

# INVERSE PROBLEMS ASSOCIATED WITH PERFECT CUBOIDS.

JOHN RAMSDEN, RUSLAN SHARIPOV

ABSTRACT. A perfect cuboid is a rectangular parallelepiped with integer edges, integer face diagonals, and integer space diagonal. Such cuboids have not yet been found, but nor has their existence been disproved. Perfect cuboids are described by a certain system of Diophantine equations possessing an intrinsic  $S_3$  symmetry. Recently these equations were factorized with respect to this  $S_3$  symmetry and the factor equations were transformed into  $E$ -form. As appears, the transformed factor equations are explicitly solvable. Based on this solution, polynomial inverse problems are formulated in the present paper.

## 1. INTRODUCTION.

Perfect cuboids are described by the following four polynomial equations:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 - L^2 &= 0, & x_2^2 + x_3^2 - d_1^2 &= 0, \\ x_3^2 + x_1^2 - d_2^2 &= 0, & x_1^2 + x_2^2 - d_3^2 &= 0. \end{aligned} \tag{1.1}$$

Here  $x_1, x_2, x_3$  are edges of a cuboid,  $d_1, d_2, d_3$  are its face diagonals, and  $L$  is its space diagonal. For the history of perfect cuboids the reader is referred to [1–44].

Recently in [45] the symmetry approach to the equations (1.1) was initiated. It is based on an intrinsic  $S_3$  symmetry of these equations. Indeed, let's consider the following action of the group  $S_3$  upon the variables  $x_1, x_2, x_3, d_1, d_2, d_3$  and  $L$ :

$$\sigma(x_i) = x_{\sigma i}, \quad \sigma(d_i) = d_{\sigma i}, \quad \sigma(L) = L. \tag{1.2}$$

The first equation (1.1) is invariant with respect to the transformations (1.2). The other three equations are not invariant, but the system as a whole is again invariant, i. e. it possesses  $S_3$  symmetry based on the transformations (1.2). In [46] the equations (1.1) were factorized with respect to their  $S_3$  symmetry and the following system of eight factor equations was derived:

$$x_1^2 + x_2^2 + x_3^2 - L^2 = 0, \tag{1.3}$$

$$(x_2^2 + x_3^2 - d_1^2) + (x_3^2 + x_1^2 - d_2^2) + (x_1^2 + x_2^2 - d_3^2) = 0, \tag{1.4}$$

$$d_1(x_2^2 + x_3^2 - d_1^2) + d_2(x_3^2 + x_1^2 - d_2^2) + d_3(x_1^2 + x_2^2 - d_3^2) = 0, \tag{1.5}$$

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$$x_1 (x_2^2 + x_3^2 - d_1^2) + x_2 (x_3^2 + x_1^2 - d_2^2) + x_3 (x_1^2 + x_2^2 - d_3^2) = 0, \quad (1.6)$$

$$\begin{aligned} x_1 d_1 (x_2^2 + x_3^2 - d_1^2) + x_2 d_2 (x_3^2 + x_1^2 - d_2^2) + \\ + x_3 d_3 (x_1^2 + x_2^2 - d_3^2) = 0, \end{aligned} \quad (1.7)$$

$$x_1^2 (x_2^2 + x_3^2 - d_1^2) + x_2^2 (x_3^2 + x_1^2 - d_2^2) + x_3^2 (x_1^2 + x_2^2 - d_3^2) = 0, \quad (1.8)$$

$$d_1^2 (x_2^2 + x_3^2 - d_1^2) + d_2^2 (x_3^2 + x_1^2 - d_2^2) + d_3^2 (x_1^2 + x_2^2 - d_3^2) = 0, \quad (1.9)$$

$$\begin{aligned} x_1^2 d_1^2 (x_2^2 + x_3^2 - d_1^2) + x_2^2 d_2^2 (x_3^2 + x_1^2 - d_2^2) + \\ + x_3^2 d_3^2 (x_1^2 + x_2^2 - d_3^2) = 0. \end{aligned} \quad (1.10)$$

Each solution of the equations (1.1) is a solution of the equations (1.3), (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), (1.10). But generally speaking, the converse is not true. Fortunately, in [47] the following theorem was proved.

**Theorem 1.1.** *Each integer or rational solution of the equations (1.3) through (1.10) such that  $x_1 > 0$ ,  $x_2 > 0$ ,  $x_3 > 0$ ,  $d_1 > 0$ ,  $d_2 > 0$ , and  $d_3 > 0$  is an integer or rational solution for the equations (1.1).*

Due to Theorem 1.1 the factor equations (1.3) through (1.10) can be applied for studying perfect cuboids.

Note that each of the equations (1.3) through (1.10) is invariant with respect to the transformations (1.2). The left hand sides of these equations are multisymmetric polynomials in the sense of the following definition.

**Definition 1.1.** A polynomial  $p \in \mathbb{Q}[x_1, x_2, x_3, d_1, d_2, d_3, L]$  is called multisymmetric if it is invariant with respect to the action (1.2) of the group  $S_3$ .

Multisymmetric polynomials constitute a subring within the polynomial ring  $\mathbb{Q}[x_1, x_2, x_3, d_1, d_2, d_3, L]$ . For the sake of brevity we introduce the matrix

$$M = \begin{vmatrix} x_1 & x_2 & x_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

and denote  $Q[x_1, x_2, x_3, d_1, d_2, d_3, L] = \mathbb{Q}[M, L]$ . Similarly, the subring of multisymmetric polynomials is denoted through  $\text{Sym}\mathbb{Q}[M, L]$ . For the general theory of multisymmetric polynomials the reader is referred to [48–68]. For our purposes we need the following theorem from this theory.

**Theorem 1.2.** *Each multisymmetric polynomial  $p \in \text{Sym}\mathbb{Q}[x_1, x_2, x_3, d_1, d_2, d_3, L]$  can be expressed through the following elementary multisymmetric polynomials:*

$$\begin{aligned} e_{[1,0]} &= x_1 + x_2 + x_3, & e_{[0,1]} &= d_1 + d_2 + d_3, \\ e_{[2,0]} &= x_1 x_2 + x_2 x_3 + x_3 x_1, & e_{[0,2]} &= d_1 d_2 + d_2 d_3 + d_3 d_1, \\ e_{[3,0]} &= x_1 x_2 x_3, & e_{[0,3]} &= d_1 d_2 d_3, \end{aligned} \quad (1.11)$$

$$\begin{aligned} e_{[2,1]} &= x_1 x_2 d_3 + x_2 x_3 d_1 + x_3 x_1 d_2, \\ e_{[1,1]} &= x_1 d_2 + d_1 x_2 + x_2 d_3 + d_2 x_3 + x_3 d_1 + d_3 x_1, \\ e_{[1,2]} &= x_1 d_2 d_3 + x_2 d_3 d_1 + x_3 d_1 d_2. \end{aligned} \quad (1.12)$$

Theorem 1.2 is known as the Fundamental Theorem on Elementary Multisymmetric Polynomials. Its proof can be found in [54]. If we denote

$$\mathbb{Q}[E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12}, L] = \mathbb{Q}[E, L],$$

then Theorem 1.2 means that for each multisymmetric polynomial  $p$  from the ring  $\text{Sym}\mathbb{Q}[M, L]$  there is some polynomial  $q \in \mathbb{Q}[E, L]$  of ten independent variables such that  $p$  is produced from  $q$  by substituting the elementary multisymmetric polynomials (1.11) and (1.12) for  $E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}$ , and  $E_{12}$  into its arguments. The substitution procedure can be understood as a mapping:

$$\varphi: \mathbb{Q}[E, L] \rightarrow \text{Sym}\mathbb{Q}[M, L]. \quad (1.13)$$

The mapping (1.13) is a ring homomorphism. It is surjective, which follows from Theorem 1.2, but it is not bijective, i. e. it has a nonzero kernel  $K = \text{Ker } \varphi$ . The kernel  $K$  is an ideal in the ring  $\mathbb{Q}[E, L]$ . It was calculated as

$$K = \langle q_1, \dots, q_7 \rangle, \quad (1.14)$$

where  $q_1, \dots, q_7$  are seven polynomials being a basis of  $K$ . These polynomials are given by the explicit formulas (2.4) through (2.10) in [46]. The ideal (1.14) has a Gröbner basis comprising fourteen polynomials:

$$K = \langle \tilde{q}_1, \tilde{q}_2, \tilde{q}_3, \tilde{q}_4, \tilde{q}_5, \tilde{q}_6, \tilde{q}_7, \tilde{q}_8, \tilde{q}_9, \tilde{q}_{10}, \tilde{q}_{11}, \tilde{q}_{12}, \tilde{q}_{13}, \tilde{q}_{14} \rangle. \quad (1.15)$$

The polynomials  $q_1$  through  $q_{14}$  from (1.15) were calculated in [46] with the use of the Maxima symbolic computations package, but explicit formulas for them were not given. These formulas were given later in Appendix to [69]. For the theory of Gröbner bases and their applications the reader is referred to [70].

Returning to the factor equations (1.3) through (1.10) and noting that their left hand sides are multisymmetric polynomials, one can apply Theorem 1.2 to them. This was done in [69] and the  $E$ -forms<sup>1</sup> of the factor equations were derived in [69]. Here are the transformed factor equations:

$$E_{10}^2 - 2 E_{20} - L^2 = 0, \quad (1.16)$$

$$2 E_{02} - 4 E_{20} - E_{01}^2 + 2 E_{10}^2 = 0, \quad (1.17)$$

$$E_{10} E_{11} - 3 E_{03} - E_{21} + 3 E_{01} E_{02} - E_{20} E_{01} - E_{01}^3 = 0, \quad (1.18)$$

$$E_{01} E_{11} - E_{12} - 3 E_{30} + E_{10} E_{02} + E_{20} E_{10} - E_{01}^2 E_{10} = 0, \quad (1.19)$$

$$\begin{aligned} -E_{10} E_{21} - E_{01} E_{12} - E_{01} E_{30} - E_{01}^3 E_{10} + E_{01}^2 E_{11} - \\ - E_{02} E_{11} + E_{11} E_{20} - E_{10} E_{03} + 2 E_{10} E_{01} E_{02} = 0. \end{aligned} \quad (1.20)$$

$$\begin{aligned} 4 E_{01} E_{10} E_{11} - 3 E_{01}^2 E_{10}^2 + 2 E_{10}^2 E_{02} + 2 E_{20} E_{01}^2 - 2 E_{10} E_{12} - \\ - 2 E_{02} E_{20} - 2 E_{01} E_{21} - E_{11}^2 - 12 E_{10} E_{30} + 6 E_{20}^2 = 0, \end{aligned} \quad (1.21)$$

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<sup>1</sup> The  $E$ -form of a polynomial is its preimage under the mapping (1.13).

$$4 E_{01} E_{10} E_{11} - 4 E_{10}^2 E_{02} - 4 E_{20} E_{01}^2 - 2 E_{10} E_{12} + 10 E_{02} E_{20} - \\ - 2 E_{01} E_{21} - E_{11}^2 - 12 E_{01} E_{03} - 3 E_{01}^4 - 6 E_{02}^2 + 12 E_{01}^2 E_{02} = 0, \quad (1.22)$$

$$9 E_{01} E_{03} E_{20} - 7 E_{01}^2 E_{02} E_{20} + 2 E_{02} E_{10} E_{12} - 2 E_{01}^2 E_{10} E_{12} + \\ + 3 E_{03} E_{10} E_{11} + 4 E_{01}^3 E_{10} E_{11} - 7 E_{01} E_{02} E_{10} E_{11} - 6 E_{01} E_{03} E_{10}^2 + \\ + 8 E_{01}^2 E_{02} E_{10}^2 + 3 E_{01} E_{11} E_{30} - 2 E_{01} E_{20} E_{21} + E_{10} E_{12} E_{20} - \\ - E_{02} E_{10}^2 E_{20} + E_{01} E_{10} E_{11} E_{20} + 9 E_{02} E_{10} E_{30} - 2 E_{02} E_{20}^2 + \\ + 2 E_{01}^2 E_{20} - E_{11}^2 E_{20} - 3 E_{12} E_{30} + E_{02} E_{11}^2 - E_{01}^2 E_{11}^2 - \\ - 2 E_{02}^2 E_{10}^2 + 2 E_{01}^4 E_{20} + 2 E_{02}^2 E_{20} - 3 E_{03} E_{21} - \\ - 2 E_{01}^3 E_{21} + 5 E_{01} E_{02} E_{21} - 6 E_{01}^2 E_{10} E_{30} - 3 E_{01}^4 E_{10}^2 = 0. \quad (1.23)$$

In [69] the factor equations (1.16), (1.17), (1.18), (1.19), (1.20), (1.21), (1.22), and (1.23) were complemented with fourteen kernel equations

$$\begin{aligned} \tilde{q}_1 &= 0, & \tilde{q}_2 &= 0, & \tilde{q}_3 &= 0, \\ \tilde{q}_4 &= 0, & \tilde{q}_5 &= 0, & \tilde{q}_6 &= 0, \\ \tilde{q}_7 &= 0, & \tilde{q}_8 &= 0, & \tilde{q}_9 &= 0, \\ \tilde{q}_{10} &= 0, & \tilde{q}_{11} &= 0, & \tilde{q}_{12} &= 0, \\ \tilde{q}_{13} &= 0, & \tilde{q}_{14} &= 0. \end{aligned} \quad (1.24)$$

As a result a huge system of twenty two polynomial equations with respect to ten variables was obtained. In [69] this system was analyzed and luckily was reduced to a single equation. Here is this equation:

$$(2 E_{11})^2 + (E_{01}^2 + L^2 - E_{10}^2)^2 - 8 E_{01}^2 L^2 = 0. \quad (1.25)$$

Equation (1.25) turns out to be explicitly solvable, which is also a lucky event. Its general solution was discovered in [71]. This general solution of (1.25), upon eliminating  $L$  by homogeneity, consists of one two-parameter solution and a series of one-parameter solutions. The main goal of the present paper is to propagate these explicit solutions back to the equations (1.16) through (1.24) and formulate polynomial inverse problems relating these solutions with perfect cuboids.

## 2. RATIONAL PERFECT CUBOIDS.

Let  $x_1, x_2, x_3, d_1, d_2, d_3$  be edges and face diagonals of some perfect cuboid and let  $L$  be its space diagonal. Then, dividing these numbers by  $L$ , we get a cuboid whose edges and face diagonals are given by rational numbers, while the space diagonal is equal to unity. Such a cuboid is called a rational perfect cuboid. Conversely, if we have a rational perfect cuboid with unit space diagonal, we can take the common denominator of the rational numbers  $x_1, x_2, x_3, d_1, d_2, d_3$  for  $L$  and then, multiplying these numbers by  $L$ , we get a perfect cuboid with integer edges and face diagonals whose space diagonal is equal to the integer number  $L$ . Thus, integer perfect cuboids and rational perfect cuboids with unit space diagonal are equivalent to each other.

The equivalence of perfect cuboids and rational perfect cuboids was already used in [41] for deriving three cuboid conjectures. In the present paper we use this fact

by setting  $L = 1$  and thus reducing the number of variables in the equations (1.16) through (1.23) and in the equation (1.25).

### 3. THE TWO-PARAMETER CASE.

Let's substitute  $L = 1$  into (1.25). As a result, using the notation  $E_{11} = x$ ,  $E_{01} = y$ ,  $E_{10} = z$ , we get the equation coinciding with the equation (1.1) in [71]:

$$(2 E_{11})^2 + (E_{01}^2 + 1 - E_{10}^2)^2 = 8 E_{01}^2. \quad (3.1)$$

This equation was solved in [71]. Theorem 2.1 from [71] yields the following two-parameter solution of the equation (3.1):

$$E_{11} = -\frac{b(c^2 + 2 - 4c)}{b^2 c^2 + 2b^2 - 3b^2 c + c - b c^2 + 2b}, \quad (3.2)$$

$$E_{01} = -\frac{b(c^2 + 2 - 2c)}{b^2 c^2 + 2b^2 - 3b^2 c + c - b c^2 + 2b}, \quad (3.3)$$

$$E_{10} = -\frac{b^2 c^2 + 2b^2 - 3b^2 c - c}{b^2 c^2 + 2b^2 - 3b^2 c + c - b c^2 + 2b}. \quad (3.4)$$

The numbers  $b$  and  $c$  are two rational parameters in (3.2), (3.3), and (3.4).

Apart from  $E_{11}$ ,  $E_{10}$ , and  $E_{01}$  there are six other variables  $E_{20}$ ,  $E_{30}$ ,  $E_{02}$ ,  $E_{03}$ ,  $E_{21}$ ,  $E_{12}$  in the equations (1.16) through (1.23). These variables are expressed through  $E_{11}$ ,  $E_{10}$ ,  $E_{01}$ , and  $L$  by means of the formulas (4.1), (4.3), (4.6), (4.7), (5.1), and (5.2) from [69]. Here are the formulas for  $E_{20}$  and  $E_{02}$ :

$$E_{20} = \frac{1}{2} E_{10}^2 - \frac{1}{2} L^2, \quad E_{02} = \frac{1}{2} E_{01}^2 - L^2. \quad (3.5)$$

Substituting  $L = 1$ , (3.3), and (3.4) into (3.5), we derive

$$E_{20} = \frac{b}{2} (b c^2 - 2c - 2b) (2b c^2 - c^2 - 6bc + 2 + 4b) \times \\ \times (bc - 1 - b)^{-2} (bc - c - 2b)^{-2}, \quad (3.6)$$

$$E_{02} = \frac{1}{2} (28b^2 c^2 - 16b^2 c - 2c^2 - 4b^2 - b^2 c^4 + \\ + 4b^3 c^4 - 12b^3 c^3 + 4b c^3 + 24b^3 c - 8bc - 2b^4 c^4 + \\ + 12b^4 c^3 - 26b^4 c^2 - 8b^2 c^3 + 24b^4 c - 16b^3 - 8b^4) \times \\ \times (bc - 1 - b)^{-2} (bc - c - 2b)^{-2}. \quad (3.7)$$

The formulas for  $E_{21}$  and  $E_{12}$  are taken from (5.1) and (5.2) in [69] respectively:

$$E_{21} = \frac{2 E_{10}^3 E_{11} + 2 E_{01}^2 E_{10} E_{11} - E_{01} E_{10}^4 + E_{01}^5}{8 (E_{01}^2 + E_{10}^2)} + \\ + \frac{6 E_{10} E_{11} L^2 - 2 E_{01} E_{10}^2 L^2 - 8 E_{01}^3 L^2 + 3 E_{01} L^4}{8 (E_{01}^2 + E_{10}^2)}, \quad (3.8)$$

$$\begin{aligned}
E_{12} = & \frac{E_{01}^4 E_{10} - 2 E_{01}^3 E_{11} - 2 E_{01} E_{10}^2 E_{11} - E_{10}^5}{8(E_{01}^2 + E_{10}^2)} + \\
& + \frac{6 E_{10}^3 L^2 - 6 E_{01} E_{11} L^2 + 3 E_{10} L^4}{8(E_{01}^2 + E_{10}^2)}. \tag{3.9}
\end{aligned}$$

Substituting  $L = 1$ , (3.2), (3.3), and (3.4) into (3.8) and (3.9), we derive

$$\begin{aligned}
E_{21} = & \frac{b}{2} (5 c^6 b - 2 c^6 b^2 + 52 c^5 b^2 - 16 c^5 b - 2 c^7 b^2 + 2 b^4 c^8 + \\
& + 142 b^4 c^6 - 26 b^4 c^7 - 426 b^4 c^5 - 61 b^3 c^6 + 100 b^3 c^5 + 14 c^7 b^3 - \\
& - c^8 b^3 - 20 b c^2 - 8 b^2 c^2 - 16 b^2 c - 128 b^2 c^4 - 200 b^3 c^3 + \\
& + 244 b^3 c^2 + 32 b c^3 - 112 b^3 c + 768 b^4 c^4 - 852 b^4 c^3 + 568 b^4 c^2 + \\
& + 104 b^2 c^3 - 208 b^4 c + 8 c^4 - 4 c^3 + 16 b^3 + 32 b^4 - 2 c^5) \times \\
& \times (b^2 c^4 - 6 b^2 c^3 + 13 b^2 c^2 - 12 b^2 c + 4 b^2 + c^2)^{-1} \times \\
& \times (b c - 1 - b)^{-2} (b c - c - 2 b)^{-2}, \tag{3.10}
\end{aligned}$$

$$\begin{aligned}
E_{12} = & (16 b^6 + 32 b^5 - 6 c^5 b^2 + 2 c^5 b - 62 b^5 c^6 + 62 b^6 c^6 - \\
& - 180 b^6 c^5 + 18 b^5 c^7 - 12 b^6 c^7 - 2 b^5 c^8 + b^6 c^8 + 248 b^5 c^2 + \\
& + 248 b^6 c^2 - 96 b^6 c + 321 b^6 c^4 - 180 b^5 c^3 - 144 b^5 c - 360 b^6 c^3 + \\
& + b^4 c^8 + 8 b^4 c^6 - 6 b^4 c^7 + 18 b^4 c^5 + 7 b^3 c^6 + 90 b^5 c^5 - 14 b^3 c^5 - \\
& - c^7 b^3 + 17 b^2 c^4 + 28 b^3 c^3 - 28 b^3 c^2 - 4 b c^3 + 8 b^3 c - 57 b^4 c^4 + \\
& + 36 b^4 c^3 + 32 b^4 c^2 - 12 b^2 c^3 - 48 b^4 c - c^4 + 16 b^4) \times \\
& \times (b^2 c^4 - 6 b^2 c^3 + 13 b^2 c^2 - 12 b^2 c + 4 b^2 + c^2)^{-1} \times \\
& \times (b c - 1 - b)^{-2} (b c - c - 2 b)^{-2}. \tag{3.11}
\end{aligned}$$

The formulas for  $E_{30}$  and  $E_{03}$  are taken from (4.6) and (4.7) in [69]:

$$E_{30} = -\frac{1}{3} E_{12} - \frac{1}{6} E_{10} E_{01}^2 - \frac{1}{2} E_{10} L^2 + \frac{1}{6} E_{10}^3 + \frac{1}{3} E_{01} E_{11}, \tag{3.12}$$

$$E_{03} = -\frac{1}{3} E_{21} - \frac{1}{6} E_{01} E_{10}^2 - \frac{5}{6} E_{01} L^2 + \frac{1}{6} E_{01}^3 + \frac{1}{3} E_{10} E_{11}. \tag{3.13}$$

The formulas (3.12) and (3.13) comprise  $E_{21}$  and  $E_{12}$  from (3.10) and (3.11). Applying  $L = 1$ , (3.10), (3.11), (3.2), (3.3), and (3.4) to (3.12) and (3.13), we get

$$\begin{aligned}
E_{03} = & \frac{b}{2} (b^2 c^4 - 5 b^2 c^3 + 10 b^2 c^2 - 10 b^2 c + 4 b^2 + 2 b c + \\
& + 2 c^2 - b c^3) (2 b^2 c^4 - 12 b^2 c^3 + 26 b^2 c^2 - 24 b^2 c + \\
& + 8 b^2 - c^4 b + 3 b c^3 - 6 b c + 4 b + c^3 - 2 c^2 + 2 c) \times \\
& \times ((b^2 c^4 - 6 b^2 c^3 + 13 b^2 c^2 - 12 b^2 c + 4 b^2 + c^2)^{-1} \times \\
& \times (b c - 1 - b)^{-2} (-c + b c - 2 b)^{-2}, \tag{3.14}
\end{aligned}$$

$$\begin{aligned}
 E_{30} &= c b^2 (1 - c) (c - 2) (b c^2 - 4 b c + 2 + 4 b) \times \\
 &\quad \times (2 b c^2 - c^2 - 4 b c + 2 b) \times \\
 &\quad \times (b^2 c^4 - 6 b^2 c^3 + 13 b^2 c^2 - 12 b^2 c + 4 b^2 + c^2)^{-1} \times \\
 &\quad \times (b c - 1 - b)^{-2} (-c + b c - 2 b)^{-2}.
 \end{aligned} \tag{3.15}$$

Note that the quantities  $E_{10}$ ,  $E_{20}$ , and  $E_{30}$  given by the formulas (3.4), (3.6), (3.15) are the values of three elementary multisymmetric polynomials in the left column of (1.11). One can easily see that these polynomials coincide with the regular symmetric polynomials of three variables  $x_1$ ,  $x_2$ , and  $x_3$  (see [72]). Therefore, if we write the cubic equation  $(x - x_1)(x - x_2)(x - x_3) = 0$ , it expands to

$$x^3 - E_{10} x^2 + E_{20} x - E_{30} = 0. \tag{3.16}$$

Similarly, the quantities  $E_{01}$ ,  $E_{02}$ , and  $E_{03}$  given by the formulas (3.3), (3.7), and (3.14) are the values of the elementary multisymmetric polynomials in the right column of (1.11). These three polynomials coincide with regular symmetric polynomials of three variables  $d_1$ ,  $d_2$ , and  $d_3$ . For this reason, if we write the cubic equation  $(d - d_1)(d - d_2)(d - d_3) = 0$ , this equation expands to

$$d^3 - E_{01} d^2 + E_{02} d - E_{03} = 0. \tag{3.17}$$

The remaining three quantities  $E_{11}$ ,  $E_{21}$ , and  $E_{12}$  given by the formulas (3.2), (3.10), (3.11) are the values of three elementary multisymmetric polynomials in (1.12). They lead to the following auxiliary polynomial equations:

$$\begin{aligned}
 x_1 x_2 d_3 + x_2 x_3 d_1 + x_3 x_1 d_2 &= E_{21}, \\
 x_1 d_2 + d_1 x_2 + x_2 d_3 + d_2 x_3 + x_3 d_1 + d_3 x_1 &= E_{11}, \\
 x_1 d_2 d_3 + x_2 d_3 d_1 + x_3 d_1 d_2 &= E_{12}.
 \end{aligned} \tag{3.18}$$

Now we can formulate the following two inverse cuboid problems.

**Problem 3.1.** *Find all pairs of rational numbers  $b$  and  $c$  for which the cubic equations (3.16) and (3.17) with the coefficients given by the formulas (3.4), (3.6), (3.15), (3.3), (3.7), (3.14) possess positive rational roots  $x_1$ ,  $x_2$ ,  $x_3$ ,  $d_1$ ,  $d_2$ ,  $d_3$  obeying the auxiliary polynomial equations (3.18) whose right hand sides are given by the formulas (3.2), (3.10), (3.11).*

**Problem 3.2.** *Find at least one pair of rational numbers  $b$  and  $c$  for which the cubic equations (3.16) and (3.17) with the coefficients given by the formulas (3.4), (3.6), (3.15), (3.3), (3.7), (3.14) possess positive rational roots  $x_1$ ,  $x_2$ ,  $x_3$ ,  $d_1$ ,  $d_2$ ,  $d_3$  obeying the auxiliary polynomial equations (3.18) whose right hand sides are given by the formulas (3.2), (3.10), (3.11).*

#### 4. THE FIRST ONE-PARAMETER CASE.

The four one-parameter cases considered below correspond to the special solutions of Theorem 2.1 in [71]. The first one-parameter case corresponds to the following choice of sign options in this theorem:

$$x = y, \qquad y = z + 1. \tag{4.1}$$

Equations (4.1) are easily resolved in the one-parameter form:

$$x = c, \quad y = c, \quad z = c - 1. \quad (4.2)$$

Taking into account the notations  $E_{11} = x$ ,  $E_{01} = y$ ,  $E_{10} = z$ , we derive the following one-parameter solution of the equation (3.1) from (4.2):

$$E_{11} = c, \quad E_{01} = c, \quad E_{10} = c - 1. \quad (4.3)$$

Substituting  $L = 1$  and (4.3) into (3.5), (3.8), (3.9), (3.12), and (3.13), we get

$$\begin{aligned} E_{20} &= \frac{c^2 - 2c}{2}, & E_{02} &= \frac{c^2 - 2}{2}, \\ E_{21} &= \frac{c^2 - 2c}{2}, & E_{12} &= 1, \\ E_{30} &= 0, & E_{03} &= \frac{c^2 - 2c}{2}. \end{aligned} \quad (4.4)$$

In the formulas (4.4) we find  $E_{30} = 0$ , i. e. the last term  $E_{30}$  of the cubic equation (3.16) is zero. Then one of its roots  $x_1$ ,  $x_2$ , or  $x_3$  is zero. But a non-degenerate cuboid cannot have a zero edge. As a result we have the following theorem.

**Theorem 4.1.** *There are no perfect cuboids associated with the one-parameter solution (4.3) of the equation (3.1).*

## 5. THE SECOND ONE-PARAMETER CASE.

In the second one-parameter case we choose the following sign options in Theorem 2.1 from [71]:  $x = y$  and  $y = -z - 1$ . Then instead of (4.3) we derive

$$E_{11} = c, \quad E_{01} = c, \quad E_{10} = -c - 1. \quad (5.1)$$

Substituting  $L = 1$  and (5.1) into (3.5), (3.8), (3.9), (3.12), and (3.13), we get

$$\begin{aligned} E_{20} &= \frac{c^2 + 2c}{2}, & E_{02} &= \frac{c^2 - 2}{2}, \\ E_{21} &= -\frac{c^2 + 2c}{2}, & E_{12} &= 1, \\ E_{30} &= 0, & E_{03} &= -\frac{c^2 + 2c}{2}. \end{aligned} \quad (5.2)$$

In (5.2) we again see  $E_{30} = 0$ . Therefore we can formulate the following theorem.

**Theorem 5.1.** *There are no perfect cuboids associated with the one-parameter solution (5.1) of the equation (3.1).*

## 6. THE THIRD AND THE FOURTH ONE-PARAMETER CASES.

These two one-parameter cases correspond to the last two sign options in Theorem 2.1 from [71]. These two sign options are given by the formulas  $x = -y$  and



$y = \pm(z + 1)$ . They yield the following one-parameter solutions for (3.1):

$$E_{11} = c, \quad E_{01} = -c, \quad E_{10} = \pm c - 1. \quad (6.1)$$

The formulas  $E_{11} = c$  and  $E_{01} = -c$  mean that  $E_{11}$  and  $E_{01}$  are of opposite signs or both of them are zero. On the other hand, we have two equalities

$$\begin{aligned} x_1 d_2 + d_1 x_2 + x_2 d_3 + d_2 x_3 + x_3 d_1 + d_3 x_1 &= E_{11}, \\ d_1 + d_2 + d_3 &= E_{01} \end{aligned}$$

derived from (1.11) and (1.12). The left hand sides of these equalities are positive numbers since edges and face diagonals of a perfect cuboid are positive. The contradiction obtained leads to the following theorem.

**Theorem 6.1.** *There are no perfect cuboids associated with the one-parameter solutions (6.1) of the equation (3.1).*

## 7. CONCLUSIONS.

Theorems 4.1, 5.1, and 6.1 show that the inverse problem 3.1 formulated in Section 3 is equivalent to finding all perfect cuboids, while the problem 3.2 is equivalent to finding at least one perfect cuboid.

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CSR PLC, CAMBRIDGE BUSINESS PARK, CAMBRIDGE, CB4 0WZ, UK  
 E-mail address: [jhnrmsdn@yahoo.co.uk](mailto:jhnrmsdn@yahoo.co.uk)

BASHKIR STATE UNIVERSITY, 32 ZAKI VALIDI STREET, 450074 UFA, RUSSIA  
 E-mail address: [r-sharipov@mail.ru](mailto:r-sharipov@mail.ru)