A NOTE ON THE SECOND CUBOID CONJECTURE. PART I.

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Abstract. The problem of finding perfect Euler cuboids or proving their non-existence is an old unsolved problem in mathematics. The second cuboid conjecture is one of the three propositions suggested as intermediate stages in proving the non-existence of perfect Euler cuboids. It is associated with a certain polynomial Diophantine equation of the order 10. In this paper a structural theorem for the solutions of this Diophantine equation is proved and some examples of its application are considered.

1. Introduction.

Let's denote through $Q_{pq}(t)$ the following polynomial of the order 10 depending on two integer parameters p and q:

$$\begin{aligned}Q_{pq}(t) = t^{10} + (2\,q^2 + p^2)\,(3\,q^2 - 2\,p^2)\,t^8 + (q^8 + 10\,p^2\,q^6 +{}\\ + 4\,p^4\,q^4 - 14\,p^6\,q^2 + p^8)\,t^6 - p^2\,q^2\,(p^8 + 10\,q^2\,p^6 + 4\,q^4\,p^4 -{}\\ - 14\,q^6\,p^2 + q^8)\,t^4 - p^6\,q^6\,(2\,p^2 + q^2)\,(3\,p^2 - 2\,q^2)\,t^2 - q^{10}\,p^{10}.\end{aligned}\tag{1.1}$$

Conjecture 1.1 (second cuboid conjecture). *For any positive coprime integers $p \neq q$ the polynomial $Q_{pq}(t)$ in (1.1) is irreducible in the ring $\mathbb{Z}[t]$.*

The second cuboid conjecture 1.1 and the polynomial $Q_{pq}(t)$ in it were introduced in [1]. They are associated with the problem of constructing a perfect Euler cuboid (see [2] and [3–37] for more details). Let's write the equation

$$Q_{pq}(t) = 0.\tag{1.2}$$

The equation (1.2) can be understood as a Diophantine equation of the order 10 with two integer parameters p and q. The second cuboid conjecture 1.1 implies the following theorem.

Theorem 1.1. *For any positive coprime integers $p \neq q$ the polynomial Diophantine equation (1.2) has no integer solutions.*

Note that a similar theorem associated with the first cuboid conjecture was formulated and proved in [38].

The theorem 1.1 is a weaker proposition than the conjecture 1.1 itself. However, even proving this proposition in the case of the second cuboid conjecture is rather

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difficult. Below in section 4 we formulate and prove a structural theorem for the solutions of the Diophantine equation (1.2), if any, and in section 5 we use it in order to prove the theorem 1.1 for some particular values p and q.

2. The inversion symmetry and parity.

The polynomial $Q_{pq}(t)$ in (1.1) possesses some special properties. They are expressed by the following formulas which can be verified by direct calculations:

$$\begin{aligned} &Q_{pq}(t)=-\frac{Q_{qp}(p^2\,q^2/t)\,t^{10}}{p^{10}\,q^{10}}, &&Q_{pq}(t)=Q_{pq}(-t),\\ &Q_{qp}(t)=-\frac{Q_{pq}(p^2\,q^2/t)\,t^{10}}{p^{10}\,q^{10}}, &&Q_{qp}(t)=Q_{qp}(-t).\end{aligned}\tag{2.1}$$

Note that in (2.1) we have two polynomials $Q_{pq}(t)$ and $Q_{qp}(t)$. The polynomial $Q_{qp}(t)$ is produced from (1.1) by exchanging parameters p and q:

$$\begin{aligned} &Q_{qp}(t)=t^{10}+(2\,p^2+q^2)\,(3\,p^2-2\,q^2)\,t^8+(p^8+10\,q^2\,p^6+\\ &+4\,q^4\,p^4-14\,q^6\,p^2+q^8)\,t^6-q^2\,p^2\,(q^8+10\,p^2\,q^6+4\,p^4\,q^4+\\ &-14\,p^6\,q^2+p^8)\,t^4-q^6\,p^6\,(2\,q^2+p^2)\,(3\,q^2-2\,p^2)\,t^2-p^{10}\,q^{10}.\end{aligned}\tag{2.2}$$

Two of the four symmetries in (2.1) contain the inversion of t. For this reason they are called inversion symmetries. The other two symmetries in (2.1) mean that the polynomials (1.1) and (2.2) are even with respect to their argument t.

3. Some prerequisites.

Assume that the polynomial $Q_{pq}(t)$ has an integer root $t=A_0$. Since $p\neq 0$ and $q\neq 0$, we have $A_0\neq 0$. Then due to the inversion symmetries in (2.1) the polynomial $Q_{qp}(t)$ has an integer root $t=B_0$, where

$$B_0=\frac{p^2\,q^2}{A_0}.\tag{3.1}$$

Since $p\neq 0$ and $q\neq 0$, from (3.1) we derive $B_0\neq 0$. Applying the parity symmetry from (2.1), we conclude that the polynomial $Q_{pq}(t)$ has the other integer root $t=-A_0$, while $Q_{qp}(t)$ has the other integer root $t=-B_0$. As a result the polynomials $Q_{pq}(t)$ and $Q_{qp}(t)$ split into factors

$$Q_{pq}(t)=(t^2-A_0^2)\;C_8(t),\qquad\qquad Q_{qp}(t)=(t^2-B_0^2)\;D_8(t)\tag{3.2}$$

with $A_0>0$ and $B_0>0$. Here $C_8(t)$ and $D_8(t)$ are eighth order polynomials complementary to $t^2-A_0^2$ and $t^2-B_0^2$. Applying (2.1) to (3.2) we derive

$$C_8(t)=C_8(-t),\qquad\qquad D_8(t)=D_8(-t).\tag{3.3}$$

Due to (3.3) the polynomials $C_8(t)$ and $D_8(t)$ are given by the formulas

$$\begin{aligned} &C_8(t)=t^8+C_6\,t^6+C_4\,t^4+C_2\,t^2+C_0,\\ &D_8(t)=t^8+D_6\,t^6+D_4\,t^4+D_2\,t^2+D_0.\end{aligned}\tag{3.4}$$

The coefficients of the polynomials (3.4) are integer numbers.

Now let's apply the inversion symmetries from (2.1) to (3.2). As a result we get

$$C_8(t)=\frac{D_8(p^2\,q^2/t)\,t^8}{p^6\,q^6\,A_0^2},\qquad\qquad D_8(t)=\frac{C_8(p^2\,q^2/t)\,t^8}{p^6\,q^6\,B_0^2}.\tag{3.5}$$

Applying the symmetries (3.5) to (3.4), we derive a series of relationships for the coefficients of the polynomials $C_8(t)$ and $D_8(t)$:

$$\begin{aligned}
&C_0\,A_0^2=p^{10}\,q^{10}, &&C_2\,A_0^2=p^6\,q^6\,D_6,\\
&C_4\,A_0^2=p^2\,q^2\,D_4, &&C_6\,A_0^2\,p^2\,q^2=D_2,\\
&A_0^2\,p^6\,q^6=D_0, &&D_0\,B_0^2=p^{10}\,q^{10},\\
&D_2\,B_0^2=p^6\,q^6\,C_6, &&D_4\,B_0^2=p^2\,q^2\,C_4,\\
&D_6\,B_0^2\,p^2\,q^2=C_2, &&B_0^2\,p^6\,q^6=C_0.
\end{aligned}\tag{3.6}$$

The equations (3.6) are excessive. Due to (3.1) some of them are equivalent to some others. For this reason we can eliminate excessive variables:

$$\begin{aligned}
&D_0=p^6\,q^6\,A_0^2, &&D_2=C_6\,p^2\,q^2\,A_0^2,\\
&C_0=p^6\,q^6\,B_0^2, &&C_2=D_6\,p^2\,q^2\,B_0^2.
\end{aligned}\tag{3.7}$$

Substituting (3.7) into the formulas (3.4) for $C_8(t)$ and $D_8(t)$, we get

$$\begin{aligned}
&C_8(t)=t^8+C_6\,t^6+C_4\,t^4+D_6\,p^2\,q^2\,B_0^2\,t^2+p^6\,q^6\,B_0^2,\\
&D_8(t)=t^8+D_6\,t^6+D_4\,t^4+C_6\,p^2\,q^2\,A_0^2\,t^2+p^6\,q^6\,A_0^2.
\end{aligned}\tag{3.8}$$

Unlike C_0, D_0, C_2, and D_2 in (3.7), the coefficients C_4 and D_4 in (3.8) are not expressed through other coefficients. However, they are not independent. They are related with each other by means of the equation

$$A_0\,C_4=B_0\,D_4.\tag{3.9}$$

The equation (3.9) is derived from (3.6) by means of the formula (3.1).

Having derived the formulas (3.8), we substitute them back into the relationships (3.2). As a result we derive the following formulas:

$$\begin{aligned}
Q_{pq}(t)=t^{10}&+(C_6-A_0^2)\,t^8+(C_4-A_0^2\,C_6)\,t^6+(D_6\,p^2\,q^2\,B_0^2-\\
&-A_0^2\,C_4)\,t^4+q^2\,p^2\,B_0^2\,(p^4\,q^4-A_0^2\,D_6)\,t^2-A_0^2\,p^6\,q^6\,B_0^2,
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
Q_{qp}=t^{10}&+(D_6-B_0^2)\,t^8+(D_4-B_0^2\,D_6)\,t^6+(C_6\,p^2\,q^2\,A_0^2-\\
&-B_0^2\,D_4)\,t^4+p^2\,q^2\,A_0^2\,(p^4\,q^4-B_0^2\,C_6)\,t^2-A_0^2\,p^6\,q^6\,B_0^2.
\end{aligned}\tag{3.11}$$

Comparing the formula (3.10) with (1.1) and comparing the formula (3.11) with (2.2), we derive ten equations for the coefficients of the polynomials (3.8). Two of them are equivalent to the equation (3.1) written as

$$A_0\,B_0=p^2\,q^2.\tag{3.12}$$

The other eight equations are written as follows:

$$
\begin{aligned}
&C_6 - A_0^2 = (2\,q^2 + p^2)\,(3\,q^2 - 2\,p^2),\\
&D_6 - B_0^2 = (2\,p^2 + q^2)\,(3\,p^2 - 2\,q^2),
\end{aligned}
\tag{3.13}
$$

$$
\begin{aligned}
&C_4 - A_0^2\,C_6 = p^8 - 14\,p^6\,q^2 + 4\,p^4\,q^4 + 10\,p^2\,q^6 + q^8,\\
&D_4 - B_0^2\,D_6 = q^8 - 14\,q^6\,p^2 + 4\,q^4\,p^4 + 10\,q^2\,p^6 + p^8,
\end{aligned}
\tag{3.14}
$$

$$
\begin{aligned}
&A_0^2\,C_4 - D_6\,p^2\,q^2\,B_0^2 = p^2\,q^2\,(q^8 - 14\,q^6\,p^2 + 4\,q^4\,p^4 + 10\,q^2\,p^6 + p^8),\\
&B_0^2\,D_4 - C_6\,p^2\,q^2\,A_0^2 = p^2\,q^2\,(p^8 - 14\,p^6\,q^2 + 4\,p^4\,q^4 + 10\,p^2\,q^6 + q^8),
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
&B_0^2\,(A_0^2\,D_6 - p^4\,q^4) = p^4\,q^4\,(2\,p^2 + q^2)\,(3\,p^2 - 2\,q^2),\\
&A_0^2\,(B_0^2\,C_6 - p^4\,q^4) = p^4\,q^4\,(2\,q^2 + p^2)\,(3\,q^2 - 2\,p^2).
\end{aligned}
\tag{3.16}
$$

The equations (3.12), (3.13), (3.14), (3.15), (3.16) are excessive. Indeed, the equations (3.16) follow from (3.12) and (3.13). Similarly, the equations (3.15) can be derived from (3.14) with the use of (3.9) and (3.12). As for the equations (3.13) and (3.14), when complemented with the equations (3.9) and (3.12), they constitute a system of Diophantine equations with respect to C_4, D_4, C_6, D_6, A_0 and B_0. The results of the above calculations are summarized in the following lemma.

Lemma 3.1. *For $p \neq 0$ and $q \neq 0$ the polynomial $Q_{pq}(t)$ in (1.1) has integer roots if and only if the system of Diophantine equations (3.9), (3.12), (3.13), and (3.14) is solvable with respect to the integer variables C_4, D_4, C_6, D_6, $A_0 > 0$ and $B_0 > 0$.*

4. The structural theorem.

Below we continue studying the equations (3.9), (3.12), (3.13), (3.14) implicitly assuming $p \neq q$ to be two positive coprime integer numbers. Let $p_1, \ldots, p_m$ be the prime factors of p and let $q_1, \ldots, q_n$ be the prime factors of q:

$$
p = p_1^{\alpha_1} \cdot \ldots \cdot p_m^{\alpha_m}, \qquad\qquad q = q_1^{\beta_1} \cdot \ldots \cdot q_n^{\beta_n}.
\tag{4.1}
$$

Usually the multiplicities $\alpha_1, \ldots, \alpha_m$ and $\beta_1, \ldots, \beta_n$ in (4.1) are positive numbers. However, in order to cover two special cases $p = 1$ and $q = 1$ we assume them to be non-negative numbers. Since p and q are assumed coprime, i. e.

$$
\gcd(p, q) = 1,
\tag{4.2}
$$

the prime factors $p_1, \ldots, p_m$ and $q_1, \ldots, q_n$ in (4.1) are distinct, i. e. $p_i \neq q_j$.

Lemma 4.1. *For any solution of the Diophantine equations (3.9), (3.12), (3.13), and (3.14) with $A_0 > 0$ and $B_0 > 0$ if $A_0 \neq 1$, each prime factor r of A_0 is a prime factor of p or a prime factor of q, i. e. $r = p_i$ or $r = q_j$.*

Lemma 4.2. *For any solution of the Diophantine equations (3.9), (3.12), (3.13), and (3.14) with $A_0 > 0$ and $B_0 > 0$ if $B_0 \neq 1$, each prime factor r of B_0 is a prime factor of p or a prime factor of q, i. e. $r = p_i$ or $r = q_j$.*

The lemmas 4.1 and 4.2 are immediate from (4.2) and (3.12). Due to the lemmas 4.1 and 4.2 we can write the following expansions for A_0 and B_0:

$$\begin{aligned} A_0&=p_1^{\mu_1}\cdot\ldots\cdot p_m^{\mu_m}\,q_1^{\nu_1}\cdot\ldots\cdot q_n^{\nu_n},\\ B_0&=p_1^{\eta_1}\cdot\ldots\cdot p_m^{\eta_m}\,q_1^{\tau_1}\cdot\ldots\cdot q_n^{\tau_n}.\end{aligned}\tag{4.3}$$

The multiplicities $\mu_1,\,\ldots\,,\,\mu_m$, $\nu_1,\,\ldots\,,\,\nu_n$, $\eta_1,\,\ldots\,,\,\eta_m$, and $\tau_1,\,\ldots\,,\,\tau_n$ in the expansions (4.3) obey the following relationships:

$$\mu_i+\eta_i=2\,\alpha_i,\qquad\qquad \nu_i+\tau_i=2\,\beta_i.\tag{4.4}$$

The formulas (4.4) are easily derived by substituting the expansions (4.1) and (4.3) into the equation (3.12).

Now let's return back to the equations (3.13) and (3.14). It is easy to see that the equations (3.13) can be explicitly resolved with respect to C_6 and D_6:

$$\begin{aligned} C_6&=A_0^2+(2\,q^2+p^2)\,(3\,q^2-2\,p^2),\\ D_6&=B_0^2+(2\,p^2+q^2)\,(3\,p^2-2\,q^2).\end{aligned}\tag{4.5}$$

Upon substituting (4.5) into the equations (3.14) we can explicitly resolve the equations (3.14) with respect to the variables C_4 and D_4:

$$\begin{aligned} C_4=A_0^4&+(2\,q^2+p^2)\,(3\,q^2-2\,p^2)\,A_0^2+\\ &+p^8-14\,p^6\,q^2+4\,p^4\,q^4+10\,p^2\,q^6+q^8,\end{aligned}\tag{4.6}$$

$$\begin{aligned} D_4=B_0^4&+(2\,p^2+q^2)\,(3\,p^2-2\,q^2)\,B_0^2+\\ &+q^8-14\,q^6\,p^2+4\,q^4\,p^4+10\,q^2\,p^6+p^8.\end{aligned}\tag{4.7}$$

And finally, we can substitute (4.6) and (4.7) into the equation (3.9). As a result we derive the following equation for the variables A_0 and B_0:

$$\begin{aligned} A_0\,\bigl(A_0^4&+(2\,q^2+p^2)\,(3\,q^2-2\,p^2)\,A_0^2+\\ &+p^8-14\,p^6\,q^2+4\,p^4\,q^4+10\,p^2\,q^6+q^8\bigr)=\\ =B_0\,\bigl(B_0^4&+(2\,p^2+q^2)\,(3\,p^2-2\,q^2)\,B_0^2+\\ &+q^8-14\,q^6\,p^2+4\,q^4\,p^4+10\,q^2\,p^6+p^8\bigr).\end{aligned}\tag{4.8}$$

Summarizing these calculations, we can formulate the following lemma.

Lemma 4.3. *The system of four Diophantine equations* (3.9)*,* (3.12)*,* (3.13)*,* (3.14) *is equivalent to the system of two Diophantine equations* (3.12) *and* (4.8)*.*

Lemma 4.4. *For any solution of the Diophantine equations* (3.9)*,* (3.12)*,* (3.13)*, and* (3.14) *with $A_0>0$ and $B_0>0$ if $\mu_i>0$ and $\eta_i>0$ in* (4.3)*, then $\mu_i=\eta_i=\alpha_i$.*

Proof. The proof is by contradiction. Assume that $\mu_i>0$, $\eta_i>0$, and $\mu_i\neq\eta_i$. Then from (4.6) and (4.7), applying (4.1), (4.2), (4.3), and (4.4), we derive

$$C_4\equiv q^8\pmod{p_i},\qquad\qquad D_4\equiv q^8\pmod{p_i}.\tag{4.9}$$

Moreover, the formula (4.3) yields the relationships

$$A_0 = A'_0\,p_i^{\mu_i}, \qquad\qquad B_0 = B'_0\,p_i^{\eta_i}, \tag{4.10}$$

where $A'_0 \not\equiv 0\,(\mathrm{mod}\ p_i)$ and $B'_0 \not\equiv 0\,(\mathrm{mod}\ p_i)$. Our assumption $\mu_i \neq \eta_i$ means that $\mu_i > \eta_i$ or $\mu_i < \eta_i$. If $\mu_i > \eta_i$, then substituting (4.10) into (3.9), we derive

$$A'_0\,C_4\,p_i^{\mu_i-\eta_i} = B'_0\,D_4. \tag{4.11}$$

Due to (4.10), (4.9) and (4.2) the left hand side of (4.11) is zero modulo p_i, while the right hand side of (4.11) is nonzero modulo p_i, which is contradictory.

Similarly, if $\mu_i < \eta_i$, substituting (4.10) into (3.9), we derive

$$A'_0\,C_4 = B'_0\,D_4\,p_i^{\eta_i-\mu_i}. \tag{4.12}$$

In this case the left hand side of (4.12) is nonzero modulo p_i, while the right hand side of (4.12) is zero modulo p_i, which is also contradictory.

The contradictions obtained prove that $\mu_i = \eta_i$. The equalities $\mu_i = \alpha_i$ and $\eta_i = \alpha_i$ are immediate from $\mu_i = \eta_i$ due to (4.4). The lemma 4.4 is proved. □

Lemma 4.5. *For any solution of the Diophantine equations* (3.9)*,* (3.12)*,* (3.13)*, and* (3.14) *with $A_0 > 0$ and $B_0 > 0$ if $\nu_i > 0$ and $\tau_i > 0$ in* (4.3)*, then $\nu_i = \tau_i = \beta_i$.*

The lemma 4.5 is analogous to the lemma 4.4. Its proof is similar to the above proof of the lemma 4.4.

Now, relying on the lemmas 4.4 and 4.5, we define the following integer numbers:

$$a_p = \prod_{\eta_i=0} p_i^{\alpha_i}, \qquad b_p = \prod_{\mu_i=0} p_i^{\alpha_i}, \qquad c_p = \prod_{\substack{\mu_i>0\\ \eta_i>0}} p_i^{\alpha_i}, \tag{4.13}$$

$$a_q = \prod_{\tau_i=0} q_i^{\beta_i}, \qquad b_q = \prod_{\nu_i=0} q_i^{\beta_i}, \qquad c_q = \prod_{\substack{\nu_i>0\\ \tau_i>0}} q_i^{\beta_i}. \tag{4.14}$$

Note that the numbers (4.13) and (4.14) are pairwise mutually coprime, i. e.

$$\begin{aligned}
&\gcd(a_p, b_p) = 1, &&\gcd(a_p, c_p) = 1, &&\gcd(a_p, a_q) = 1,\\
&\gcd(a_p, b_q) = 1, &&\gcd(a_p, c_q) = 1, &&\gcd(b_p, c_p) = 1,\\
&\gcd(b_p, a_q) = 1, &&\gcd(b_p, b_q) = 1, &&\gcd(b_p, c_q) = 1,\\
&\gcd(c_p, a_q) = 1, &&\gcd(c_p, b_q) = 1, &&\gcd(c_p, c_q) = 1,\\
&\gcd(a_q, b_q) = 1, &&\gcd(a_q, c_q) = 1, &&\gcd(b_q, c_q) = 1.
\end{aligned} \tag{4.15}$$

If $\mu_i = 0$, then $\eta_i = 2\,\alpha_i$, and if $\eta_i = 0$, then $\mu_i = 2\,\alpha_i$. Similarly, if $\nu_i = 0$, then $\tau_i = 2\,\beta_i$ and if $\tau_i = 0$, then $\nu_i = 2\,\beta_i$. These implications are derived from (4.4). The lemmas 4.4 and 4.5 say that if $\mu_i > 0$ and $\eta_i > 0$, then $\mu_i = \eta_i = \alpha_i$, and if $\nu_i > 0$ and $\tau_i > 0$, then $\nu_i = \tau_i = \beta_i$. As a result from (4.13) and (4.14) we derive

$$\begin{aligned}
&A_0 = a_p^2\,c_p\,a_q^2\,c_q, &&B_0 = b_p^2\,c_p\,b_q^2\,c_q,\\
&p = a_p\,b_p\,c_p, &&q = a_q\,b_q\,c_q.
\end{aligned} \tag{4.16}$$

The result expressed by the formulas (4.15) and (4.16) is rather important. For this reason it is formulated as a lemma.

Lemma 4.6. *For any solution of the Diophantine equations (3.9), (3.12), (3.13), and (3.14) with $A_0 > 0$ and $B_0 > 0$ there are six positive pairwise mutually coprime integer numbers a_p, b_p, c_p, a_q, b_q, c_q such that the numbers A_0, B_0, p, q are expressed through them by means of the formulas (4.16).*

Let's substitute (4.16) into the equation (4.10). As a result we obtain the following equation with respect to the numbers a_p, b_p, c_p, a_q, b_q, c_q:

$$\begin{gathered}
a_p^{10}\,a_q^{10}\,c_p^4\,c_q^4+6\,a_p^6\,a_q^{10}\,c_p^2\,c_q^6\,b_q^4-a_p^8\,a_q^8\,c_p^4\,c_q^4\,b_q^2\,b_p^2-\\
-2\,a_p^{10}\,a_q^6\,c_p^6\,c_q^2\,b_p^4+4\,a_p^6\,a_q^6\,b_p^4\,c_p^4\,b_q^4\,c_q^4+a_p^{10}\,a_q^2\,b_p^8\,c_p^8+\\
+a_p^2\,a_q^{10}\,b_q^8\,c_q^8+10\,a_p^4\,a_q^8\,b_p^2\,c_p^2\,b_q^6\,c_q^6-14\,a_p^8\,a_q^4\,b_p^6\,c_p^6\,b_q^2\,c_q^2=\\
=b_p^{10}\,b_q^{10}\,c_p^4\,c_q^4+6\,b_p^{10}\,b_q^6\,c_p^6\,c_q^2\,a_p^4-b_p^8\,b_q^8\,c_p^4\,c_q^4\,a_q^2\,a_p^2-\\
-2\,b_p^6\,b_q^{10}\,c_p^2\,c_q^6\,a_q^4+4\,b_p^6\,b_q^6\,a_p^4\,c_p^4\,a_q^4\,c_q^4+b_p^2\,b_q^{10}\,a_q^8\,c_q^8+\\
+b_p^{10}\,b_q^2\,a_p^8\,c_p^8+10\,b_p^8\,b_q^4\,a_p^6\,c_p^6\,a_q^2\,c_q^2-14\,b_p^4\,b_q^8\,a_p^2\,c_p^2\,a_q^6\,c_q^6.
\end{gathered}\tag{4.17}$$

Lemma 4.7. *For a given pair of positive coprime integer numbers $p \neq q$ the system of Diophantine equations (3.9), (3.12), (3.13), and (3.14) is resolvable if and only if there are six positive integer numbers a_p, b_p, c_p, a_q, b_q, c_q obeying the equation (4.17), obeying the coprimality conditions (4.15), and such that p and q are expressed through them by means of the formulas $p = a_p\,b_p\,c_p$ and $q = a_q\,b_q\,c_q$.*

The lemma 4.7 follows from the lemmas 4.3 and 4.6 due to the calculations in deriving the equation (4.17) from the equation (4.8). Combining the lemmas 3.1 and 4.7, now we derive the following structural theorem for the solutions of the Diophantine equation (1.2).

Theorem 4.1. *For a given pair of positive coprime integer numbers $p \neq q$ the Diophantine equation (1.2) is resolvable with respect to the variable t if and only if there are six positive integer numbers a_p, b_p, c_p, a_q, b_q, c_q obeying the equation (4.17), obeying the coprimality conditions (4.15), and such that p and q are expressed through them by means of the formulas $p = a_p\,b_p\,c_p$ and $q = a_q\,b_q\,c_q$. Under these conditions the equation (1.2) has at least two solutions given by the formulas*

$$t = a_p^2\,c_p\,a_q^2\,c_q, \qquad\qquad t = -a_p^2\,c_p\,a_q^2\,c_q. \tag{4.18}$$

The structural theorem 4.1 is the main result of this paper. The formulas (4.18) in this theorem are immediate from (4.16).

5. Some applications of the structural theorem.

Let's choose $q = 1$ and assume that p is some prime number. Then p and q are coprime, i.e. the relationship (4.2) is fulfilled. Applying the formula $q = a_q\,b_q\,c_q$ from the theorem 4.1 to this case, we get

$$a_q = 1 \qquad\qquad b_q = 1 \qquad\qquad c_q = 1. \tag{5.1}$$

Similarly, applying the formula $p=a_p\,b_p\,c_p$ from the theorem 4.1 and taking into account that p is prime, we derive three options

$$\begin{aligned} &a_p=p, && b_p=1, && c_p=1; \quad (5.2)\\ &a_p=1, && b_p=p, && c_p=1; \quad (5.3)\\ &a_p=1, && b_p=1, && c_p=p. \quad (5.4)\end{aligned}$$

Substituting (5.1) and (5.2) into (4.17), we obtain the following equation for p:

$$16\,p^2-16\,p^8=0. \tag{5.5}$$

The left hand side of the equation (5.5) factorizes as

$$-16\,p^2\,(p-1)\,(p+1)\,(p^2+p+1)\,(p^2-p+1)=0. \tag{5.6}$$

Therefore the only integer solutions of the equation (5.6) are

$$p=-1, \qquad\qquad p=1. \tag{5.7}$$

The first of them is negative. The second one is positive, but $p=1$ contradicts the inequality $p\neq q$ in the theorem 4.1 since $q=1$ in our present case.

The second option is given by the formulas (5.3). Substituting (5.1) and (5.3) into (4.17), we obtain another equation for p:

$$-8\,p^{10}-8\,p^8-16\,p^6+16\,p^4+8\,p^2+8=0. \tag{5.8}$$

The left hand side of the equation (5.8) factorizes as follows:

$$-8\,(p-1)\,(p+1)\,(p^8+2\,p^6+4\,p^4+2\,p^2+1)=0. \tag{5.9}$$

Again the only integer solutions of the equation (5.9) are given by the formulas (5.7). The first of these two solutions is negative, while the second contradicts the inequality $p\neq q$ in the theorem 4.1.

And finally, the third option is given by the formulas (5.4). Substituting (5.1) and (5.4) into (4.17), we obtain the third equation for p:

$$-32\,p^6+32\,p^2=0. \tag{5.10}$$

The left hand side of the equation (5.10) factorizes as follows:

$$-32\,p^2\,(p-1)\,(p+1)\,(p^2+1)=0. \tag{5.11}$$

The integer solutions of the equation (5.11) are given by the formula (5.7). The first of them is negative, while the second one contradicts the inequality $p\neq q$. Thus, none of the equations (5.6), (5.9), and (5.11) has a solution suitable for the theorem 4.1. Applying this theorem, we can formulate the following result.

Theorem 5.1. *For $q=1$ and for any positive prime integer p the polynomial Diophantine equation (1.2) has no integer solutions.*

Exchanging p and q in the theorem 5.1, we obtain another theorem.

Theorem 5.2. *For $p = 1$ and for any positive prime integer q the polynomial Diophantine equation* (1.2) *has no integer solutions.*

The proof of the theorem 5.2 is similar to the above proof of the theorem 5.1. The theorems 5.1 and 5.2 prove the theorem 1.1 in two special cases where p is prime and $q = 1$ and where q is prime and $p = 1$. Other examples of applying the structural theorem 4.1 will be given in a separate paper.

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