^2 + 1)(u^2 + z^2)}{(1 + u^2)^2 (1 + z^2)^2}. \quad (4.2)$$

Similarly, substituting (3.2) and (3.6) into the second equation (4.1), we get

$$(d_2)^2 = \frac{((1 + u^2)(1 + z^2) + 2z(1 - u^2))((1 + u^2)(1 + z^2) - 2z(1 - u^2))}{(1 + u^2)^2 (1 + z^2)^2}. \quad (4.3)$$

Relying on (4.2), we define the following two quantities  $\xi$  and  $a$ :

$$\xi = u^2 z^2 + 1, \quad a = \frac{d_3 (1 + u^2) (1 + z^2)}{2(u^2 z^2 + 1)}. \quad (4.4)$$

Similarly, relying on (4.2), we define other two quantities  $\zeta$  and  $b$ :

$$\begin{aligned} \zeta &= (1 + u^2)(1 + z^2) + 2z(1 - u^2), \\ b &= \frac{d_2 (1 + u^2) (1 + z^2)}{(1 + u^2)(1 + z^2) + 2z(1 - u^2)}. \end{aligned} \quad (4.5)$$

The formulas (4.4) and (4.5) are consistent since the denominators of the fractions in them are positive. For  $a$  and  $b$  from (4.2), (4.3), (4.4), and (4.5), we derive.

$$a^2 = \frac{u^2 + z^2}{u^2 z^2 + 1}, \quad (4.6)$$

$$b^2 = \frac{(1 + u^2)(1 + z^2) - 2z(1 - u^2)}{(1 + u^2)(1 + z^2) + 2z(1 - u^2)}. \quad (4.7)$$

Since  $d_2 > 0$  and  $d_3 > 0$  (see (2.8), (2.2) and (2.3)), the quantities  $a$  and  $b$  are positive. Therefore, the formulas (4.6) and (4.7) define two positive functions

$$a = a(u, z), \quad b = b(u, z). \quad (4.8)$$

The domain of the functions (4.8) is outlined by the inequalities (3.3) and (3.7):

$$D_{uz} = \{(u, z) \in \mathbb{R}^2 : 0 < u < 1 \text{ and } 0 < z < 1\}. \quad (4.9)$$

The functions (4.8) defined in the domain (4.9) constitute a mapping

$$f: D_{uz} \rightarrow \mathbb{R}^2. \quad (4.10)$$

Let's denote through  $D_{ab}$  the image of the domain  $D_{uz}$  under the mapping (4.10):

$$D_{ab} = \text{Im } f = f(D_{uz}). \quad (4.11)$$

The domain (4.11) is shown in Fig. 4.1. It is an open triangle with one curvilinear side. The curvilinear side of the triangle  $D_{ab}$  is the graph of the function

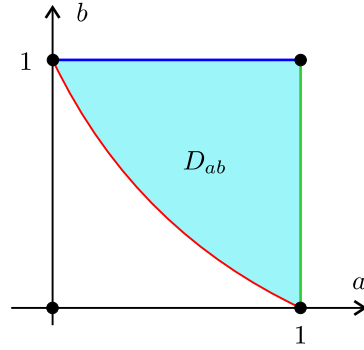


Fig 4.1

$$b(a) = -1 + \frac{2}{a+1}.$$

Using the formulas (4.6) and (4.7), one can prove that the mapping (4.10) sets up a bijective correspondence of the points of  $D_{uz}$  with the points of  $D_{ab}$ :

$$f: D_{uz} \rightarrow D_{ab}.$$

The inverse mapping

$$f^{-1}: D_{ab} \rightarrow D_{uz}$$

establishing the backward correspondence of the points of  $D_{ab}$  with those of  $D_{uz}$  is given by two algebraic functions

$$u = u(a, b), \quad z = z(a, b). \quad (4.12)$$

Let's consider the second formula (4.4) and the second formula (4.5). We can write these two formulas in the following way:

$$d_2 = \frac{(1+u^2)(1+z^2) + 2z(1-u^2)}{(1+u^2)(1+z^2)} b, \quad (4.13)$$

$$d_3 = \frac{2(u^2 z^2 + 1)}{(1+u^2)(1+z^2)} a.$$

Substituting (4.12) into the formulas (3.2), (3.6), and (4.13), we can represent  $x_1, x_2, x_3$  and  $d_1, d_2, d_3$  as functions of two variables  $a$  and  $b$ :

$$\begin{aligned} x_1 &= x_1(a, b), & x_2 &= x_2(a, b), & x_3 &= x_3(a, b), \\ d_1 &= d_1(a, b), & d_2 &= d_2(a, b), & d_3 &= d_3(a, b). \end{aligned} \quad (4.14)$$

**Definition 4.1.** The functions (4.14) sharing the common domain  $D_{ab}$  constitute a parametrization for the problem of a perfect Euler cuboid. They extend the rational parametrization given by the functions (3.2) and (3.6).

**Theorem 4.1.** *The equations (2.6) and (2.7) providing a perfect Euler cuboid are fulfilled identically by the functions (4.14).*

The theorem 4.1 is analogous to the theorem 3.1. One can see that it is already proved by the above considerations.

## 5. THE CHARACTERISTIC EQUATION.

Unlike (3.2) and (3.6), the functions (4.14) are not explicit. Below we derive an algorithm for evaluating them. For this purpose let's return back to the formulas (4.4), (4.5), (4.6), and (4.7). From (4.4) and (4.6) we derive the equations

$$\xi = u^2 z^2 + 1, \quad \xi a^2 = u^2 + z^2. \quad (5.1)$$

Similarly, from (4.5) and (4.7) we derive the equations

$$\begin{aligned} \zeta &= (1 + u^2)(1 + z^2) + 2z(1 - u^2), \\ \zeta b^2 &= (1 + u^2)(1 + z^2) - 2z(1 - u^2). \end{aligned} \quad (5.2)$$

Subtracting both equations (5.1) from each of the equations (5.2), we get

$$\begin{cases} \zeta - \xi(1 + a^2) = 2z(1 - u^2), \\ \zeta b^2 - \xi(1 + a^2) = -2z(1 - u^2). \end{cases} \quad (5.3)$$

The equations (5.3) constitute a system of two linear algebraic equations with respect to the variables  $\xi$  and  $\zeta$ . Solving them, we derive

$$\xi = \frac{2z(1 - u^2)(1 + b^2)}{(1 - b^2)(1 + a^2)}, \quad \zeta = \frac{4z(1 - u^2)}{1 - b^2}. \quad (5.4)$$

Now let's substitute  $\theta$  for  $z^2$  into (5.1). Then the equations (5.1) turn to a system of two linear algebraic equations with respect to the variables  $\xi$  and  $\theta$ :

$$\begin{cases} \xi - u^2 \theta = 1, \\ \xi a^2 - \theta = u^2. \end{cases} \quad (5.5)$$

Solving the system of linear equations (5.5), we obtain

$$\xi = \frac{(1 - u^2)(1 + u^2)}{1 - a^2 u^2}, \quad \theta = \frac{a^2 - u^2}{1 - a^2 u^2}. \quad (5.6)$$

In (5.4) and (5.6) we have two expressions for  $\xi$ . Equating them we derive the following expression for  $z$  expressing it through  $a$ ,  $b$  and  $u$ :

$$z = \frac{(1 + u^2)(1 - b^2)(1 + a^2)}{2(1 + b^2)(1 - a^2 u^2)}. \quad (5.7)$$

Substituting (5.7) into the second equation (5.4), we derive:

$$\zeta = \frac{2(1 + u^2)(1 - u^2)(1 + a^2)}{(1 + b^2)(1 - a^2 u^2)}. \quad (5.8)$$

Note that the formulas (5.6), (5.7), and (5.8) are similar to each other. They express  $z$ ,  $\xi$ ,  $\zeta$ , and  $\theta$  through  $a$ ,  $b$ , and  $u$ . But only two of the three variables  $a$ ,  $b$ , and  $u$  are independent. Due to (4.12) the variable  $u$  is uniquely expressed

through  $a$  and  $b$  within the domain  $D_{ab}$  shown in Fig. 4.1. In order to evaluate this expression let's recall that we have the following equation:

$$\theta = z^2. \quad (5.9)$$

Applying (5.6) and (5.8) to (5.9), we write (5.9) as

$$\frac{(1+u^2)^2(1-b^2)^2(1+a^2)^2}{4(1+b^2)^2(1-a^2u^2)^2} = \frac{a^2-u^2}{1-a^2u^2}. \quad (5.10)$$

The denominators of the fractions in the equation (5.10) are nonzero within the domain  $D_{ab}$ . For this reason it can be brought to a polynomial equation:

$$\begin{aligned} & u^4 a^4 b^4 + (6 a^4 u^2 b^4 - 2 u^4 a^4 b^2 - 2 u^4 a^2 b^4) + (4 u^2 b^4 a^2 + \\ & + 4 a^4 u^2 b^2 - 12 u^4 a^2 b^2 + u^4 a^4 + u^4 b^4 + a^4 b^4) + (6 a^4 u^2 + 6 u^2 b^4 - \\ & - 8 a^2 b^2 u^2 - 2 u^4 a^2 - 2 u^4 b^2 - 2 a^4 b^2 - 2 b^4 a^2) + (u^4 + b^4 + \\ & + a^4 + 4 a^2 u^2 + 4 b^2 u^2 - 12 b^2 a^2) + (6 u^2 - 2 a^2 - 2 b^2) + 1 = 0. \end{aligned} \quad (5.11)$$

**Theorem 5.1.** *The equation (5.11) defines the function  $u = u(a, b)$  from (4.12) in an implicit form. It is called the characteristic equation.*

**Theorem 5.2.** *A perfect Euler cuboid does exist if and only if the characteristic equation (5.11) has a rational solution such that  $0 < u < 1$ , while  $a$  and  $b$  are the coordinates of some point within the open domain  $D_{ab}$  shown in Fig. 4.1.*

The inhomogeneous polynomial equation (5.11) can be transformed to a homogeneous equation by adding one more variable  $c$ :

$$\begin{aligned} & u^4 a^4 b^4 + 6 a^4 u^2 b^4 c^2 - 2 u^4 a^4 b^2 c^2 - 2 u^4 a^2 b^4 c^2 + 4 u^2 b^4 a^2 c^4 + \\ & + 4 a^4 u^2 b^2 c^4 - 12 u^4 a^2 b^2 c^4 + u^4 a^4 c^4 + u^4 b^4 c^4 + a^4 b^4 c^4 + \\ & + 6 a^4 u^2 c^6 + 6 u^2 b^4 c^6 - 8 a^2 b^2 u^2 c^6 - 2 u^4 a^2 c^6 - 2 u^4 b^2 c^6 - \\ & - 2 a^4 b^2 c^6 - 2 b^4 a^2 c^6 + u^4 c^8 + b^4 c^8 + a^4 c^8 + 4 a^2 u^2 c^8 + \\ & + 4 b^2 u^2 c^8 - 12 b^2 a^2 c^8 + 6 u^2 c^{10} - 2 a^2 c^{10} - 2 b^2 c^{10} + c^{12} = 0. \end{aligned} \quad (5.12)$$

**Theorem 5.3.** *A perfect Euler cuboid does exist if and only if the Diophantine equation (5.12) has a solution such that  $c > 0$  and  $0 < u/c < 1$ , while  $a/c$  and  $b/c$  are the coordinates of some point within the open domain  $D_{ab}$  shown in Fig. 4.1.*

The theorems 5.2 and 5.3 constitute the main result of this paper. They can be used in numeric search for a perfect Euler cuboid.

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