A BIQUADRATIC DIOPHANTINE EQUATION ASSOCIATED WITH PERFECT CUBOIDS.

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Abstract. A perfect Euler cuboid is a rectangular parallelepiped with integer edges and integer face diagonals whose space diagonal is also integer. Such cuboids are not yet discovered and their non-existence is also not proved. Perfect Euler cuboids are described by a system of four Diophantine equation possessing a natural S_3 symmetry. Recently these equations were factorized with respect to this S_3 symmetry and the factor equations were derived. In the present paper the factor equations are transformed to E-form and then reduced to a single biquadratic equation.

1. Introduction.

Perfect cuboids are described by four polynomial equations

$$p_0=0, \qquad p_1=0, \qquad p_2=0, \qquad p_3=0, \tag{1.1}$$

where p_0, p_1, p_2, p_3 are the following polynomials of seven variables:

$$\begin{aligned} &p_0=x_1^2+x_2^2+x_3^2-L^2, && p_1=x_2^2+x_3^2-d_1^2,\\ &p_2=x_3^2+x_1^2-d_2^2, && p_3=x_1^2+x_2^2-d_3^2. \end{aligned} \tag{1.2}$$

Here x_1, x_2, x_3 are edges of a cuboid, d_1, d_2, d_3 are its face diagonals, and L is its space diagonal. Though the equations (1.1) with the polynomials (1.2) look very simple, the search for perfect cuboids has the long history since 1719 (see [1–44]).

Recently in [45] the symmetry approach to the equations (1.1) was initiated. It is based on the intrinsic S_3 symmetry of these equations. Let the permutation group S_3 act upon the variables x_1, x_2, x_3, d_1, d_2, d_3, L according to the rules

$$\sigma(x_i)=x_{\sigma i}, \qquad \sigma(d_i)=d_{\sigma i}, \qquad \sigma(L)=L. \tag{1.3}$$

If the variables x_1, x_2, x_3 and d_1, d_2, d_3 are arranged into the matrix

$$M=\begin{Vmatrix} x_1 & x_2 & x_3\\ d_1 & d_2 & d_3 \end{Vmatrix}, \tag{1.4}$$

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then, according to (1.3), the group S_3 acts upon the matrix (1.4) by permuting its columns. Applying the rules (1.3) to the polynomials (1.2), we derive

$$\sigma(p_i)=p_{\sigma i}, \qquad\qquad \sigma(p_0)=p_0. \tag{1.5}$$

The polynomials p_0, p_1, p_2, p_3 in (1.2) and in (1.5) are elements of the polynomial ring $\mathbb{Q}[x_1,x_2,x_3,d_1,d_2,d_3,L]$. For the sake of brevity we denote this ring through

$$\mathbb{Q}[M,L]=\mathbb{Q}[x_1,x_2,x_3,d_1,d_2,d_3,L], \tag{1.6}$$

where M is the matrix given by the formula (1.4).

Definition 1.1. A polynomial $p\in\mathbb{Q}[M,L]$ is called multisymmetric if it is invariant with respect to the action (1.3) of the group S_3.

General multisymmetric polynomials, which are also known as vector symmetric polynomials, diagonally symmetric polynomials, McMahon polynomials etc, were initially studied in [46–52] (see also later publications [53–66]).

Multisymmetric polynomials constitute a subring in the ring (1.6). We denote this subring through $\operatorname{Sym}\mathbb{Q}[M,L]$. The formulas (1.5) show that the polynomial p_0 belongs to the subring $\operatorname{Sym}\mathbb{Q}[M,L]$, i. e. it is multisymmetric, while the polynomials p_1, p_2, p_3 are not multisymmetric. Nevertheless, the system of equations (1.1) in whole is invariant with respect to the action of the group S_3.

The polynomials p_0, p_1, p_2, p_3 generate an ideal in the ring $\mathbb{Q}[M,L]$. In [67] this ideal was called the *perfect cuboid ideal* and was denoted through

$$I_{\text{PC}}=\langle p_0,p_1,p_2,p_3\rangle. \tag{1.7}$$

Each polynomial equation $p=0$ with $p\in I_{\text{PC}}$ follows from the equations (1.1). Therefore such an equation is called a *perfect cuboid equation*.

The symmetry approach to the equations (1.1) initiated in [45] leads to studying the following ideal in the ring of multisymmetric polynomials $\operatorname{Sym}\mathbb{Q}[M,L]$:

$$I_{\text{PC\_sym}}=I_{\text{PC}}\cap\operatorname{Sym}\mathbb{Q}[M,L]. \tag{1.8}$$

Definition 1.2. A polynomial equation of the form $p=0$ with $p\in I_{\text{PC\_sym}}$ is called an S_3 *factor equation* for the perfect cuboid equations (1.1).

The ideal (1.7) was initially studied in [68]. There it was denoted through I_{sym}. However, in this paper we use the notation (1.8) taken from [67]. In [68], when studying the ideal $I_{\text{PC\_sym}}$, the following eight polynomials were introduced:

$$\tilde{p}_1=p_0=x_1^2+x_2^2+x_3^2-L^2. \tag{1.9}$$

$$\begin{aligned}\tilde{p}_2=p_1+p_2+p_3&=(x_2^2+x_3^2-d_1^2)\,+\\&+\,(x_3^2+x_1^2-d_2^2)+(x_1^2+x_2^2-d_3^2),\end{aligned} \tag{1.10}$$

$$\begin{aligned}\tilde{p}_3=d_1\,p_1+d_2\,p_2+d_3\,p_3&=d_1\,(x_2^2+x_3^2-d_1^2)\,+\\&+\,d_2\,(x_3^2+x_1^2-d_2^2)+d_3\,(x_1^2+x_2^2-d_3^2),\end{aligned} \tag{1.11}$$

$$\begin{aligned}\tilde{p}_4 = x_1\,p_1 + x_2\,p_2 + x_3\,p_3 = x_1\,(x_2^2 + x_3^2 - d_1^2) \,+ \\ +\; x_2\,(x_3^2 + x_1^2 - d_2^2) + x_3\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.12}$$

$$\begin{aligned}\tilde{p}_5 = x_1\,d_1\,p_1 + x_2\,d_2\,p_2 + x_3\,d_3\,p_3 = x_1\,d_1\,(x_2^2 + x_3^2 - d_1^2) \,+ \\ +\; x_2\,d_2\,(x_3^2 + x_1^2 - d_2^2) + x_3\,d_3\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.13}$$

$$\begin{aligned}\tilde{p}_6 = x_1^2\,p_1 + x_2^2\,p_2 + x_3^2\,p_3 = x_1^2\,(x_2^2 + x_3^2 - d_1^2) \,+ \\ +\; x_2^2\,(x_3^2 + x_1^2 - d_2^2) + x_3^2\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.14}$$

$$\begin{aligned}\tilde{p}_7 = d_1^2\,p_1 + d_2^2\,p_2 + d_3^2\,p_3 = d_1^2\,(x_2^2 + x_3^2 - d_1^2) \,+ \\ +\; d_2^2\,(x_3^2 + x_1^2 - d_2^2) + d_3^2\,(x_1^2 + x_2^2 - d_3^2),\end{aligned} \tag{1.15}$$

$$\begin{aligned}\tilde{p}_8 = x_1^2\,d_1^2\,p_1 + x_2^2\,d_2^2\,p_2 + x_3^2\,d_3^2\,p_3 = x_1^2\,d_1^2\,(x_2^2 + x_3^2 - d_1^2) \,+ \\ +\; x_2^2\,d_2^2\,(x_3^2 + x_1^2 - d_2^2) + x_3^2\,d_3^2\,(x_1^2 + x_2^2 - d_3^2).\end{aligned} \tag{1.16}$$

The polynomials $\tilde{p}_0$, $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{p}_3$, $\tilde{p}_4$, $\tilde{p}_5$, $\tilde{p}_6$, $\tilde{p}_7$, $\tilde{p}_8$ in (1.9), (1.10), (1.11), (1.12), (1.13), (1.14), (1.15), (1.16) are multisymmetric and they are generated by the polynomials p_0, p_1, p_2, p_3 from (1.2). For this reason they belong to the ideal (1.8). Moreover, there is the following theorem.

Theorem 1.1. *The ideal $I_{\text{PC\_sym}}$ in* (1.8) *is finitely generated within the ring* $\operatorname{Sym}\mathbb{Q}[M, L]$. *Eight polynomials* (1.9), (1.10), (1.11), (1.12), (1.13), (1.14), (1.15), (1.16) *belong to the ideal $I_{\text{PC\_sym}}$ and constitute a basis of this ideal.*

The theorem 1.1 was proved in [68]. Using the polynomials (1.9), (1.10), (1.11), (1.12), (1.13), (1.14), (1.15), (1.16), in [67] the following equations were written:

$$\begin{aligned}&\tilde{p}_1 = 0, &&\tilde{p}_2 = 0, &&\tilde{p}_3 = 0, &&\tilde{p}_4 = 0,\\ &\tilde{p}_5 = 0, &&\tilde{p}_6 = 0, &&\tilde{p}_7 = 0, &&\tilde{p}_8 = 0.\end{aligned} \tag{1.17}$$

The equations (1.17) are factor equations of the perfect cuboid equations (1.1) in the sense of the definition 1.2. The theorem 1.1 means that the equations (1.17) constitute a complete system of the factor equations.

The equations (1.17) are derived from the equations (1.1). Therefore each solution of the equations (1.1) is a solution for the factor equations (1.17). Generally speaking, the converse proposition is not valid. However, in [67] the following theorem was proved.

Theorem 1.2. *Each integer or rational solution of the factor equations* (1.17) *such that $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, $d_1 > 0$, $d_2 > 0$, and $d_3 > 0$ is an integer or rational solution for the equations* (1.1).

The theorem 1.2 means that the factor equations (1.17) are equivalent to the original equations (1.1) with respect to the main problem of finding perfect cuboids or proving their non-existence. In this paper we continue studying the factor equations (1.17) by transforming them into so-called E-form. As a result below in section 4 we derive a single biquadratic equation from the equations (1.17).

E-form of multisymmetric polynomials.

Multisymmetric polynomials from the ring $\operatorname{Sym}\mathbb{Q}[M,L]$ are similar to regular symmetric polynomials (see [69]). Like in the case of regular symmetric polynomials, there are elementary symmetric polynomials in $\operatorname{Sym}\mathbb{Q}[M,L]$:

$$\begin{aligned} &e_{[1,0]}=x_1+x_2+x_3, && e_{[0,1]}=d_1+d_2+d_3,\\ &e_{[2,0]}=x_1\,x_2+x_2\,x_3+x_3\,x_1, && e_{[0,2]}=d_1\,d_2+d_2\,d_3+d_3\,d_1,\\ &e_{[3,0]}=x_1\,x_2\,x_3, && e_{[0,3]}=d_1\,d_2\,d_3,\end{aligned}\tag{2.1}$$

$$\begin{aligned} &e_{[2,1]}=x_1\,x_2\,d_3+x_2\,x_3\,d_1+x_3\,x_1\,d_2,\\ &e_{[1,1]}=x_1\,d_2+d_1\,x_2+x_2\,d_3+d_2\,x_3+x_3\,d_1+d_3\,x_1,\\ &e_{[1,2]}=x_1\,d_2\,d_3+x_2\,d_3\,d_1+x_3\,d_1\,d_2.\end{aligned}\tag{2.2}$$

The role of the polynomials (2.1) and (2.2) is described by the following theorem.

Theorem 2.1. *The elementary multisymmetric polynomials* (2.1) *and* (2.2) *generate the ring* $\operatorname{Sym}\mathbb{Q}[M,L]$*, i. e. each multisymmetric polynomial* $p\in\operatorname{Sym}\mathbb{Q}[M,L]$ *can be expressed as a polynomial with rational coefficients through these elementary multisymmetric polynomials.*

The theorem 2.1 is known as the fundamental theorem for elementary multisymmetric polynomials. Its proof can be found in [52].

Let's denote through $\mathbb{Q}[E,L]$ the following polynomial ring of ten independent variables E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12}, L:

$$\mathbb{Q}[E,L]=\mathbb{Q}[E_{10},E_{20},E_{30},E_{01},E_{02},E_{03},E_{21},E_{11},E_{12},L].\tag{2.3}$$

Like (1.6), the notation (2.3) is used for the sake of brevity. In terms of the notation (2.3) the theorem 2.1 can be formulated as follows.

Theorem 2.2. *For each multisymmetric polynomial* $p\in\operatorname{Sym}\mathbb{Q}[M,L]$ *there is some polynomial* $q\in\mathbb{Q}[E,L]$ *such that* p *is produced from* q *by substituting* $e_{[1,0]}$, $e_{[2,0]}$, $e_{[3,0]}$, $e_{[0,1]}$, $e_{[0,2]}$, $e_{[0,3]}$, $e_{[2,1]}$, $e_{[1,1]}$, $e_{[1,2]}$ *for* E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12} *into the arguments of the polynomial* q*:*

$$p=q(e_{[1,0]},e_{[2,0]},e_{[3,0]},e_{[0,1]},e_{[0,2]},e_{[0,3]},e_{[2,1]},e_{[1,1]},e_{[1,2]},L).\tag{2.4}$$

The substitution procedure (2.4) determines a mapping:

$$\varphi\colon\ \mathbb{Q}[E,L]\longrightarrow\operatorname{Sym}\mathbb{Q}[M,L].\tag{2.5}$$

It is easy to see that the mapping (2.5) is a ring homomorphism. Such a homomorphism is called a *substitution homomorphism*. The theorem 2.2 means that the substitution homomorphism (2.5) is surjective.

Definition 2.1. A polynomial $q=q(E_{10},E_{20},E_{30},E_{01},E_{02},E_{03},E_{21},E_{11},E_{12},L)$ such that $p=\varphi(q)$ is called an E-form of a polynomial $p\in\operatorname{Sym}\mathbb{Q}[M,L]$.

Unfortunately the substitution homomorphism (2.5) is not bijective. Therefore an E-form of a polynomial $p\in\operatorname{Sym}\mathbb{Q}[M,L]$ is not unique. It is defined up to a polynomial from the kernel of the homomorphism (2.5). The kernel $K = \operatorname{Ker}\varphi$ was studied in [68]. There it was shown that the ideal $K = \operatorname{Ker}\varphi$ has a Gröbner basis consisting of 14 polynomials, i. e. it is presented as

$$K = \langle \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5, \tilde{q}_6, \tilde{q}_7, \tilde{q}_8, \tilde{q}_9, \tilde{q}_{10}, \tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{13}, \tilde{q}_{14} \rangle. \tag{2.6}$$

For the definition of Gröbner bases and for their applications the reader is referred to the book [70]. The polynomials $\tilde{q}_1, \ldots, \tilde{q}_{14}$ in (2.6) were calculated with the use of symbolic computations, but they were not given in an explicit form in [68]. Instead, another basis of the ideal K consisting of seven polynomials was suggested:

$$K = \langle q_1, q_2, q_3, q_4, q_5, q_6, q_7 \rangle. \tag{2.7}$$

The explicit formulas for the polynomials $q_1, \ldots, q_7$ from (2.7) are available in [68]. The explicit expressions for $\tilde{q}_1, \ldots, \tilde{q}_{14}$ are given in Appendix.

3. E-form of the factor equations.

Now we are ready to proceed with studying the factor equations (1.17). For this purpose we transform them into the E-form as declared in the definition 2.1. The first equation (1.17) transformed into the E-form is very simple:

$$E_{10}^2 - 2\,E_{20} - L^2 = 0. \tag{3.1}$$

The second equation (1.17) is a little bit more complicated:

$$2\,E_{02} - 4\,E_{20} - E_{01}^2 + 2\,E_{10}^2 = 0. \tag{3.2}$$

Here are the third and the fourth equations (1.17) transformed into the E-form:

$$\begin{gathered} E_{10}\,E_{11} - 3\,E_{03} - E_{21} + 3\,E_{01}\,E_{02} - E_{20}\,E_{01} - E_{01}^3 = 0,\\ E_{01}\,E_{11} - E_{12} - 3\,E_{30} + E_{10}\,E_{02} + E_{20}\,E_{10} - E_{01}^2\,E_{10} = 0. \end{gathered} \tag{3.3}$$

Then we transform the fifth equation (1.17). As a result we get

$$\begin{gathered} -E_{10}\,E_{21} - E_{01}\,E_{12} - E_{01}\,E_{30} - E_{01}^3\,E_{10} + E_{01}^2\,E_{11} -\\ -\ E_{02}\,E_{11} + E_{11}\,E_{20} - E_{10}\,E_{03} + 2\,E_{10}\,E_{01}\,E_{02} = 0. \end{gathered} \tag{3.4}$$

The next step is to transform the sixth and the seventh equations (1.17). Upon doing it we multiply both equations by 3. Then we have

$$\begin{gathered} 4\,E_{01}\,E_{10}\,E_{11} - 3\,E_{01}^2\,E_{10}^2 + 2\,E_{10}^2\,E_{02} + 2\,E_{20}\,E_{01}^2 - 2\,E_{10}\,E_{12} -\\ -2\,E_{02}\,E_{20} - 2\,E_{01}\,E_{21} - E_{11}^2 - 12\,E_{10}\,E_{30} + 6\,E_{20}^2 = 0. \end{gathered} \tag{3.5}$$

$$\begin{gathered} 4\,E_{01}\,E_{10}\,E_{11} - 4\,E_{10}^2\,E_{02} - 4\,E_{20}\,E_{01}^2 - 2\,E_{10}\,E_{12} + 10\,E_{02}\,E_{20} -\\ -2\,E_{01}\,E_{21} - E_{11}^2 - 12\,E_{01}\,E_{03} - 3\,E_{01}^4 - 6\,E_{02}^2 + 12\,E_{01}^2\,E_{02} = 0. \end{gathered} \tag{3.6}$$

The last step is to transform the eighth equation (1.17). This equation is the most complicated of all eight. Upon transforming it we multiply this equation by 3:

$$
\begin{gathered}
9\,E_{01}\,E_{03}\,E_{20}-7\,E_{01}^2\,E_{02}\,E_{20}+2\,E_{02}\,E_{10}\,E_{12}-2\,E_{01}^2\,E_{10}\,E_{12}+\\
+\,3\,E_{03}\,E_{10}\,E_{11}+4\,E_{01}^3\,E_{10}\,E_{11}-7\,E_{01}\,E_{02}\,E_{10}\,E_{11}-6\,E_{01}\,E_{03}\,E_{10}^2+\\
+\,8\,E_{01}^2\,E_{02}\,E_{10}^2+3\,E_{01}\,E_{11}\,E_{30}-2\,E_{01}\,E_{20}\,E_{21}+E_{10}\,E_{12}\,E_{20}-\\
-\,E_{02}\,E_{10}^2\,E_{20}+E_{01}\,E_{10}\,E_{11}\,E_{20}+9\,E_{02}\,E_{10}\,E_{30}-2\,E_{02}\,E_{20}^2+\\
+\,2\,E_{01}^2\,E_{20}^2-E_{11}^2\,E_{20}-3\,E_{12}\,E_{30}+E_{02}\,E_{11}^2-E_{01}^2\,E_{11}^2-\\
-\,2\,E_{02}^2\,E_{10}^2+2\,E_{01}^4\,E_{20}+2\,E_{02}^2\,E_{20}-3\,E_{03}\,E_{21}-\\
-\,2\,E_{01}^3\,E_{21}+5\,E_{01}\,E_{02}\,E_{21}-6\,E_{01}^2\,E_{10}\,E_{30}-3\,E_{01}^4\,E_{10}^2=0.
\end{gathered}
\tag{3.7}
$$

The transformed factor equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) should be complemented with the following 14 kernel equations:

$$
\begin{aligned}
&\tilde q_1=0, &&\tilde q_2=0, &&\tilde q_3=0,\\
&\tilde q_4=0, &&\tilde q_5=0, &&\tilde q_6=0,\\
&\tilde q_7=0, &&\tilde q_8=0, &&\tilde q_9=0,\\
&\tilde q_{10}=0, &&\tilde q_{11}=0, &&\tilde q_{12}=0,\\
&\tilde q_{13}=0, &&\tilde q_{14}=0.
\end{aligned}
\tag{3.8}
$$

The kernel polynomials $\tilde q_1,\,\ldots\,,\,\tilde q_{14}$ are rather huge. The explicit expressions for them are given in Appendix in a machine readable form.

The equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7) complemented with the equations (3.8) constitute a huge system of 22 polynomial equations with integer coefficients with respect to 10 variables. Despite being a system of polynomial equations with integer coefficients, we can consider real solutions of this system. These real solutions constitute a real algebraic variety in $\mathbb{R}^{10}$. We denote this variety through $E^{10}_{\text{PC\_sym}}$. Integer points of this variety, provided they do exist, are associated with perfect cuboids. The real algebraic variety $E^{10}_{\text{PC\_sym}}\subset\mathbb{R}^{10}$ is studied below in section 4.

4. Reduction of the factor equations.

Let's consider the equation (3.1). This equation is linear with respect to the variable E_{20}. Resolving it with respect to this variable, we get

$$
E_{20}=\frac{1}{2}\,E_{10}^2-\frac{1}{2}\,L^2.
\tag{4.1}
$$

Substituting (4.1) into (3.2), we derive

$$
2\,L^2+2\,E_{02}-E_{01}^2=0.
\tag{4.2}
$$

The equation (4.2) is similar to (3.1). This equation is linear with respect to the variable E_{02}. Resolving it with respect to this variable, we get

$$
E_{02}=\frac{1}{2}\,E_{01}^2-L^2.
\tag{4.3}
$$

Now we substitute (4.1) and (4.3) into (3.3). This yields the equations

$$-3\,E_{03}-E_{21}-\frac{5}{2}\,E_{01}\,L^2+\frac{1}{2}\,E_{01}^3+E_{10}\,E_{11}-\frac{1}{2}\,E_{01}\,E_{10}^2=0, \tag{4.4}$$

$$-3\,E_{30}-E_{12}-\frac{3}{2}\,E_{10}\,L^2-\frac{1}{2}\,E_{01}^2\,E_{10}+E_{01}\,E_{11}+\frac{1}{2}\,E_{10}^3=0. \tag{4.5}$$

The equation (4.4) is linear with respect to E_{03}, while the equation (4.5) is linear with respect to E_{30}. Resolving these equations, we derive

$$E_{03}=-\frac{1}{3}\,E_{21}-\frac{1}{6}\,E_{01}\,E_{10}^2-\frac{5}{6}\,E_{01}\,L^2+\frac{1}{6}\,E_{01}^3+\frac{1}{3}\,E_{10}\,E_{11}, \tag{4.6}$$

$$E_{30}=-\frac{1}{3}\,E_{12}-\frac{1}{6}\,E_{10}\,E_{01}^2-\frac{1}{2}\,E_{10}\,L^2+\frac{1}{6}\,E_{10}^3+\frac{1}{3}\,E_{01}\,E_{11}. \tag{4.7}$$

Note that the formulas (4.1), (4.3), (4.6), and (4.7) were already derived in [45].

Using the formulas (4.1), (4.3), (4.6), (4.7), we can eliminate the variables E_{20}, E_{02}, E_{30}, E_{03} from the remaining factor equations (3.4), (3.5), (3.6), (3.7) and from the kernel equations (3.8). As a result the number of variables reduces from 10 to 6, while the total number of equations reduced from 22 to 18. The reduced system of 18 equations determines a real algebraic variety in $\mathbb{R}^6$. We denote this variety through $E^6_{\text{PC\_sym}}$. The algebraic variety $E^6_{\text{PC\_sym}}\subset\mathbb{R}^6$ is the projection of the variety $E^{10}_{\text{PC\_sym}}\subset\mathbb{R}^{10}$ onto the subspace $\mathbb{R}^6\subset\mathbb{R}^{10}$:

$$\pi_6\colon\ E^{10}_{\text{PC\_sym}}\to E^6_{\text{PC\_sym}}. \tag{4.8}$$

The right hand sides of the formulas (4.1), (4.3), (4.6), (4.7) are polynomials with respect to six variables E_{10}, E_{01}, E_{21}, E_{11}, E_{12}, and L. For this reason the mapping (4.8) is bijective, i. e. there is the inverse mapping

$$\pi_6^{-1}\colon\ E^6_{\text{PC\_sym}}\to E^{10}_{\text{PC\_sym}}. \tag{4.9}$$

As it was said above, the equations (3.4), (3.5), (3.6), (3.7) are transformed with the use of (4.1), (4.3), (4.6), (4.7). Here is the explicit form of the transformed equation (3.4) multiplied by the number $-3/2$:

$$E_{10}\,E_{21}+E_{01}\,E_{12}=\frac{1}{4}\,E_{11}\,E_{10}^2+\frac{1}{4}\,E_{01}^2\,E_{11}+\frac{3}{4}\,E_{11}\,L^2-E_{10}\,E_{01}\,L^2. \tag{4.10}$$

Two terms of the equation (4.10) are left in its left hand side, the other four terms are brought to the right hand side of this equation.

Upon transforming the equations (3.5) and (3.6) with the use of (4.1), (4.3), (4.6), (4.7) we do not write them separately, but compose their sum and their difference. The difference of the equations (3.5) and (3.6) transformed by means of (4.1), (4.3), (4.6), (4.7) and then multiplied by $1/4$ looks like

$$-E_{01}\,E_{21}+E_{10}\,E_{12}=\frac{1}{8}\,E_{10}^4-\frac{1}{8}\,E_{01}^4-\frac{3}{4}\,E_{10}^2\,L^2+E_{01}^2\,L^2-\frac{3}{8}\,L^4. \tag{4.11}$$

Like in (4.10), Two terms of the equation (4.11) are left in its left hand side, the other five terms are moved to the right hand side of this equation.

The sum of the equations (3.5) and (3.6) transformed by means of (4.1), (4.3), (4.6), (4.7) and then multiplied by the number -2 is written as

$$4\,E_{11}^2+E_{10}^4+E_{01}^4-2\,E_{10}^2\,E_{01}^2-2\,L^2\,E_{10}^2-6\,E_{01}^2\,L^2+L^4=0. \tag{4.12}$$

And finally, the equation (3.7) transformed by means of (4.1), (4.3), (4.6), (4.7) and then multiplied by the number 4 is written as

$$\begin{aligned}&8\,E_{10}\,E_{11}\,E_{21}+8\,E_{01}\,E_{11}\,E_{12}-4\,E_{21}^2-4\,E_{12}^2-8\,E_{10}\,E_{12}\,L^2=\\&=2\,E_{10}^2\,E_{11}^2+2\,E_{01}^2\,E_{10}^2\,L^2+2\,E_{01}^2\,E_{11}^2-E_{01}^4\,L^2-2\,E_{10}^4\,L^2-\\&-8\,E_{01}\,E_{10}\,E_{11}\,L^2+8\,E_{10}^2\,L^4+6\,E_{01}^2\,L^4-2\,E_{11}^2\,L^2-2\,L^6.\end{aligned} \tag{4.13}$$

It is easy to see that the transformed equation (4.13) looks much more simple than the original equation (3.7).

The kernel equations (3.8) also get more simple when transformed by means of (4.1), (4.3), (4.6), (4.7). But they are still rather huge for to write them explicitly.

5. Further transformations.

Note that the equations (4.10) and (4.11) are linear with respect to the variables E_{21} and E_{12}. They can be resolved with respect to these variables as

$$\begin{aligned}E_{21}=&\frac{2\,E_{10}^3\,E_{11}+2\,E_{01}^2\,E_{10}\,E_{11}-E_{01}\,E_{10}^4+E_{01}^5}{8\,(E_{01}^2+E_{10}^2)}+\\&+\frac{6\,E_{10}\,E_{11}\,L^2-2\,E_{01}\,E_{10}^2\,L^2-8\,E_{01}^3\,L^2+3\,E_{01}\,L^4}{8\,(E_{01}^2+E_{10}^2)},\end{aligned} \tag{5.1}$$

$$\begin{aligned}E_{12}=&\frac{E_{01}^4\,E_{10}-2\,E_{01}^3\,E_{11}-2\,E_{01}\,E_{10}^2\,E_{11}-E_{10}^5}{8\,(E_{01}^2+E_{10}^2)}+\\&+\frac{6\,E_{10}^3\,L^2-6\,E_{01}\,E_{11}\,L^2+3\,E_{10}\,L^4}{8\,(E_{01}^2+E_{10}^2)}.\end{aligned} \tag{5.2}$$

The equation (4.12) can be resolved with respect to the square of E_{11}:

$$E_{11}^2=\frac{1}{2}\,E_{01}^2\,E_{10}^2-\frac{1}{4}\,E_{10}^4-\frac{1}{4}\,E_{01}^4+\frac{1}{2}\,E_{10}^2\,L^2+\frac{3}{2}\,E_{01}^2\,L^2-\frac{1}{4}\,L^4. \tag{5.3}$$

Using (5.1) and (5.2), we can eliminate E_{21} and E_{12} from the remaining factor equation (4.13) and from the kernel equations (3.8) which are already transformed with the use of (4.1), (4.3), (4.6), (4.7). Then we can apply (5.3) and eliminate squares of E_{11}, cubes of E_{11}, and all other higher degrees of E_{11} from (4.13) and (3.8). As a result of such transformations the equations (4.13) and (3.8) luckily turn to trivial identities $0=0$.

Note that the formulas (5.1) and (5.2) have nontrivial denominators. Therefore we should be careful in interpreting our lucky result. The determinants in (5.1) and (5.2) turn to zero at the points where $E_{10}=0$ and $E_{01}=0$ simultaneously. Hence we can formulate the following theorem.

Theorem 5.1. *At the locus of points in $\mathbb{R}^{10}$ where $E_{10} \neq 0$ or $E_{01} \neq 0$ the system of 22 polynomial equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), and (3.8) is equivalent to the single equation (4.12).*

The equation (4.12) is a polynomial equation with respect to four variables E_{10}, E_{01}, E_{11}, and L. It defines a real algebraic variety in $\mathbb{R}^4$. Let's denote it through $E^4_{\text{PC\_sym}}$. Like $E^{10}_{\text{PC\_sym}}$ and $E^6_{\text{PC\_sym}}$, the variety $E^4_{\text{PC\_sym}}$ can have special points where $E_{10} = 0$ and $E_{01} = 0$ simultaneously. Let's pin out these special points from each of the three varieties. As a result we get three Zariski open subsets $\tilde{E}^{10}_{\text{PC\_sym}}$, $\tilde{E}^6_{\text{PC\_sym}}$, and $\tilde{E}^4_{\text{PC\_sym}}$ within $E^{10}_{\text{PC\_sym}}$, $E^6_{\text{PC\_sym}}$, and $E^4_{\text{PC\_sym}}$ respectively.

Generally speaking, the algebraic variety $E^4_{\text{PC\_sym}}$ should not be a projection of $E^6_{\text{PC\_sym}}$. However, its subset $\tilde{E}^4_{\text{PC\_sym}}$ is the projection of the subset $\tilde{E}^6_{\text{PC\_sym}}$ from $\mathbb{R}^6$ onto the subspace $\mathbb{R}^4 \subset \mathbb{R}^6$. We have the mapping

$$\pi_4\colon\ \tilde{E}^6_{\text{PC\_sym}} \to \tilde{E}^4_{\text{PC\_sym}}. \tag{5.4}$$

Due to (5.1) and (5.2) the mapping (5.4) is bijective, i. e. it has the inverse mapping

$$\pi_4^{-1}\colon\ \tilde{E}^4_{\text{PC\_sym}} \to \tilde{E}^6_{\text{PC\_sym}}. \tag{5.5}$$

The mappings (5.4) and (5.5) are similar to the mappings (4.8) and (4.9).

The equation (4.12) was derived from the equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), and (3.8) without use of the denominators in (5.1) and (5.2). For this reason each integer solution of the equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) induces an integer solution of the equation (4.12). The backward formulas (4.1), (4.3), (4.6), (4.7), (5.1), (5.2) comprise fractions. Nevertheless, we can prove the following theorem.

Theorem 5.2. *Each integer or rational solution of the equation (4.12) such that $E_{10}^2 + E_{01}^2 \neq 0$ induces an integer solution of the complete system of 22 equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8).*

Proof. The equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) are weighted homogeneous. If we have a solution of them, then applying the transformations

$$\begin{aligned}
&E_{10} \mapsto \alpha\, E_{10}, && E_{01} \mapsto \alpha\, E_{01},\\
&E_{20} \mapsto \alpha^2\, E_{20}, && E_{02} \mapsto \alpha^2\, E_{02},\\
&E_{30} \mapsto \alpha^3\, E_{30}, && E_{03} \mapsto \alpha^3\, E_{03},\\
&E_{21} \mapsto \alpha^3\, E_{21}, && E_{12} \mapsto \alpha^3\, E_{12},\\
&E_{11} \mapsto \alpha^2\, E_{11}, && L \mapsto \alpha\, L,
\end{aligned} \tag{5.6}$$

we get another solution for them. Assume that we have an integer or a rational solution of the equation (4.12) such that $E_{10}^2 + E_{01}^2 \neq 0$. Then by means of the formulas (4.1), (4.3), (4.6), (4.7), (5.1), (5.2) we get a solution for the equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8) such that E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12}, and L are integer or rational numbers. Let Q be a common denominator for all of these ten numbers. It is sufficient to apply the transformations (5.6) with $\alpha = Q$ and obtain an integer solution for the equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8). □

6. Back to perfect cuboids.

If a perfect Euler cuboid does exist, it induces an integer solution for the equations (1.1) and an integer solution for the factor equations (1.17) such that $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, $d_1 > 0$, $d_2 > 0$, $d_1 > 0$, and $L > 0$. Via the formulas (2.1) and (2.2) the latter one induces a positive integer solution for the equations (3.1), (3.2), (3.3), (3.4), (3.5), (3.6), (3.7), (3.8), and finally, an integer solution for the equation (4.12) with $E_{10} > 0$, $E_{01} > 0$, $E_{11} > 0$ and $L > 0$. Combining this fact with the theorem 5.2 we formulate the following result.

Theorem 6.1. *The existence of positive rational solutions of the equation* (4.12) *is a necessary condition for the existence of perfect Euler cuboids.*

The backward path from the equation (4.12) to perfect cuboids is not so straightforward. On this path we need to solve the inverse problem of calculating the values of x_1, x_2, x_3, d_1, d_2, d_3 through the known values of the elementary multisymmetric polynomials in (2.1) and (2.2). As I know, this problem is poorly studied. Therefore the theorem 6.1 is formulated as a necessary, but not a sufficient, condition.

The theorem 6.1 is the main result of this paper. It could be useful in computerized search for perfect cuboids or in proving their non-existence.

7. Similarity to Heron's problem.

The ancient Greek mathematician Heron of Alexandria (10–70 C.E.) has discovered the formula expressing the area S of a triangle through its sides a, b, and c. This Heron's formula is written as follows:

$$S = \sqrt{p\,(p-a)\,(p-b)\,(p-c)},\ \ \text{where}\ \ p = \frac{a+b+c}{2}. \tag{7.1}$$

The formula (7.1) can be written as an equation with respect to a, b, c, and S:

$$(4\,S)^2 + (a^2 + b^2 - c^2)^2 - 4\,a^2\,b^2 = 0. \tag{7.2}$$

The equation (7.2) is associated with the well known Heron's problem — find all triangles whose sides and whose area are integer numbers (see [71]).

Now let's consider the equation (4.12). This equation can be rewritten as

$$(2\,E_{11})^2 + (E_{01}^2 + L^2 - E_{10}^2)^2 - 8\,E_{01}^2\,L^2 = 0. \tag{7.3}$$

The equation (7.3) is very similar to the equation (7.2). Both equations belong to the class of biquadratic Diophantine equations.

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Appendix.

Here are the formulas for the polynomials $\tilde{q}_1, \ldots, \tilde{q}_{14}$ written in a machine readable form convenient for to copy-paste into some symbolic computations package:

```
q̃1:=-3*E02*E21+E01^2*E21-9*E03*E20+4*E01*E02*E20-E01^3*E20+3*E11*E12
-2*E01*E10*E12-E01*E11^2+E01^2*E10*E11+3*E03*E10^2-E01*E02*E10^2;
```

```
q̃2:=9*E02*E30-3*E01^2*E30-3*E11*E21+2*E01*E10*E21+3*E12*E20-4*E02
*E10*E20+E01^2*E10*E20-E10^2*E12+E10*E11^2-E01*E10^2*E11+E02*E10^3;
```

```
q̃3:=-9*E02^2*E30+6*E01^2*E02*E30-E01^4*E30-3*E02*E12*E20+E01^2*E12
*E20-9*E03*E11*E20+4*E01*E02*E11*E20-E01^3*E11*E20+6*E01*E03*E10*E20
+4*E02^2*E10*E20-5*E01^2*E02*E10*E20+E01^4*E10*E20+3*E11^2*E12-4*E01
*E10*E11*E12+E02*E10^2*E12+E01^2*E10^2*E12-E01*E11^3-E02*E10*E11^2
+2*E01^2*E10*E11^2+3*E03*E10^2*E11-E01^3*E10^2*E11-2*E01*E03*E10^3
-E02^2*E10^3+E01^2*E02*E10^3;
```

```
q̃4:=-27*E03*E21+E01^3*E21-18*E01*E03*E20+12*E02^2*E20+E01^2*E02*E20
-E01^4*E20+9*E12^2+3*E01*E11*E12-6*E02*E10*E12-2*E01^2*E10*E12-3*E02
*E11^2-E01^2*E11^2+9*E03*E10*E11+3*E01*E02*E10*E11+E01^3*E10*E11+3
*E01*E03*E10^2-3*E02^2*E10^2-E01^2*E02*E10^2;
```

```
q̃5:=-81*E03*E30+18*E01*E02*E30-3*E01^3*E30+9*E12*E21-E01^2*E10*E21
-6*E01*E12*E20+12*E02*E11*E20-3*E01^2*E11*E20+36*E03*E10*E20-16*E01
*E02*E10*E20+4*E01^3*E10*E20-6*E10*E11*E12+5*E01*E10^2*E12-3*E11^3
+7*E01*E10*E11^2-3*E02*E10^2*E11-4*E01^2*E10^2*E11-9*E03*E10^3+4
*E01*E02*E10^3;
```

```
q̃6:=-9*E12*E30+3*E01*E11*E30-3*E02*E10*E30+3*E21^2-2*E01*E20*E21+4
*E02*E20^2-E01^2*E20^2+E10*E12*E20-E11^2*E20+E01*E10*E11*E20-E02
*E10^2*E20;
```

```
q̃7:=-27*E02*E03*E30+9*E01^2*E03*E30+3*E01*E02^2*E30-E01^3*E02*E30-9
*E03*E12*E20+E01*E02*E12*E20-3*E01*E03*E11*E20+4*E02^2*E11*E20-E01^2
*E02*E11*E20+12*E02*E03*E10*E20-E01^2*E03*E10*E20-4*E01*E02^2*E10
*E20+E01^3*E02*E10*E20+3*E11*E12^2-2*E01*E10*E12^2-2*E02*E10*E11*E12
+3*E03*E10^2*E12+E01*E02*E10^2*E12-E02*E11^3+2*E01*E02*E10*E11^2+E01
*E03*E10^2*E11-E02^2*E10^2*E11-E01^2*E02*E10^2*E11-3*E02*E03*E10^3
+E01*E02^2*E10^3;
```

```
q̃8:=27*E02*E12*E30-9*E01^2*E12*E30-81*E03*E11*E30+18*E01*E02*E11*E30
-3*E01^3*E11*E30+54*E01*E03*E10*E30-9*E02^2*E10*E30-9*E01^2*E02*E10
*E30+2*E01^4*E10*E30+9*E12^2*E20-6*E01*E11*E12*E20-15*E02*E10*E12
```

```
*E20+7*E01^2*E10*E12*E20+12*E02*E11^2*E20-3*E01^2*E11^2*E20+27*E03
*E10*E11*E20-20*E01*E02*E10*E11*E20+5*E01^3*E10*E11*E20-18*E01*E03
*E10^2*E20+4*E02^2*E10^2*E20+7*E01^2*E02*E10^2*E20-2*E01^4*E10^2*E20
-3*E10^2*E12^2+2*E01*E10^2*E11*E12+4*E02*E10^3*E12-2*E01^2*E10^3*E12
-3*E11^4+8*E01*E10*E11^3-4*E02*E10^2*E11^2-7*E01^2*E10^2*E11^2-6*E03
*E10^3*E11+6*E01*E02*E10^3*E11+2*E01^3*E10^3*E11+4*E01*E03*E10^4
-E02^2*E10^4-2*E01^2*E02*E10^4;
```

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q̃9:=-27*E03^2*E20+18*E01*E02*E03*E20-4*E01^3*E03*E20-4*E02^3*E20
+E01^2*E02^2*E20-3*E02*E12^2+E01^2*E12^2+9*E03*E11*E12-E01*E02*E11
*E12-6*E01*E03*E10*E12+2*E02^2*E10*E12-3*E01*E03*E11^2+E02^2*E11^2
-3*E02*E03*E10*E11+4*E01^2*E03*E10*E11-E01*E02^2*E10*E11+9*E03^2
*E10^2-4*E01*E02*E03*E10^2+E02^3*E10^2;
```

```
q̃10:=-243*E02*E03*E20*E30+81*E01^2*E03*E20*E30+81*E01*E02^2*E20*E30
-45*E01^3*E02*E20*E30+6*E01^5*E20*E30-81*E02*E11*E12*E30+27*E01^2
*E11*E12*E30+54*E01*E02*E10*E12*E30-18*E01^3*E10*E12*E30+243*E03
*E11^2*E30-54*E01*E02*E11^2*E30+9*E01^3*E11^2*E30-324*E01*E03*E10
*E11*E30+27*E02^2*E10*E11*E30+63*E01^2*E02*E10*E11*E30-12*E01^4*E10
*E11*E30+81*E02*E03*E10^2*E30+81*E01^2*E03*E10^2*E30-45*E01*E02^2
*E10^2*E30-3*E01^3*E02*E10^2*E30+2*E01^5*E10^2*E30-81*E03*E12*E20^2
+27*E01*E02*E12*E20^2-6*E01^3*E12*E20^2+27*E01*E03*E11*E20^2+36
*E02^2*E11*E20^2-33*E01^2*E02*E11*E20^2+6*E01^4*E11*E20^2+108*E02
*E03*E10*E20^2-45*E01^2*E03*E10*E20^2-60*E01*E02^2*E10*E20^2+39
*E01^3*E02*E10*E20^2-6*E01^5*E10*E20^2+27*E02*E10*E11*E12*E20-9
*E01^2*E10*E11*E12*E20+54*E03*E10^2*E12*E20-36*E01*E02*E10^2*E12*E20
+10*E01^3*E10^2*E12*E20-45*E02*E11^3*E20+15*E01^2*E11^3*E20-81*E03
*E10*E11^2*E20+108*E01*E02*E10*E11^2*E20-33*E01^3*E10*E11^2*E20+90
*E01*E03*E10^2*E11*E20-33*E02^2*E10^2*E11*E20-59*E01^2*E02*E10^2*E11
*E20+20*E01^4*E10^2*E11*E20-63*E02*E03*E10^3*E20-9*E01^2*E03*E10^3
*E20+43*E01*E02^2*E10^3*E20-5*E01^3*E02*E10^3*E20-2*E01^5*E10^3*E20
-6*E02*E10^3*E11*E12+2*E01^2*E10^3*E11*E12-9*E03*E10^4*E12+7*E01*E02
*E10^4*E12-2*E01^3*E10^4*E12+9*E11^5-30*E01*E10*E11^4+15*E02*E10^2
*E11^3+35*E01^2*E10^2*E11^3+18*E03*E10^3*E11^2-34*E01*E02*E10^3
*E11^2-16*E01^3*E10^3*E11^2-21*E01*E03*E10^4*E11+6*E02^2*E10^4*E11
+21*E01^2*E02*E10^4*E11+2*E01^4*E10^4*E11+9*E02*E03*E10^5+4*E01^2
*E03*E10^5-7*E01*E02^2*E10^5-2*E01^3*E02*E10^5;
```

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q̃11:=81*E02^2*E12*E30-54*E01^2*E02*E12*E30+9*E01^4*E12*E30-243*E02
*E03*E11*E30+81*E01^2*E03*E11*E30+54*E01*E02^2*E11*E30-27*E01^3*E02
*E11*E30+3*E01^5*E11*E30+162*E01*E02*E03*E10*E30-54*E01^3*E03*E10
*E30-27*E02^3*E10*E30-18*E01^2*E02^2*E10*E30+15*E01^4*E02*E10*E30-2
*E01^6*E10*E30-243*E03^2*E20^2+162*E01*E02*E03*E20^2-36*E01^3*E03
*E20^2-36*E02^3*E20^2+9*E01^2*E02^2*E20^2+81*E03*E11*E12*E20-27*E01
*E02*E11*E12*E20+6*E01^3*E11*E12*E20-54*E01*E03*E10*E12*E20-27*E02^2
*E10*E12*E20+36*E01^2*E02*E10*E12*E20-7*E01^4*E10*E12*E20-27*E01*E03
*E11^2*E20+45*E02^2*E11^2*E20-21*E01^2*E02*E11^2*E20+3*E01^4*E11^2
*E20+54*E02*E03*E10*E11*E20+9*E01^2*E03*E10*E11*E20-69*E01*E02^2*E10
*E11*E20+35*E01^3*E02*E10*E11*E20-5*E01^5*E10*E11*E20+162*E03^2
*E10^2*E20-144*E01*E02*E03*E10^2*E20+30*E01^3*E03*E10^2*E20+33*E02^3
```

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*E10^2*E20+14*E01^2*E02^2*E10^2*E20-13*E01^4*E02*E10^2*E20+2*E01^6
*E10^2*E20-27*E03*E10^2*E11*E12+9*E01*E02*E10^2*E11*E12-2*E01^3
*E10^2*E11*E12+18*E01*E03*E10^3*E12+6*E02^2*E10^3*E12-10*E01^2*E02
*E10^3*E12+2*E01^4*E10^3*E12-9*E02*E11^4+3*E01^2*E11^4+24*E01*E02
*E10*E11^3-8*E01^3*E10*E11^3+9*E01*E03*E10^2*E11^2-15*E02^2*E10^2
*E11^2-17*E01^2*E02*E10^2*E11^2+7*E01^4*E10^2*E11^2-9*E02*E03*E10^3
*E11-6*E01^2*E03*E10^3*E11+21*E01*E02^2*E10^3*E11-2*E01^5*E10^3*E11
-27*E03^2*E10^4+24*E01*E02*E03*E10^4-4*E01^3*E03*E10^4-6*E02^3*E10^4
-5*E01^2*E02^2*E10^4+2*E01^4*E02*E10^4;
```

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q̃12:=-27*E02^3*E30^2+27*E01^2*E02^2*E30^2-9*E01^4*E02*E30^2+E01^6
*E30^2-81*E02*E03*E11*E20*E30+27*E01^2*E03*E11*E20*E30+27*E01*E02^2
*E11*E20*E30-15*E01^3*E02*E11*E20*E30+2*E01^5*E11*E20*E30+54*E01*E02
*E03*E10*E20*E30-18*E01^3*E03*E10*E20*E30+18*E02^3*E10*E20*E30-36
*E01^2*E02^2*E10*E20*E30+16*E01^4*E02*E10*E20*E30-2*E01^6*E10*E20
*E30+27*E03*E11^3*E30-9*E01*E02*E11^3*E30+2*E01^3*E11^3*E30-54*E01
*E03*E10*E11^2*E30+18*E01^2*E02*E10*E11^2*E30-4*E01^4*E10*E11^2*E30
+27*E02*E03*E10^2*E11*E30+27*E01^2*E03*E10^2*E11*E30-9*E01*E02^2
*E10^2*E11*E30-7*E01^3*E02*E10^2*E11*E30+2*E01^5*E10^2*E11*E30-18
*E01*E02*E03*E10^3*E30-2*E01^3*E03*E10^3*E30-4*E02^3*E10^3*E30+10
*E01^2*E02^2*E10^3*E30-2*E01^4*E02*E10^3*E30-27*E03^2*E20^3+18*E01
*E02*E03*E20^3-4*E01^3*E03*E20^3-4*E02^3*E20^3+E01^2*E02^2*E20^3+9
*E02^2*E11^2*E20^2-6*E01^2*E02*E11^2*E20^2+E01^4*E11^2*E20^2+27*E02
*E03*E10*E11*E20^2-9*E01^2*E03*E10*E11*E20^2-21*E01*E02^2*E10*E11
*E20^2+13*E01^3*E02*E10*E11*E20^2-2*E01^5*E10*E11*E20^2+27*E03^2
*E10^2*E20^2-36*E01*E02*E03*E10^2*E20^2+10*E01^3*E03*E10^2*E20^2
+E02^3*E10^2*E20^2+12*E01^2*E02^2*E10^2*E20^2-7*E01^4*E02*E10^2
*E20^2+E01^6*E10^2*E20^2-6*E02*E11^4*E20+2*E01^2*E11^4*E20-9*E03*E10
*E11^3*E20+19*E01*E02*E10*E11^3*E20-6*E01^3*E10*E11^3*E20+18*E01*E03
*E10^2*E11^2*E20-6*E02^2*E10^2*E11^2*E20-18*E01^2*E02*E10^2*E11^2
*E20+6*E01^4*E10^2*E11^2*E20-15*E02*E03*E10^3*E11*E20-7*E01^2*E03
*E10^3*E11*E20+13*E01*E02^2*E10^3*E11*E20+3*E01^3*E02*E10^3*E11*E20
-2*E01^5*E10^3*E11*E20-9*E03^2*E10^4*E20+16*E01*E02*E03*E10^4*E20-2
*E01^3*E03*E10^4*E20-7*E01^2*E02^2*E10^4*E20+2*E01^4*E02*E10^4*E20
+E11^6-4*E01*E10*E11^5+2*E02*E10^2*E11^4+6*E01^2*E10^2*E11^4+2*E03
*E10^3*E11^3-6*E01*E02*E10^3*E11^3-4*E01^3*E10^3*E11^3-4*E01*E03
*E10^4*E11^2+E02^2*E10^4*E11^2+6*E01^2*E02*E10^4*E11^2+E01^4*E10^4
*E11^2+2*E02*E03*E10^5*E11+2*E01^2*E03*E10^5*E11-2*E01*E02^2*E10^5
*E11-2*E01^3*E02*E10^5*E11+E03^2*E10^6-2*E01*E02*E03*E10^6+E01^2
*E02^2*E10^6;
```

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q̃13:=-729*E02^3*E12*E30^2+729*E01^2*E02^2*E12*E30^2-243*E01^4*E02
*E12*E30^2+27*E01^6*E12*E30^2+2187*E02^2*E03*E11*E30^2-1458*E01^2
*E02*E03*E11*E30^2+243*E01^4*E03*E11*E30^2-486*E01*E02^3*E11*E30^2
+405*E01^3*E02^2*E11*E30^2-108*E01^5*E02*E11*E30^2+9*E01^7*E11*E30^2
-1458*E01*E02^2*E03*E10*E30^2+972*E01^3*E02*E03*E10*E30^2-162*E01^5
*E03*E10*E30^2+243*E02^4*E10*E30^2+81*E01^2*E02^3*E10*E30^2-189
*E01^4*E02^2*E10*E30^2+63*E01^6*E02*E10*E30^2-6*E01^8*E10*E30^2+4374
*E02*E03^2*E20^2*E30-1458*E01^2*E03^2*E20^2*E30-2916*E01*E02^2*E03
*E20^2*E30+1620*E01^3*E02*E03*E20^2*E30-216*E01^5*E03*E20^2*E30+324
```

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*E02^4*E20^2*E30+54*E01^2*E02^3*E20^2*E30-162*E01^4*E02^2*E20^2*E30
+48*E01^6*E02*E20^2*E30-4*E01^8*E20^2*E30+486*E02^3*E10*E12*E20*E30
-486*E01^2*E02^2*E10*E12*E20*E30+162*E01^4*E02*E10*E12*E20*E30-18
*E01^6*E10*E12*E20*E30-2187*E03^2*E11^2*E20*E30+1458*E01*E02*E03
*E11^2*E20*E30-324*E01^3*E03*E11^2*E20*E30-405*E02^3*E11^2*E20*E30
+162*E01^2*E02^2*E11^2*E20*E30-27*E01^4*E02*E11^2*E20*E30+3*E01^6
*E11^2*E20*E30+2916*E01*E03^2*E10*E11*E20*E30-1458*E02^2*E03*E10*E11
*E20*E30-972*E01^2*E02*E03*E10*E11*E20*E30+270*E01^4*E03*E10*E11*E20
*E30+864*E01*E02^3*E10*E11*E20*E30-486*E01^3*E02^2*E10*E11*E20*E30
+108*E01^5*E02*E10*E11*E20*E30-10*E01^7*E10*E11*E20*E30-2916*E02
*E03^2*E10^2*E20*E30+2916*E01*E02^2*E03*E10^2*E20*E30-1080*E01^3*E02
*E03*E10^2*E20*E30+108*E01^5*E03*E10^2*E20*E30-378*E02^4*E10^2*E20
*E30-270*E01^2*E02^3*E10^2*E20*E30+306*E01^4*E02^2*E10^2*E20*E30-86
*E01^6*E02*E10^2*E20*E30+8*E01^8*E10^2*E20*E30-108*E02^3*E10^3*E12
*E30+108*E01^2*E02^2*E10^3*E12*E30-36*E01^4*E02*E10^3*E12*E30+4
*E01^6*E10^3*E12*E30+81*E02^2*E11^4*E30-54*E01^2*E02*E11^4*E30+9
*E01^4*E11^4*E30-216*E01*E02^2*E10*E11^3*E30+144*E01^3*E02*E10*E11^3
*E30-24*E01^5*E10*E11^3*E30+729*E03^2*E10^2*E11^2*E30-486*E01*E02
*E03*E10^2*E11^2*E30+108*E01^3*E03*E10^2*E11^2*E30+135*E02^3*E10^2
*E11^2*E30+162*E01^2*E02^2*E10^2*E11^2*E30-135*E01^4*E02*E10^2*E11^2
*E30+23*E01^6*E10^2*E11^2*E30-972*E01*E03^2*E10^3*E11*E30+324*E02^2
*E03*E10^3*E11*E30+432*E01^2*E02*E03*E10^3*E11*E30-108*E01^4*E03
*E10^3*E11*E30-252*E01*E02^3*E10^3*E11*E30+36*E01^3*E02^2*E10^3*E11
*E30+36*E01^5*E02*E10^3*E11*E30-8*E01^7*E10^3*E11*E30+486*E02*E03^2
*E10^4*E30+162*E01^2*E03^2*E10^4*E30-540*E01*E02^2*E03*E10^4*E30+108
*E01^3*E02*E03*E10^4*E30+72*E02^4*E10^4*E30+78*E01^2*E02^3*E10^4*E30
-54*E01^4*E02^2*E10^4*E30+8*E01^6*E02*E10^4*E30+729*E03^2*E12*E20^3
-486*E01*E02*E03*E12*E20^3+108*E01^3*E03*E12*E20^3+81*E01^2*E02^2
*E12*E20^3-36*E01^4*E02*E12*E20^3+4*E01^6*E12*E20^3-243*E01*E03^2
*E11*E20^3-324*E02^2*E03*E11*E20^3+378*E01^2*E02*E03*E11*E20^3-72
*E01^4*E03*E11*E20^3+108*E01*E02^3*E11*E20^3-123*E01^3*E02^2*E11
*E20^3+40*E01^5*E02*E11*E20^3-4*E01^7*E11*E20^3-1701*E02*E03^2*E10
*E20^3+648*E01^2*E03^2*E10*E20^3+1350*E01*E02^2*E03*E10*E20^3-828
*E01^3*E02*E03*E10*E20^3+120*E01^5*E03*E10*E20^3-108*E02^4*E10*E20^3
-117*E01^2*E02^3*E10*E20^3+148*E01^4*E02^2*E10*E20^3-44*E01^6*E02
*E10*E20^3+4*E01^8*E10*E20^3-729*E03^2*E10^2*E12*E20^2+486*E01*E02
*E03*E10^2*E12*E20^2-108*E01^3*E03*E10^2*E12*E20^2-81*E02^3*E10^2
*E12*E20^2+9*E01^4*E02*E10^2*E12*E20^2-E01^6*E10^2*E12*E20^2+405*E02
*E03*E11^3*E20^2-135*E01^2*E03*E11^3*E20^2-135*E01*E02^2*E11^3*E20^2
+75*E01^3*E02*E11^3*E20^2-10*E01^5*E11^3*E20^2+729*E03^2*E10*E11^2
*E20^2-1296*E01*E02*E03*E10*E11^2*E20^2+378*E01^3*E03*E10*E11^2
*E20^2+135*E02^3*E10*E11^2*E20^2+216*E01^2*E02^2*E10*E11^2*E20^2-141
*E01^4*E02*E10*E11^2*E20^2+19*E01^6*E10*E11^2*E20^2-729*E01*E03^2
*E10^2*E11*E20^2+567*E02^2*E03*E10^2*E11*E20^2+648*E01^2*E02*E03
*E10^2*E11*E20^2-225*E01^4*E03*E10^2*E11*E20^2-342*E01*E02^3*E10^2
*E11*E20^2+60*E01^3*E02^2*E10^2*E11*E20^2+36*E01^5*E02*E10^2*E11
*E20^2-7*E01^7*E10^2*E11*E20^2+1539*E02*E03^2*E10^3*E20^2-270*E01^2
*E03^2*E10^3*E20^2-1404*E01*E02^2*E03*E10^3*E20^2+540*E01^3*E02*E03
*E10^3*E20^2-42*E01^5*E03*E10^3*E20^2+123*E02^4*E10^3*E20^2+184
```

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*E01^2*E02^3*E10^3*E20^2-147*E01^4*E02^2*E10^3*E20^2+31*E01^6*E02
*E10^3*E20^2-2*E01^8*E10^3*E20^2+243*E03^2*E10^4*E12*E20-162*E01*E02
*E03*E10^4*E12*E20+36*E01^3*E03*E10^4*E12*E20+36*E02^3*E10^4*E12*E20
-9*E01^2*E02^2*E10^4*E12*E20-81*E03*E11^5*E20+27*E01*E02*E11^5*E20-6
*E01^3*E11^5*E20+270*E01*E03*E10*E11^4*E20-27*E02^2*E10*E11^4*E20-72
*E01^2*E02*E10*E11^4*E20+17*E01^4*E10*E11^4*E20-270*E02*E03*E10^2
*E11^3*E20-270*E01^2*E03*E10^2*E11^3*E20+162*E01*E02^2*E10^2*E11^3
*E20+22*E01^3*E02*E10^2*E11^3*E20-12*E01^5*E10^2*E11^3*E20-405*E03^2
*E10^3*E11^2*E20+810*E01*E02*E03*E10^3*E11^2*E20-75*E02^3*E10^3
*E11^2*E20-222*E01^2*E02^2*E10^3*E11^2*E20+63*E01^4*E02*E10^3*E11^2
*E20-3*E01^6*E10^3*E11^2*E20+459*E01*E03^2*E10^4*E11*E20-216*E02^2
*E03*E10^4*E11*E20-522*E01^2*E02*E03*E10^4*E11*E20+84*E01^4*E03
*E10^4*E11*E20+160*E01*E02^3*E10^4*E11*E20+51*E01^3*E02^2*E10^4*E11
*E20-36*E01^5*E02*E10^4*E11*E20+4*E01^7*E10^4*E11*E20-459*E02*E03^2
*E10^5*E20+450*E01*E02^2*E03*E10^5*E20-84*E01^3*E02*E03*E10^5*E20-40
*E02^4*E10^5*E20-75*E01^2*E02^3*E10^5*E20+36*E01^4*E02^2*E10^5*E20-4
*E01^6*E02*E10^5*E20-27*E03^2*E10^6*E12+18*E01*E02*E03*E10^6*E12-4
*E01^3*E03*E10^6*E12-4*E02^3*E10^6*E12+E01^2*E02^2*E10^6*E12+27*E03
*E10^2*E11^5-9*E01*E02*E10^2*E11^5+2*E01^3*E10^2*E11^5-90*E01*E03
*E10^3*E11^4+6*E02^2*E10^3*E11^4+26*E01^2*E02*E10^3*E11^4-6*E01^4
*E10^3*E11^4+45*E02*E03*E10^4*E11^3+105*E01^2*E03*E10^4*E11^3-31*E01
*E02^2*E10^4*E11^3-21*E01^3*E02*E10^4*E11^3+6*E01^5*E10^4*E11^3+54
*E03^2*E10^5*E11^2-126*E01*E02*E03*E10^5*E11^2-42*E01^3*E03*E10^5
*E11^2+10*E02^3*E10^5*E11^2+42*E01^2*E02^2*E10^5*E11^2-2*E01^6*E10^5
*E11^2-63*E01*E03^2*E10^6*E11+24*E02^2*E03*E10^6*E11+86*E01^2*E02
*E03*E10^6*E11-20*E01*E02^3*E10^6*E11-15*E01^3*E02^2*E10^6*E11+4
*E01^5*E02*E10^6*E11+45*E02*E03^2*E10^7+6*E01^2*E03^2*E10^7-46*E01
*E02^2*E03*E10^7+4*E02^4*E10^7+9*E01^2*E02^3*E10^7-2*E01^4*E02^2
*E10^7;
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q̃14:=81*E03^2*E30-27*E01*E02*E03*E30+3*E01^3*E03*E30+3*E02^3*E30+9
*E01*E03*E12*E20-3*E02^2*E12*E20-9*E02*E03*E11*E20+3*E01^2*E03*E11
*E20-54*E03^2*E10*E20+30*E01*E02*E03*E10*E20-7*E01^3*E03*E10*E20-4
*E02^3*E10*E20+E01^2*E02^2*E10*E20-3*E12^3+E01^2*E10*E12^2+9*E03*E10
*E11*E12-E01*E02*E10*E11*E12-9*E01*E03*E10^2*E12+2*E02^2*E10^2*E12+3
*E03*E11^3-9*E01*E03*E10*E11^2+E02^2*E10*E11^2+7*E01^2*E03*E10^2*E11
-E01*E02^2*E10^2*E11+15*E03^2*E10^3-7*E01*E02*E03*E10^3+E02^3*E10^3.
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