ON THE EQUIVALENCE OF CUBOID EQUATIONS AND THEIR FACTOR EQUATIONS.

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Abstract. An Euler cuboid is a rectangular parallelepiped with integer edges and integer face diagonals. An Euler cuboid is called perfect if its space diagonal is also integer. Some Euler cuboids are already discovered. As for perfect cuboids, none of them is currently known and their non-existence is not yet proved. Euler cuboids and perfect cuboids are described by certain systems of Diophantine equations. These equations possess an intrinsic S_3 symmetry. Recently they were factorized with respect to this S_3 symmetry and the factor equations were derived. In the present paper the factor equations are shown to be equivalent to the original cuboid equations regarding the search for perfect cuboids and in selecting Euler cuboids.

1. Introduction.

The search for perfect cuboids has the long history. The reader can follow this history since 1719 in the references [1–44]. In order to write the cuboidal Diophantine equations we use the following polynomials:

$$\begin{aligned} &p_0 = x_1^2 + x_2^2 + x_3^2 - L^2, && p_1 = x_2^2 + x_3^2 - d_1^2,\\ &p_2 = x_3^2 + x_1^2 - d_2^2, && p_3 = x_1^2 + x_2^2 - d_3^2. \end{aligned} \tag{1.1}$$

Here x_1, x_2, x_3 are edges of a cuboid, d_1, d_2, d_3 are its face diagonals, and L is its space diagonal. An Euler cuboid is described by a system of three Diophantine equations. In terms of the polynomials (1.1) these equations are written as

$$p_1 = 0, \qquad p_2 = 0, \qquad p_3 = 0. \tag{1.2}$$

In the case of a perfect cuboid the number of equations is greater by one, i. e. instead of the equations (1.2) we write the following system of four equations:

$$p_0 = 0, \qquad p_1 = 0, \qquad p_2 = 0, \qquad p_3 = 0. \tag{1.3}$$

The permutation group S_3 acts upon the cuboid variables x_1, x_2, x_3, d_1, d_2, d_3, and L according to the rules expressed by the formulas

$$\sigma(x_i) = x_{\sigma i}, \qquad \sigma(d_i) = d_{\sigma i}, \qquad \sigma(L) = L. \tag{1.4}$$

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The variables x_1, x_2, x_3 and d_1, d_2, d_3 are usually arranged into a matrix:

$$M = \begin{Vmatrix} x_1 & x_2 & x_3 \\ d_1 & d_2 & d_3 \end{Vmatrix}. \tag{1.5}$$

The rules (1.4) means that S_3 acts upon the matrix (1.5) by permuting its columns. Applying the rules (1.4) to the polynomials (1.1), we derive

$$\sigma(p_i) = p_{\sigma i}, \qquad\qquad \sigma(p_0) = p_0. \tag{1.6}$$

The polynomials p_0, p_1, p_2, p_3 in (1.1) are treated as elements of the polynomial ring $\mathbb{Q}[x_1, x_2, x_3, d_1, d_2, d_3, L]$. For the sake of brevity we denote

$$\mathbb{Q}[x_1, x_2, x_3, d_1, d_2, d_3, L] = \mathbb{Q}[M, L], \tag{1.7}$$

where M is the matrix given by the formula (1.5).

Definition 1.1. A polynomial $p \in \mathbb{Q}[M, L]$ is called multisymmetric if it is invariant with respect to the action (1.4) of the group S_3.

Multisymmetric polynomials constitute a subring in the ring (1.7). We denote this subring through $\operatorname{Sym}\mathbb{Q}[M, L]$. The formulas (1.6) show that the polynomial p_0 belongs to the subring $\operatorname{Sym}\mathbb{Q}[M, L]$, i. e. it is multisymmetric, while the polynomials p_1, p_2, p_3 are not multisymmetric. Nevertheless, the system of equations (1.2) in whole is invariant with respect to the action of the group S_3. The same is true for the system of equations (1.3).

The polynomials p_1, p_2, p_3 generate an ideal in the ring $\mathbb{Q}[M, L]$. It is natural to call it the *cuboid ideal* and denote this ideal through

$$I_{\mathrm{C}} = \bigl\langle p_1, p_2, p_3 \bigr\rangle. \tag{1.8}$$

Similarly, one can define the *perfect cuboid ideal*

$$I_{\mathrm{PC}} = \bigl\langle p_0, p_1, p_2, p_3 \bigr\rangle. \tag{1.9}$$

Each polynomial equation $p = 0$ with $p \in I_{\mathrm{C}}$ follows from the equations (1.2). Therefore such an equation is called a *cuboid equation*. Similarly, each polynomial equation $p = 0$ with $p \in I_{\mathrm{PC}}$ follows from the equations (1.3). Such an equation is called a *perfect cuboid equation*.

The symmetry approach to the equations (1.2) and (1.3) initiated in [45] leads to studying the following ideals in the ring of multisymmetric polynomials:

$$I_{\mathrm{C\_sym}} = I_{\mathrm{C}} \cap \operatorname{Sym}\mathbb{Q}[M, L], \qquad\qquad I_{\mathrm{PC\_sym}} = I_{\mathrm{PC}} \cap \operatorname{Sym}\mathbb{Q}[M, L]. \tag{1.10}$$

Definition 1.2. A polynomial equation of the form $p = 0$ with $p \in I_{\mathrm{C\_sym}}$ or with $p \in I_{\mathrm{PC\_sym}}$ is called an S_3 *factor equation* for the Euler cuboid equations (1.2) or for the perfect cuboid equations (1.3) respectively.

The ideal $I_{\mathrm{PC\_sym}}$ from (1.10) was studied in [46] (there it was denoted through I_{sym}). The polynomial p_0 used as a generator in (1.9) is multisymmetric in the sense of the definition 1.1. Therefore it belongs to the ideal $I_{\text{PC\_sym}}$. In [46] this polynomial was denoted through $\tilde{p}_1$:

$$\tilde{p}_1 = p_0 = x_1^2 + x_2^2 + x_3^2 - L^2. \tag{1.11}$$

In addition to (1.11) in [46] the following seven polynomials were considered:

$$\begin{aligned}\tilde{p}_2 = p_1 + p_2 + p_3 = (x_2^2 + x_3^2 - d_1^2) +\\ +\,(x_3^2 + x_1^2 - d_2^2) + (x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.12}$$

$$\begin{aligned}\tilde{p}_3 = d_1\,p_1 + d_2\,p_2 + d_3\,p_3 = d_1\,(x_2^2 + x_3^2 - d_1^2) +\\ +\,d_2\,(x_3^2 + x_1^2 - d_2^2) + d_3\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.13}$$

$$\begin{aligned}\tilde{p}_4 = x_1\,p_1 + x_2\,p_2 + x_3\,p_3 = x_1\,(x_2^2 + x_3^2 - d_1^2) +\\ +\,x_2\,(x_3^2 + x_1^2 - d_2^2) + x_3\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.14}$$

$$\begin{aligned}\tilde{p}_5 = x_1\,d_1\,p_1 + x_2\,d_2\,p_2 + x_3\,d_3\,p_3 = x_1\,d_1\,(x_2^2 + x_3^2 - d_1^2) +\\ +\,x_2\,d_2\,(x_3^2 + x_1^2 - d_2^2) + x_3\,d_3\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.15}$$

$$\begin{aligned}\tilde{p}_6 = x_1^2\,p_1 + x_2^2\,p_2 + x_3^2\,p_3 = x_1^2\,(x_2^2 + x_3^2 - d_1^2) +\\ +\,x_2^2\,(x_3^2 + x_1^2 - d_2^2) + x_3^2\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.16}$$

$$\begin{aligned}\tilde{p}_7 = d_1^2\,p_1 + d_2^2\,p_2 + d_3^2\,p_3 = d_1^2\,(x_2^2 + x_3^2 - d_1^2) +\\ +\,d_2^2\,(x_3^2 + x_1^2 - d_2^2) + d_3^2\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.17}$$

$$\begin{aligned}\tilde{p}_8 = x_1^2\,d_1^2\,p_1 + x_2^2\,d_2^2\,p_2 + x_3^2\,d_3^2\,p_3 = x_1^2\,d_1^2\,(x_2^2 + x_3^2 - d_1^2) +\\ +\,x_2^2\,d_2^2\,(x_3^2 + x_1^2 - d_2^2) + x_3^2\,d_3^2\,(x_1^2 + x_2^2 - d_3^2).\end{aligned} \tag{1.18}$$

Theorem 1.1. *The ideal $I_{\text{PC\_sym}}$ from (1.10) is finitely generated within the ring* $\operatorname{Sym}\mathbb{Q}[M, L]$. *Eight polynomials (1.11), (1.12), (1.13), (1.14), (1.15), (1.16), (1.17), and (1.18) belong to the ideal $I_{\text{PC\_sym}}$ and constitute a basis of this ideal.*

The theorem 1.1 was proved in [46]. The ideal $I_{\text{C\_sym}}$ in (1.8) is similar to the ideal $I_{\text{PC\_sym}}$. There is the following theorem describing this ideal.

Theorem 1.2. *The ideal $I_{\text{C\_sym}}$ from (1.10) is finitely generated within the ring* $\operatorname{Sym}\mathbb{Q}[M, L]$. *Seven polynomials (1.12), (1.13), (1.14), (1.15), (1.16), (1.17), and (1.18) belong to the ideal $I_{\text{C\_sym}}$ and constitute a basis of this ideal.*

The theorem 1.2 can be proved in a way similar to the proof of the theorem 1.1 in [46]. I do not give the proof of the theorem 1.2 here for the sake of brevity.

Relying on the theorem 1.2 and using the polynomials (1.12), (1.13), (1.14), (1.15), (1.16), (1.17), (1.18), we write the system of seven factor equations

$$\begin{aligned} & &\tilde{p}_2 &= 0, &\tilde{p}_3 &= 0, &\tilde{p}_4 &= 0,\\ \tilde{p}_5 &= 0, &\tilde{p}_6 &= 0, &\tilde{p}_7 &= 0, &\tilde{p}_8 &= 0.\end{aligned} \tag{1.19}$$

The factor equations (1.19) correspond to the case of Euler cuboids. Similarly, in the case of perfect cuboids, relying on the theorem 1.1 and using the polynomials given by the formulas (1.11), (1.12), (1.13), (1.14), (1.15), (1.16), (1.17), (1.18), we write the following system of eight factor equations:

$$\begin{aligned} &\tilde{p}_1=0, &&\tilde{p}_2=0, &&\tilde{p}_3=0, &&\tilde{p}_4=0,\\ &\tilde{p}_5=0, &&\tilde{p}_6=0, &&\tilde{p}_7=0, &&\tilde{p}_8=0. \end{aligned}\tag{1.20}$$

The structure of the polynomials $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{p}_3$, $\tilde{p}_4$, $\tilde{p}_5$, $\tilde{p}_6$, $\tilde{p}_7$, $\tilde{p}_8$ in (1.11), (1.12), (1.13), (1.14), (1.15), (1.16), (1.17), (1.18) is so that each solution of the equations (1.2) is a solution for the equations (1.19). Similarly, each solution of the equations (1.3) is a solution for the equations (1.20). The main goal of this paper is to prove converse propositions. They are given by the following two theorems.

Theorem 1.3. *Each integer or rational solution of the factor equations* (1.19) *such that $x_1>0$, $x_2>0$, $x_3>0$, $d_1>0$, $d_2>0$, and $d_3>0$ is an integer or rational solution for the equations* (1.2).

Theorem 1.4. *Each integer or rational solution of the factor equations* (1.20) *such that $x_1>0$, $x_2>0$, $x_3>0$, $d_1>0$, $d_2>0$, and $d_3>0$ is an integer or rational solution for the equations* (1.3).

2. The analysis of the factor equations.

Let's consider the factor equations (1.19) associated with Euler cuboids. Due to (1.12), (1.13), (1.14), (1.15), (1.16), (1.17), and (1.18) the factor equations (1.19) can be united into a single matrix equation

$$\begin{Vmatrix} 1 & 1 & 1\\ d_1 & d_2 & d_3\\ x_1 & x_2 & x_3\\ x_1\,d_1 & x_2\,d_2 & x_3\,d_3\\ x_1^2 & x_2^2 & x_3^2\\ d_1^2 & d_2^2 & d_3^2\\ x_1^2\,d_1^2 & x_2^2\,d_2^2 & x_3^2\,d_3^2 \end{Vmatrix}\cdot\begin{Vmatrix} p_1\\ p_2\\ p_3\end{Vmatrix}=\begin{Vmatrix}0\\0\\0\\0\\0\\0\\0\end{Vmatrix}.\tag{2.1}$$

In order to study the equations (2.1) we denote through N the transposed matrix

$$N=\begin{Vmatrix} 1 & d_1 & x_1 & x_1\,d_1 & x_1^2 & d_1^2 & x_1^2\,d_1^2\\ 1 & d_2 & x_2 & x_2\,d_2 & x_2^2 & d_2^2 & x_2^2\,d_2^2\\ 1 & d_3 & x_3 & x_3\,d_3 & x_3^2 & d_3^2 & x_3^2\,d_3^2 \end{Vmatrix}.\tag{2.2}$$

If we have a solution of the equation (2.1) which is not a solution for the initial system of cuboid equations (1.2), then the equations (1.2) should not be fulfilled simultaneously. Therefore we have the vectorial inequality

$$\begin{Vmatrix} p_1\\ p_2\\ p_3\end{Vmatrix}\neq 0.\tag{2.3}$$

Applying (2.3) to (2.1), we derive that the columns of the matrix in (2.1) are linearly dependent. Then the rows of N in (2.2) are also linearly dependent, i. e.

$$\operatorname{rank} N \leqslant 2. \tag{2.4}$$

The condition (2.4) leads to several special cases which are considered below one by one. In addition to N we define the following two matrices:

$$N_1 = \begin{Vmatrix} 1 & d_1 \\ 1 & d_2 \\ 1 & d_3 \end{Vmatrix}, \qquad\qquad N_2 = \begin{Vmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \end{Vmatrix}. \tag{2.5}$$

The matrices N_1 and N_2 in (2.5) are submatrices of the matrix N.

3. The case $\operatorname{rank} N = 1$.

The first column of the matrix (2.2) is nonzero. Therefore $\operatorname{rank} N > 0$. Now we consider the case where $\operatorname{rank} N = 1$. In this case each column of the matrix N is proportional to its first column. In particular, this yields

$$\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \alpha \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}, \qquad\qquad \begin{Vmatrix} d_1 \\ d_2 \\ d_3 \end{Vmatrix} = \beta \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}. \tag{3.1}$$

The equations (3.1) lead to the equalities

$$x_1 = x_2 = x_3, \qquad\qquad d_1 = d_2 = d_3. \tag{3.2}$$

Applying (3.2) to the formulas (1.1), we derive

$$p_1 = p_2 = p_3. \tag{3.3}$$

Then we substitute (3.3) into (1.12). As a result we get

$$\tilde{p}_2 = 3\,p_1 = 3\,p_2 = 3\,p_3. \tag{3.4}$$

The relationships (3.4) mean that if the equations (1.19) are fulfilled, then in the case of $\operatorname{rank} N = 1$ the equations (1.2) are also fulfilled.

Theorem 3.1. *Each solution of the equations* (1.19) *corresponding to the case* $\operatorname{rank} N = 1$ *is a solution for the equations* (1.2).

Theorem 3.2. *Each solution of the equations* (1.20) *corresponding to the case* $\operatorname{rank} N = 1$ *is a solution for the equations* (1.3).

Due to (1.11) the equation $p_0 = 0$ in (1.3) coincides with the equation $\tilde{p}_1 = 0$ in (1.20). For this reason the theorem 3.2 is immediate from the theorem 3.1.

4. The case $\operatorname{rank} N_1 = 2$ and $\operatorname{rank} N_2 = 1$.

The condition $\operatorname{rank} N_2 = 1$ for the matrix N_2 in (2.5) means that the third column of the matrix (2.2) is proportional to the first column of this matrix. The condition $\operatorname{rank} N_1 = 2$ for the matrix N_1 in (2.5) means that the first and the second columns of the matrix (2.2) are linearly independent. Other columns are expressed as linear combinations of these two columns. As a result we can write

$$\begin{Vmatrix} x_1\\ x_2\\ x_3 \end{Vmatrix}=\alpha\cdot\begin{Vmatrix} 1\\ 1\\ 1 \end{Vmatrix},\qquad\qquad \begin{Vmatrix} d_1^2\\ d_2^2\\ d_3^2 \end{Vmatrix}=\beta\cdot\begin{Vmatrix} d_1\\ d_2\\ d_3 \end{Vmatrix}+\gamma\cdot\begin{Vmatrix} 1\\ 1\\ 1 \end{Vmatrix}. \tag{4.1}$$

It is easy to see that the conditions (4.1) are sufficient for to provide the condition $\operatorname{rank} N = 2$, which is in agreement with (2.4).

The second equality in (4.1) is very important. It means that d_1, d_2, and d_3, are roots of the following quadratic equation:

$$d^2-\beta\,d-\gamma=0. \tag{4.2}$$

The quadratic equation (4.2) has at most two roots. Let's denote them s_1 and s_2. Then we have the following subcases derived from $\operatorname{rank} N_1 = 2$ and $\operatorname{rank} N_2 = 1$:

$$\begin{Vmatrix} d_1\\ d_2\\ d_3 \end{Vmatrix}=\begin{Vmatrix} s_1\\ s_1\\ s_2 \end{Vmatrix},\qquad \begin{Vmatrix} d_1\\ d_2\\ d_3 \end{Vmatrix}=\begin{Vmatrix} s_1\\ s_2\\ s_1 \end{Vmatrix},\qquad \begin{Vmatrix} d_1\\ d_2\\ d_3 \end{Vmatrix}=\begin{Vmatrix} s_2\\ s_1\\ s_1 \end{Vmatrix}. \tag{4.3}$$

The numbers s_1 and s_2 in the formulas (4.3) are arbitrary two numbers not coinciding with each other: $s_1 \neq s_2$. They are integer numbers in the case of integer solutions and they are rational numbers in the case of rational solutions.

The three cases in (4.3) are similar to each other. Without loss of generality we can consider only one of them, e. g. the first one. Then from (4.3) we derive

$$(s_1-s_2)\cdot\begin{Vmatrix} 1\\ 1\\ 0 \end{Vmatrix}=\begin{Vmatrix} s_1\\ s_1\\ s_2 \end{Vmatrix}-s_2\cdot\begin{Vmatrix} 1\\ 1\\ 1 \end{Vmatrix}, \tag{4.4}$$

$$(s_2-s_1)\cdot\begin{Vmatrix} 0\\ 0\\ 1 \end{Vmatrix}=\begin{Vmatrix} s_1\\ s_1\\ s_2 \end{Vmatrix}-s_1\cdot\begin{Vmatrix} 1\\ 1\\ 1 \end{Vmatrix}. \tag{4.5}$$

Due to the relationships (4.4) and (4.5) the matrix equation (2.1) reduces to

$$\begin{Vmatrix} 1 & 1 & 0\\ 0 & 0 & 1 \end{Vmatrix}\cdot\begin{Vmatrix} p_1\\ p_2\\ p_3 \end{Vmatrix}=\begin{Vmatrix} 0\\ 0 \end{Vmatrix}. \tag{4.6}$$

The matrix equality (4.6) means that instead of the seven equations (1.19) we have two equations $p_1 + p_2 = 0$ and $p_3 = 0$. Substituting $x_1 = x_2 = x_3 = \alpha$, $d_1 = d_2 = s_1$, and $d_3 = s_2$ into these two equations, we derive

$$s_1^2-2\,\alpha^2=0,\qquad\qquad s_2^2-2\,\alpha^2=0. \tag{4.7}$$

The equations (4.7) can be written in the following way:

$$|s_1|=\sqrt{2}\,|\alpha|,\qquad\qquad |s_2|=\sqrt{2}\,|\alpha|. \tag{4.8}$$

Now it is easy to see that the equations (4.8) can be satisfied by three integer or rational numbers s_1, s_2, and α if and only if all of them are zero. Substituting $s_1 = s_2 = \alpha = 0$ into (4.1) and (4.3), we get

$$x_1 = x_2 = x_3 = 0, \qquad\qquad d_1 = d_2 = d_3 = 0. \tag{4.9}$$

The equalities (4.9) contradict the condition $\operatorname{rank} N_1 = 2$ for the matrix N_1 in (2.5). This contradiction yields the following two theorems.

Theorem 4.1. *The factor equations* (1.19)*, as well as the original equations* (1.2)*, have no integer or rational solution in the case of* $\operatorname{rank} N_1 = 2$ *and* $\operatorname{rank} N_2 = 1$*.*

Theorem 4.2. *The factor equations* (1.20)*, as well as the original equations* (1.3)*, have no integer or rational solution in the case of* $\operatorname{rank} N_1 = 2$ *and* $\operatorname{rank} N_2 = 1$*.*

5. The case $\operatorname{rank} N_1 = 1$ and $\operatorname{rank} N_2 = 2$.

The condition $\operatorname{rank} N_1 = 1$ for the matrix N_1 in (2.5) means that the second column of the matrix (2.2) is proportional to the first column of this matrix. The condition $\operatorname{rank} N_2 = 2$ for the matrix N_2 in (2.5) means that the first and the third columns of the matrix (2.2) are linearly independent. Other columns are expressed as linear combinations of these two columns. As a result we can write the relationships similar to the relationships (4.1):

$$\begin{Vmatrix} d_1 \\ d_2 \\ d_3 \end{Vmatrix} = \delta \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}, \qquad\qquad \begin{Vmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{Vmatrix} = \varepsilon \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} + \zeta \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}. \tag{5.1}$$

The conditions (5.1) are sufficient for to provide the condition $\operatorname{rank} N = 2$.

Like in the case of (4.1), the second condition (5.1) mean that x_1, x_2, and x_3 are roots of the quadratic equation similar to (4.2):

$$x^2 - \varepsilon\, x - \zeta = 0. \tag{5.2}$$

The quadratic equation (5.2) has at most two roots. Let's denote them r_1 and r_2. Then we have the following subcases derived from $\operatorname{rank} N_1 = 1$ and $\operatorname{rank} N_2 = 2$:

$$\begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} r_1 \\ r_1 \\ r_2 \end{Vmatrix}, \qquad \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} r_1 \\ r_2 \\ r_1 \end{Vmatrix}, \qquad \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} = \begin{Vmatrix} r_2 \\ r_1 \\ r_1 \end{Vmatrix}. \tag{5.3}$$

The numbers r_1 and r_2 in the formulas (5.3) are arbitrary two integer or rational numbers not coinciding with each other: $r_1 \neq r_2$.

The three cases in (5.3) are similar to each other. Without loss of generality we can consider only one of them, e.g. the first one. Then from (5.3) we derive

$$(r_1 - r_2) \cdot \begin{Vmatrix} 1 \\ 1 \\ 0 \end{Vmatrix} = \begin{Vmatrix} r_1 \\ r_1 \\ r_2 \end{Vmatrix} - r_2 \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}, \tag{5.4}$$

$$(r_2 - r_1) \cdot \begin{Vmatrix} 0 \\ 0 \\ 1 \end{Vmatrix} = \begin{Vmatrix} r_1 \\ r_1 \\ r_2 \end{Vmatrix} - r_1 \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}. \tag{5.5}$$

Due to the relationships (5.4) and (5.5) the matrix equation (2.1) reduces to

$$\begin{Vmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix} \cdot \begin{Vmatrix} p_1 \\ p_2 \\ p_3 \end{Vmatrix} = \begin{Vmatrix} 0 \\ 0 \end{Vmatrix}. \tag{5.6}$$

The matrix equality (5.6) means that instead of the seven equations (1.19) we have two equations $p_1+p_2 = 0$ and $p_3 = 0$. Substituting $d_1 = d_2 = d_3 = \delta$, $x_1 = x_2 = r_1$, and $x_3 = r_2$ into these two equations, we derive

$$r_1^2 + r_2^2 - \delta^2 = 0, \qquad\qquad 2\,r_1^2 - \delta^2 = 0. \tag{5.7}$$

The second equation (5.7) can be written in the following form:

$$|\delta| = \sqrt{2}\,|r_1|. \tag{5.8}$$

The equation (5.8) can be satisfied by two integer or rational numbers r_1 and δ if and only if both of them are zero. Substituting $r_1 = \delta = 0$ into the first equation (5.7), we get $r_2 = 0$. Substituting $r_1 = r_2 = \delta = 0$ into (5.1) and (5.3), we get

$$x_1 = x_2 = x_3 = 0, \qquad\qquad d_1 = d_2 = d_3 = 0. \tag{5.9}$$

The equalities (5.9) contradict the condition $\operatorname{rank} N_2 = 2$ for the matrix N_2 in (2.5). This contradiction yields the following two theorems.

Theorem 5.1. *The factor equations* (1.19)*, as well as the original equations* (1.2)*, have no integer or rational solution in the case of* $\operatorname{rank} N_1 = 1$ *and* $\operatorname{rank} N_2 = 2$*.*

Theorem 5.2. *The factor equations* (1.20)*, as well as the original equations* (1.3)*, have no integer or rational solution in the case of* $\operatorname{rank} N_1 = 1$ *and* $\operatorname{rank} N_2 = 2$*.*

6. The case $\operatorname{rank} N_1 = 2$ and $\operatorname{rank} N_2 = 2$.

In this case the columns of both matrices N_1 and N_2 in (2.5) are linearly independent. Hence each column of the matrix (2.2) can be expressed as a linear combination of the first and the second columns of this matrix or as a linear combination of the first and the third columns of this matrix. In particular, we have

$$\begin{Vmatrix} d_1^2 \\ d_2^2 \\ d_3^2 \end{Vmatrix} = \beta \cdot \begin{Vmatrix} d_1 \\ d_2 \\ d_3 \end{Vmatrix} + \gamma \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}, \qquad \begin{Vmatrix} x_1^2 \\ x_2^2 \\ x_3^2 \end{Vmatrix} = \varepsilon \cdot \begin{Vmatrix} x_1 \\ x_2 \\ x_3 \end{Vmatrix} + \zeta \cdot \begin{Vmatrix} 1 \\ 1 \\ 1 \end{Vmatrix}. \tag{6.1}$$

The relationships (6.1) mean that x_1, x_2, x_3 and d_1, d_2, d_3 are roots of two quadratic equations coinciding with (5.2) and (4.2) respectively. As a result we distinguish three subcases (4.3) with $s_1 \neq s_2$ and three subcases (5.3) with $r_1 \neq r_2$. The first subcase (4.3) should be paired with the first subcase (5.3), the second subcase (4.3) should be paired with the second subcase (5.3), and the third subcase (4.3) should be paired with the third subcase (5.3). Otherwise we would have rank $N \geqslant 3$, which contradicts the condition (2.4).

Due to the pairing of subcases we have three subcases instead of nine ones, which are a priori possible. These three subcases are similar to each other. Therefore without loss of generality we can consider only one subcase, e. g. the following one:

$$\left\|\begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array}\right\| = \left\|\begin{array}{c} s_1 \\ s_1 \\ s_2 \end{array}\right\|, \qquad \left\|\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right\| = \left\|\begin{array}{c} r_1 \\ r_1 \\ r_2 \end{array}\right\|. \tag{6.2}$$

Here $s_1 \neq s_2$ and $r_1 \neq r_2$. The relationships (6.2) lead to the relationships (4.4), (4.5), (5.4), (5.5) and then to the equations (4.6) and (5.6). The matrix equations (4.6) and (5.6) mean that instead of the seven equations (1.19) we have two equations $p_1 + p_2 = 0$ and $p_3 = 0$. Substituting $d_1 = d_2 = s_1$, $d_3 = s_2$, $x_1 = x_2 = r_1$, and $x_3 = r_2$ into these two equations, we derive

$$r_1^2 + r_2^2 - s_1^2 = 0, \qquad 2\,r_1^2 - s_2^2 = 0. \tag{6.3}$$

The second equation (6.3) can be written in the following way:

$$\sqrt{2}\,|r_1| = |s_2|. \tag{6.4}$$

The equation (6.4) can be satisfied by two integer or rational numbers r_1 and s_2 if and only if both of them are zero. Substituting $r_1 = s_2 = 0$ into (6.3), we get $|r_2| = |s_1| = \theta$. Then the equalities (6.2) are written as

$$\left\|\begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array}\right\| = \pm \left\|\begin{array}{c} \theta \\ \theta \\ 0 \end{array}\right\|, \qquad \left\|\begin{array}{c} x_1 \\ x_2 \\ x_3 \end{array}\right\| = \pm \left\|\begin{array}{c} 0 \\ 0 \\ \theta \end{array}\right\|. \tag{6.5}$$

The equalities (6.5) lead to the equalities $x_1 = x_2 = 0$ and $d_3 = 0$. The latter ones contradict the inequalities in the theorems 1.3 and 1.4. Therefore we can conclude this section with the following two theorems.

Theorem 6.1. *The factor equations* (1.19)*, as well as the original equations* (1.2)*, have no integer or rational solutions such that* $x_1 > 0$*,* $x_2 > 0$*,* $x_3 > 0$*,* $d_1 > 0$*,* $d_2 > 0$*, and* $d_3 > 0$ *in the case of* $\operatorname{rank} N_1 = 2$ *and* $\operatorname{rank} N_2 = 2$*.*

Theorem 6.2. *The factor equations* (1.20)*, as well as the original equations* (1.3)*, have no integer or rational solutions such that* $x_1 > 0$*,* $x_2 > 0$*,* $x_3 > 0$*,* $d_1 > 0$*,* $d_2 > 0$*, and* $d_3 > 0$ *in the case of* $\operatorname{rank} N_1 = 2$ *and* $\operatorname{rank} N_2 = 2$*.*

7. The ultimate result and conclusions.

The four cases considered in sections 3, 4, 5, and 6 exhaust all options compatible with the inequality (2.4). For this reason the theorems 1.3 and 1.4 follow from the theorems 3.1, 4.1, 5.1, 6.1 and the theorems 3.2, 4.2, 5.2, 6.2 respectively. The theorems 1.3 and 1.4 constitute the main result of this paper. The theorem 1.4

means that the factor equations (1.20) are equally admissible for seeking perfect cuboids or for proving their non-existence as the original equations (1.3). As for the factor equations (1.19), due to the theorem 1.3 they are equally admissible for selecting Euler cuboids as the original equations (1.2).

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