

**ON A PAIR OF CUBIC EQUATIONS
ASSOCIATED WITH PERFECT CUBOIDS.**

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ABSTRACT. A perfect cuboid is a rectangular parallelepiped with integer edges and integer face diagonals whose space diagonal is also integer. The existence of such cuboids is neither proved, nor disproved. A rational perfect cuboid is a natural companion of a perfect cuboid absolutely equivalent to the latter one. Its edges and face diagonals are rational numbers, while its space diagonal is equal to unity. Recently, based on a symmetry reduction, it was shown that edges of a rational perfect cuboid are roots of a certain cubic equation with rational coefficients depending on two rational parameters. Face diagonals of this cuboid are roots of another cubic equation whose coefficients are rational numbers depending on the same two rational parameters. In the present paper these two cubic equations are studied for reducibility. Six special cases of their reducibility over the field of rational numbers are found.

1. INTRODUCTION.

The problem of a perfect cuboid is known since 1719, but is still not resolved. For the history of this problem the reader is referred to [1–44]. Let x_1, x_2, x_3 be edges of a cuboid, d_1, d_2, d_3 be its face diagonals, and L be its space diagonal. Then the cuboid is described by the following four polynomial equations:

$$\begin{aligned} x_1^2 + x_2^2 + x_3^2 - L^2 &= 0, & x_2^2 + x_3^2 - d_1^2 &= 0, \\ x_3^2 + x_1^2 - d_2^2 &= 0, & x_1^2 + x_2^2 - d_3^2 &= 0. \end{aligned} \tag{1.1}$$

This paper continues the series of papers [45–50] applying the symmetry approach to the equations (1.1). Indeed, using three numbers x_1, x_2, x_3 , one can build the cubic equation $(x - x_1)(x - x_2)(x - x_3) = 0$ which expands to

$$x^3 - E_{10} x^2 + E_{20} x - E_{30} = 0. \tag{1.2}$$

Similarly, the equation $(d - d_1)(d - d_2)(d - d_3) = 0$ expands to

$$d^3 - E_{01} d^2 + E_{02} d - E_{03} = 0. \tag{1.3}$$

The coefficients E_{10}, E_{20} , and E_{30} of the equation (1.2) are elementary symmetric polynomials of three variables x_1, x_2, x_3 (see [51]). Similarly, the coefficients E_{01}, E_{02} , and E_{03} of the equation (1.3) are elementary symmetric polynomials of three

variables d_1, d_2, d_3 . Here are the formulas for these polynomials:

$$\begin{aligned} x_1 + x_2 + x_3 &= E_{10}, & d_1 + d_2 + d_3 &= E_{01}, \\ x_1 x_2 + x_2 x_3 + x_3 x_1 &= E_{20}, & d_1 d_2 + d_2 d_3 + d_3 d_1 &= E_{02}, \\ x_1 x_2 x_3 &= E_{30}, & d_1 d_2 d_3 &= E_{03}. \end{aligned} \quad (1.4)$$

Mixing x_1, x_2, x_3 with d_1, d_2, d_3 , one can write the following formulas:

$$\begin{aligned} x_1 x_2 d_3 + x_2 x_3 d_1 + x_3 x_1 d_2 &= E_{21}, \\ x_1 d_2 + d_1 x_2 + x_2 d_3 + d_2 x_3 + x_3 d_1 + d_3 x_1 &= E_{11}, \\ x_1 d_2 d_3 + x_2 d_3 d_1 + x_3 d_1 d_2 &= E_{12}. \end{aligned} \quad (1.5)$$

The left hand sides of the formulas (1.4) complemented with the left hand sides of the formulas (1.5) constitute the complete set of so-called elementary multisymmetric polynomials. For the theory of multisymmetric polynomials, either elementary and non-elementary, the reader is referred to [52–72].

The cuboid equations (1.1) imply some equations for $E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}$, and E_{12} in (1.4) and (1.5). These equations are called factor equations. They were studied in [46] and [47] using ideals in polynomial rings and their Gröbner bases (see the general theory in [73]). In [48] the factor equations were reduced to a single biquadratic equation for three variables E_{10}, E_{01} , and E_{11} :

$$(2 E_{11})^2 + (E_{01}^2 + L^2 - E_{10}^2)^2 - 8 E_{01}^2 L^2 = 0. \quad (1.6)$$

The other variables $E_{20}, E_{30}, E_{02}, E_{03}, E_{21}$, and E_{12} are expressed through E_{10}, E_{01} , and E_{11} by means of the following formulas:

$$E_{20} = \frac{1}{2} E_{10}^2 - \frac{1}{2} L^2, \quad E_{02} = \frac{1}{2} E_{01}^2 - L^2, \quad (1.7)$$

$$\begin{aligned} E_{21} &= \frac{2 E_{10}^3 E_{11} + 2 E_{01}^2 E_{10} E_{11} - E_{01} E_{10}^4 + E_{01}^5}{8 (E_{01}^2 + E_{10}^2)} + \\ &+ \frac{6 E_{10} E_{11} L^2 - 2 E_{01} E_{10}^2 L^2 - 8 E_{01}^3 L^2 + 3 E_{01} L^4}{8 (E_{01}^2 + E_{10}^2)}, \end{aligned} \quad (1.8)$$

$$\begin{aligned} E_{12} &= \frac{E_{01}^4 E_{10} - 2 E_{01}^3 E_{11} - 2 E_{01} E_{10}^2 E_{11} - E_{10}^5}{8 (E_{01}^2 + E_{10}^2)} + \\ &+ \frac{6 E_{10}^3 L^2 - 6 E_{01} E_{11} L^2 + 3 E_{10} L^4}{8 (E_{01}^2 + E_{10}^2)}, \end{aligned} \quad (1.9)$$

$$E_{30} = -\frac{1}{3} E_{12} - \frac{1}{6} E_{10} E_{01}^2 - \frac{1}{2} E_{10} L^2 + \frac{1}{6} E_{10}^3 + \frac{1}{3} E_{01} E_{11}, \quad (1.10)$$

$$E_{03} = -\frac{1}{3} E_{21} - \frac{1}{6} E_{01} E_{10}^2 - \frac{5}{6} E_{01} L^2 + \frac{1}{6} E_{01}^3 + \frac{1}{3} E_{10} E_{11}. \quad (1.11)$$

A general solution for the equation (1.6) was derived in [49], including a two-parameter solution and several one-parameter solutions. As it was proved in [50], the one-parameter solutions do not lead to perfect cuboids. The two-parameter solution of (1.6) is written in [49] for the case of a rational cuboid with unit space diagonal $L = 1$. This solution with the parameters b and c is given by the formulas

$$E_{11} = -\frac{b(c^2 + 2 - 4c)}{b^2 c^2 + 2b^2 - 3b^2 c + c - b c^2 + 2b}, \quad (1.12)$$

$$E_{01} = -\frac{b(c^2 + 2 - 2c)}{b^2 c^2 + 2b^2 - 3b^2 c + c - b c^2 + 2b}, \quad (1.13)$$

$$E_{10} = -\frac{b^2 c^2 + 2b^2 - 3b^2 c - c}{b^2 c^2 + 2b^2 - 3b^2 c + c - b c^2 + 2b}. \quad (1.14)$$

Substituting (1.12), (1.13), and (1.14) into the formulas (1.7), (1.8), (1.9), (1.10), (1.11) and taking into account that $L = 1$, one can find that

$$E_{20} = \frac{b}{2} (b c^2 - 2c - 2b) (2b c^2 - c^2 - 6bc + 2 + 4b) \times \\ \times (bc - 1 - b)^{-2} (bc - c - 2b)^{-2}, \quad (1.15)$$

$$E_{02} = \frac{1}{2} (28b^2 c^2 - 16b^2 c - 2c^2 - 4b^2 - b^2 c^4 + 4b^3 c^4 - 12b^3 c^3 + \\ + 4bc^3 + 24b^3 c - 8bc - 2b^4 c^4 + 12b^4 c^3 - 26b^4 c^2 - 8b^2 c^3 + \\ + 24b^4 c - 16b^3 - 8b^4) (bc - 1 - b)^{-2} (bc - c - 2b)^{-2}. \quad (1.16)$$

$$E_{21} = \frac{b}{2} (5c^6 b - 2c^6 b^2 + 52c^5 b^2 - 16c^5 b - 2c^7 b^2 + 2b^4 c^8 + \\ + 142b^4 c^6 - 26b^4 c^7 - 426b^4 c^5 - 61b^3 c^6 + 100b^3 c^5 + 14c^7 b^3 - \\ - c^8 b^3 - 20bc^2 - 8b^2 c^2 - 16b^2 c - 128b^2 c^4 - 200b^3 c^3 + \\ + 244b^3 c^2 + 32bc^3 - 112b^3 c + 768b^4 c^4 - 852b^4 c^3 + 568b^4 c^2 + \\ + 104b^2 c^3 - 208b^4 c + 8c^4 - 4c^3 + 16b^3 + 32b^4 - 2c^5) \times \\ \times (b^2 c^4 - 6b^2 c^3 + 13b^2 c^2 - 12b^2 c + 4b^2 + c^2)^{-1} \times \\ \times (bc - 1 - b)^{-2} (bc - c - 2b)^{-2}, \quad (1.17)$$

$$E_{12} = (16b^6 + 32b^5 - 6c^5 b^2 + 2c^5 b - 62b^5 c^6 + 62b^6 c^6 - \\ - 180b^6 c^5 + 18b^5 c^7 - 12b^6 c^7 - 2b^5 c^8 + b^6 c^8 + 248b^5 c^2 + \\ + 248b^6 c^2 - 96b^6 c + 321b^6 c^4 - 180b^5 c^3 - 144b^5 c - 360b^6 c^3 + \\ + b^4 c^8 + 8b^4 c^6 - 6b^4 c^7 + 18b^4 c^5 + 7b^3 c^6 + 90b^5 c^5 - 14b^3 c^5 - \\ - c^7 b^3 + 17b^2 c^4 + 28b^3 c^3 - 28b^3 c^2 - 4bc^3 + 8b^3 c - 57b^4 c^4 + \\ + 36b^4 c^3 + 32b^4 c^2 - 12b^2 c^3 - 48b^4 c - c^4 + 16b^4) \times \\ \times (b^2 c^4 - 6b^2 c^3 + 13b^2 c^2 - 12b^2 c + 4b^2 + c^2)^{-1} \times \\ \times (bc - 1 - b)^{-2} (bc - c - 2b)^{-2}. \quad (1.18)$$

$$\begin{aligned}
E_{03} = & \frac{b}{2} (b^2 c^4 - 5 b^2 c^3 + 10 b^2 c^2 - 10 b^2 c + 4 b^2 + 2 b c + \\
& + 2 c^2 - b c^3) (2 b^2 c^4 - 12 b^2 c^3 + 26 b^2 c^2 - 24 b^2 c + \\
& + 8 b^2 - c^4 b + 3 b c^3 - 6 b c + 4 b + c^3 - 2 c^2 + 2 c) \times \\
& \times ((b^2 c^4 - 6 b^2 c^3 + 13 b^2 c^2 - 12 b^2 c + 4 b^2 + c^2)^{-1} \times \\
& \times (b c - 1 - b)^{-2} (-c + b c - 2 b)^{-2},
\end{aligned} \tag{1.19}$$

$$\begin{aligned}
E_{30} = & c b^2 (1 - c) (c - 2) (b c^2 - 4 b c + 2 + 4 b) \times \\
& \times (2 b c^2 - c^2 - 4 b c + 2 b) \times \\
& \times (b^2 c^4 - 6 b^2 c^3 + 13 b^2 c^2 - 12 b^2 c + 4 b^2 + c^2)^{-1} \times \\
& \times (b c - 1 - b)^{-2} (-c + b c - 2 b)^{-2}.
\end{aligned} \tag{1.20}$$

The formulas (1.15), (1.16), (1.17), (1.18), (1.19), and (1.20) were derived in [50]. Based on these formulas, two inverse problems were formulated.

Problem 1.1. *Find all pairs of rational numbers b and c for which the cubic equations (1.2) and (1.3) with the coefficients (1.14), (1.15), (1.20), (1.13), (1.16), (1.19) possess positive rational roots $x_1, x_2, x_3, d_1, d_2, d_3$ obeying the auxiliary equations (1.5) with the right hand sides (1.17), (1.18), (1.12).*

Problem 1.2. *Find at least one pair of rational numbers b and c for which the cubic equations (1.2) and (1.3) with the coefficients (1.14), (1.15), (1.20), (1.13), (1.16), (1.19) possess positive rational roots $x_1, x_2, x_3, d_1, d_2, d_3$ obeying the auxiliary equations (1.5) with the right hand sides (1.17), (1.18), (1.12).*

The problems 1.1 and 1.2 are equivalent to finding all perfect cuboids and to finding at least one perfect cuboid respectively. In the present paper we study the cubic equations (1.2) and (1.3) for reducibility using the methods of [41], which were applied to a twelfth order Diophantine equation in that paper.

2. THE FIRST REDUCIBILITY CASE $b = 0$.

Note that the formulas (1.12) through (1.20) for the coefficients of the cubic equations (1.2) and (1.3) and for the right hand sides of the auxiliary equations (1.5) possess denominators. The simultaneous non-vanishing condition for all of their denominators is written as follows:

$$\begin{aligned}
& (b^2 c^4 - 6 b^2 c^3 + 13 b^2 c^2 - 12 b^2 c + 4 b^2 + c^2) \times \\
& \times (b c - 1 - b) (b c - c - 2 b) \neq 0.
\end{aligned} \tag{2.1}$$

The case $b = 0$ is very simple. The non-vanishing condition (2.1) in this case is written as $c \neq 0$. Substituting $b = 0$ into (1.14), (1.15), (1.20), (1.13), (1.16), (1.19), we find that the cubic equations (1.2) and (1.3) reduce to

$$x^2 (x - 1) = 0, \quad d (d - 1) (d + 1) = 0. \tag{2.2}$$

Substituting $b = 0$ into (1.17), (1.18), (1.12), we obtain

$$E_{21} = 0, \quad E_{11} = 0, \quad E_{12} = -1. \tag{2.3}$$

The equations (2.2) are already factored. We can choose their roots as follows:

$$\begin{aligned} x_1 &= 1, & x_2 &= 0, & x_3 &= 0, \\ d_1 &= 0, & d_2 &= 1, & d_3 &= -1. \end{aligned} \quad (2.4)$$

Substituting (2.3) and (2.4) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Substituting (2.4) along with $L = 1$ into (1.1), we find that the cuboid equations (1.1) are also fulfilled. However, the formulas (2.4) do not provide a perfect cuboid since its edges and face diagonals cannot be zero or negative.

Theorem 2.1. *If $b = 0$ and $c \neq 0$, then the cubic polynomials in (1.2) and (1.3) are reducible and provide three integer roots for each of the equations (1.2) and (1.3) satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

3. THE SECOND REDUCIBILITY CASE $c = 0$.

The case $c = 0$ is also simple. The non-vanishing condition (2.1) in this case turns to $b(1+b) \neq 0$. The cubic equations (1.2) and (1.3) turn to

$$\begin{aligned} x(2(1+b)^2 x^2 + 2b(1+b)x - (1+2b)) &= 0, \\ (d+1)(2(1+b)^2 d^2 - 2b(1+b)d - (1+2b)) &= 0. \end{aligned} \quad (3.1)$$

Substituting $c = 0$ into the formulas (1.17), (1.18), and (1.12), we obtain

$$E_{21} = \frac{1+2b}{2(1+b)^2}, \quad E_{11} = \frac{-1}{1+b}, \quad E_{12} = 1. \quad (3.2)$$

The equations (3.1) are already factored. Upon splitting off the linear terms they turn to quadratic equations very similar to each other:

$$\begin{aligned} 2(1+b)^2 x^2 + 2b(1+b)x - (1+2b) &= 0, \\ 2(1+b)^2 d^2 - 2b(1+b)d - (1+2b) &= 0. \end{aligned} \quad (3.3)$$

The discriminants of the quadratic equations (3.3) do coincide:

$$D = 4(b^2 + 4b + 2)(1+b)^2. \quad (3.4)$$

The formula (3.4) means that in order to solve the equations (3.3) in rational numbers one should resolve the following quadratic equation in rational numbers:

$$b^2 + 4b + 2 = \beta^2. \quad (3.5)$$

The equation (3.5) can be written as $(b+2)^2 - \beta^2 = 2$. The lemma 2.2 from [49] is applicable to this equation. Applying this lemma, we get

$$b = \frac{t^2 - 4t + 2}{2t}, \quad \beta = \frac{t^2 - 2}{2t} \quad (3.6)$$

for some rational $t \neq 0$. Note that the condition $b(1+b) \neq 0$ derived from (2.1) for $c = 0$ implies no restrictions for t since the equations

$$\frac{t^2 - 4t + 2}{2t} = 0, \quad \frac{t^2 - 4t + 2}{2t} = -1$$

have no rational roots. Now, substituting b from (3.6) into the quadratic equations (3.3), we can find their roots. Since the roots $x = 0$ and $d = -1$ of the cubic equations (3.1) are already known, we can write formulas for all their roots:

$$\begin{aligned} x_1 &= 0, & x_2 &= \frac{-t(t-2)}{(t-1)^2+1}, & x_3 &= \frac{2(t-1)}{(t-1)^2+1}, \\ d_1 &= -1, & d_2 &= \frac{-2(t-1)}{(t-1)^2+1}, & d_3 &= \frac{t(t-2)}{(t-1)^2+1}. \end{aligned} \quad (3.7)$$

Taking into account the formula for b in (3.6), we transform (3.2) to

$$E_{21} = \frac{2t(t^2 - 3t + 2)}{(t-1)^2+1}, \quad E_{11} = \frac{-2t}{(t-1)^2+1}, \quad E_{12} = 1. \quad (3.8)$$

Now, if we substitute the formulas (3.7) and (3.8) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (3.7) along with $L = 1$ into (1.1), we find that the cuboid equations (1.1) are also fulfilled. However, the formulas (3.7) do not provide a perfect cuboid since x_1 is zero and d_1 is negative.

Theorem 3.1. *If $c = 0$ and $b(1+b) \neq 0$, then the cubic polynomials in (1.2) and (1.3) are reducible over \mathbb{Q} . Moreover, if b is given by the first formula (3.6) for some rational $t \neq 0$, then each of the cubic equations (1.2) and (1.3) has three rational roots satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

4. THE THIRD REDUCIBILITY CASE $c = 1$.

The case $c = 1$ is similar to the previous case $c = 0$. The non-vanishing condition (2.1) in this case turns to $b \neq -1$. The cubic equations (1.2) and (1.3) turn to

$$\begin{aligned} x(2(1+b)^2 x^2 - (2b+2)x - b(b+2)) &= 0, \\ (d+1)(2(1+b)^2 d^2 - (2b+2)d - b(b+2)) &= 0. \end{aligned} \quad (4.1)$$

Substituting $c = 1$ into the formulas (1.17), (1.18), and (1.12), we obtain

$$E_{21} = \frac{b(b+2)}{2(1+b)^2}, \quad E_{11} = \frac{b}{1+b}, \quad E_{12} = -1. \quad (4.2)$$

The equations (4.1) are already factored. Upon splitting out linear terms they turn to quadratic equations. These two quadratic equations do coincide with each other up to the change of x for d and vice versa:

$$\begin{aligned} 2(1+b)^2 x^2 - (2b+2)x - b(b+2) &= 0, \\ 2(1+b)^2 d^2 - (2b+2)d - b(b+2) &= 0. \end{aligned} \quad (4.3)$$

Certainly, the discriminants of the coinciding equations (4.3) do also coincide:

$$D = 4(2b^2 + 4b + 1)(b + 1)^2. \quad (4.4)$$

The formula (4.4) means that in order to solve the equations (4.3) in rational numbers one should resolve the following quadratic equation in rational numbers:

$$2b^2 + 4b + 1 = \beta^2. \quad (4.5)$$

If $b = 0$, then (4.5) implies $\beta = \pm 1$, which yields two trivial rational solutions for the equation (4.5). If $b \neq 0$, we can write the equation (4.5) as

$$\frac{1}{b^2} + \frac{4}{b} + 2 = \frac{\beta^2}{b^2}. \quad (4.6)$$

The equation (4.6) is very similar to (3.5). Therefore the formulas (3.6) yield

$$\frac{1}{b} = \frac{t^2 - 4t + 2}{2t}, \quad \frac{\beta}{b} = \frac{t^2 - 2}{2t} \quad (4.7)$$

for some rational $t \neq 0$. Then the formulas (4.7) can be transformed to

$$b = \frac{2t}{(t-2)^2 - 2}, \quad \beta = \frac{t^2 - 2}{(t-2)^2 - 2}. \quad (4.8)$$

Note that the formulas (4.8) are consistent since their denominators cannot vanish for any rational t . Note also that one of the above trivial solutions with $b = 0$ and $\beta = -1$ can be obtained from (4.8) for $t = 0$. In order to cover the other trivial solution with $b = 0$ and $\beta = 1$ we need to add the following sign option to (4.8):

$$b = \frac{2t}{(t-2)^2 - 2}, \quad \beta = \pm \frac{t^2 - 2}{(t-2)^2 - 2}. \quad (4.9)$$

Due to these observations the restriction $t \neq 0$ is removed and the formulas (4.9) cover all rational solutions of the equation (4.5). The condition $b \neq -1$ derived from (2.1) for $c = 1$ implies no restrictions for t since the equation

$$\frac{2t}{(t-2)^2 - 2} = -1$$

has no rational roots. Now we can substitute b from (4.9) into the quadratic equations (4.3) and find their roots. Since the roots $x = 0$ and $d = -1$ of the cubic equations (4.1) are already known, we can write formulas for all their roots:

$$\begin{aligned} x_1 &= 0, & x_2 &= \frac{t(t-2)}{(t-1)^2 + 1}, & x_3 &= \frac{-2(t-1)}{(t-1)^2 + 1}, \\ d_1 &= -1, & d_2 &= \frac{-2(t-1)}{(t-1)^2 + 1}, & d_3 &= \frac{t(t-2)}{(t-1)^2 + 1}. \end{aligned} \quad (4.10)$$

Taking into account the formula for b in (4.9), we transform (4.2) to

$$E_{21} = \frac{2(t^2 - 3t + 2)t}{((t-1)^2 + 1)^2}, \quad E_{11} = \frac{2t}{(t-1)^2 + 1}, \quad E_{12} = -1. \quad (4.11)$$

Now, if we substitute the formulas (4.10) and (4.11) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (4.10) along with $L = 1$ into (1.1), we find that the cuboid equations are also fulfilled. But again, the formulas (4.10) do not provide a perfect cuboid since x_1 is zero and d_1 is negative.

Theorem 4.1. *If $c = 1$ and $b \neq -1$, then the cubic polynomials in (1.2) and (1.3) are reducible over \mathbb{Q} . Moreover, if b is given by the first formula (4.9) for some rational t , then each of the cubic equations (1.2) and (1.3) has three rational roots satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

5. THE FOURTH REDUCIBILITY CASE $c = 2$.

The case $c = 2$ is similar to both of the previous cases $c = 0$ and $c = 1$. The non-vanishing condition (2.1) in this case turns to $b \neq 1$. The cubic equations (1.2) and (1.3) in this case turn to the following ones:

$$\begin{aligned} x(2(b-1)^2 x^2 + 2(b-1)x - b(b-2)) &= 0, \\ (d+1)(2(b-1)^2 d^2 - 2(b-1)d - b(b-2)) &= 0. \end{aligned} \quad (5.1)$$

Substituting $c = 2$ into the formulas (1.17), (1.18), and (1.12), we obtain

$$E_{21} = -\frac{b(b-2)}{2(b-1)^2}, \quad E_{11} = -\frac{b}{b-1}, \quad E_{12} = -1. \quad (5.2)$$

The equations (5.1) are already factored. If we split off the linear terms, they turn to quadratic equations. These two quadratic equations are very similar:

$$\begin{aligned} 2(b-1)^2 x^2 + 2(b-1)x - b(b-2) &= 0, \\ 2(b-1)^2 d^2 - 2(b-1)d - b(b-2) &= 0. \end{aligned} \quad (5.3)$$

The discriminants of the quadratic equations (5.3) do coincide:

$$D = 4(2b^2 - 4b + 1)(b-1)^2. \quad (5.4)$$

The formula (5.4) means that in order to solve the equations (5.3) in rational numbers one should resolve the following quadratic equation in rational numbers:

$$2b^2 - 4b + 1 = \beta^2. \quad (5.5)$$

The equation (5.5) is similar to the equation (4.5). If $b = 0$, then it has two trivial solutions with $\beta = \pm 1$. If $b \neq 0$, we can write the equation (5.5) as

$$\left(\frac{1}{b} - 2\right)^2 - \frac{\beta^2}{b^2} = 2. \quad (5.6)$$

The lemma 2.2 from [49] is applicable to the equation (5.6). This lemma yields

$$\frac{1}{b} = \frac{t^2 + 4t + 2}{2t}, \quad \frac{\beta}{b} = \frac{t^2 - 2}{2t} \quad (5.7)$$

for some rational $t \neq 0$. Now the formulas (5.7) can be transformed to

$$b = \frac{2t}{(t+2)^2 - 2}, \quad \beta = \frac{t^2 - 2}{(t+2)^2 - 2}. \quad (5.8)$$

Like (4.8), the formulas (5.8) are consistent since their denominators cannot vanish for any rational t . One of the above trivial solutions with $b = 0$ and $\beta = -1$ can be obtained from (5.8) for $t = 0$. In order to cover the other trivial solution with $b = 0$ and $\beta = 1$ we need to add the following sign option to (5.8):

$$b = \frac{2t}{(t+2)^2 - 2}, \quad \beta = \pm \frac{t^2 - 2}{(t+2)^2 - 2}. \quad (5.9)$$

Due to the above observations the restriction $t \neq 0$ is removed and the formulas (5.9) cover all rational solutions of the equation (5.5). The condition $b - 1 \neq 0$ derived from (2.1) for $c = 2$ implies no restrictions for t since the equation

$$\frac{2t}{(t+2)^2 - 2} = 1$$

has no rational roots. Now we can substitute b from (5.9) into the quadratic equations (5.3) and find their roots. Since the roots $x = 0$ and $d = 1$ of the cubic equations (5.1) are already known, we can write formulas for all their roots:

$$\begin{aligned} x_1 &= 0, & x_2 &= \frac{2(t+1)}{(t+1)^2 + 1}, & x_3 &= \frac{t(t+2)}{(t+1)^2 + 1}, \\ d_1 &= 1, & d_2 &= \frac{-t(t+2)}{(t+1)^2 + 1}, & d_3 &= \frac{-2(t+1)}{(t+1)^2 + 1}. \end{aligned} \quad (5.10)$$

Taking into account the formula for b in (5.9), the formula (5.2) is transformed to

$$E_{21} = \frac{2(t^2 + 3t + 2)t}{((t+1)^2 + 1)^2}, \quad E_{11} = \frac{2t}{(t+1)^2 + 1}, \quad E_{12} = -1. \quad (5.11)$$

If we substitute the formulas (5.10) and (5.11) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (5.10) into (1.1) along with $L = 1$, we find that the cuboid equations are also fulfilled. But, like in the previous cases, the formulas (5.10) do not provide a perfect cuboid since x_1 is zero.

Theorem 5.1. *If $c = 2$ and $b \neq 1$, then the cubic polynomials in (1.2) and (1.3) are reducible over \mathbb{Q} . Moreover, if b is given by the first formula (5.9) for some rational t , then each of the cubic equations (1.2) and (1.3) has three rational roots satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

6. SOME OTHER REDUCIBILITY CASES.

Note that in each of the previous four cases the cubic equation (1.2) has the root $x = 0$. The necessary and sufficient condition for that is written as

$$P(x) = E_{30} = 0, \quad (6.1)$$

where $P(x)$ is the cubic polynomial in the left hand side of the equation (1.2). Looking at the formulas (6.1) and (1.20), we see that along with the conditions $b = 0$, $c = 0$, $c = 1$, $c = 2$, which were already considered in the previous cases, there are the following two conditions for vanishing E_{30} :

$$bc^2 - 4bc + 4b + 2 = 0, \quad (6.2)$$

$$2bc^2 - 4bc + 2b - c^2 = 0. \quad (6.3)$$

Let's denote through $Q(d)$ the cubic polynomial in the left hand side of the equation (1.3). Then one can easily derive the following formulas:

$$\begin{aligned} Q(-1) &= -(c-1)^2 (bc^2 - 4bc + 4b + 2)^2 b^2 c^2 \times \\ &\times (b^2 c^4 - 6b^2 c^3 + 13b^2 c^2 - 12b^2 c + 4b^2 + c^2)^{-1} \times \\ &\times (bc - 1 - b)^{-2} (bc - c - 2b)^{-2}, \end{aligned} \quad (6.4)$$

$$\begin{aligned} Q(1) &= (c-2)^2 (2bc^2 - 4bc + 2b - c^2)^2 b^2 \times \\ &\times (b^2 c^4 - 6b^2 c^3 + 13b^2 c^2 - 12b^2 c + 4b^2 + c^2)^{-1} \times \\ &\times (bc - 1 - b)^{-2} (bc - c - 2b)^{-2}. \end{aligned} \quad (6.5)$$

Comparing (6.2) with (6.4), we see that the condition (6.2) implies $Q(-1) = 0$. Similarly, comparing (6.3) with (6.5), we find that the condition (6.3) implies $Q(1) = 0$. These observations yield the following theorems.

Theorem 6.1. *If the condition (6.2) is fulfilled, then both cubic equations (1.2) and (1.3) are reducible. In this case the first of them has the root $x = 0$, while the second has the root $d = -1$.*

Theorem 6.2. *If the condition (6.3) is fulfilled, then both cubic equations (1.2) and (1.3) are reducible. In this case the first of them has the root $x = 0$, while the second has the root $d = 1$.*

7. THE FIFTH REDUCIBILITY CASE.

The fifth reducibility case is defined by the condition (6.2). The equality (6.2) is linear with respect to b . It can be resolved as follows:

$$b = \frac{-2}{(c-2)^2}. \quad (7.1)$$

Substituting (7.1) into the formulas (1.14), (1.15), and (1.20), we find that the first cubic equation (1.2) reduces to the following one:

$$x((c^2 - 2c + 2)x + 2(c-1))((c^2 - 2c + 2)x - c(c-2)) = 0. \quad (7.2)$$

It is easy to see that the equation (7.2) has three rational roots

$$x_1 = 0, \quad x_2 = \frac{-2(c-1)}{(c-1)^2+1}, \quad x_3 = \frac{c(c-2)}{(c-1)^2+1}. \quad (7.3)$$

Now let's substitute (7.1) into the formulas (1.13), (1.16), and (1.19). As a result we find that the second cubic equation (1.3) reduces to the following one:

$$(d+1)((c^2-2c+2)d-2(c-1))((c^2-2c+2)d-c(c-2)) = 0. \quad (7.4)$$

Again, it is easy to see that the equation (7.4) has three rational roots

$$d_1 = -1, \quad d_2 = \frac{c(c-2)}{(c-1)^2+1}, \quad d_3 = \frac{2(c-1)}{(c-1)^2+1}. \quad (7.5)$$

Now let's substitute (7.1) into the formulas (1.17), (1.18), and (1.12). This yields

$$\begin{aligned} E_{21} &= \frac{2(c-2)(c-1)c}{((c-1)^2+1)^2}, \\ E_{11} &= \frac{2((c-2)^2-2)(c-2)}{((c-1)^2+1)^2}, \\ E_{12} &= \frac{-((c-2)^2-2)(c^2-2)}{((c-1)^2+1)^2}. \end{aligned} \quad (7.6)$$

If we substitute the formulas (7.3), (7.5), and (7.6) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (7.3) and (7.5) along with $L = 1$ into the equations (1.1), we find that the cuboid equations are also fulfilled. But, like in the previous cases, the formulas (7.3) and (7.5) do not provide a perfect cuboid since x_1 is zero.

Note that (7.1) provides the restriction $c \neq 2$. Substituting (7.1) into (2.1), we get the condition $(c-1)^2+1 \neq 0$ which is fulfilled for all rational c . Summarizing the results of this section, we can formulate the following theorem.

Theorem 7.1. *If $c \neq 2$ and $b(c-2)^2 = -2$, then the cubic polynomials in (1.2) and (1.3) are reducible over the field of rational numbers \mathbb{Q} . Moreover, each of the cubic equations (1.2) and (1.3) has three rational roots satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

8. THE SIXTH REDUCIBILITY CASE.

The sixth reducibility case is defined by the condition (6.3). The equality (6.3) is linear with respect to b . It can be resolved as follows:

$$b = \frac{c^2}{2(c-1)^2}. \quad (8.1)$$

Substituting (8.1) into the formulas (1.14), (1.15), and (1.20), we find that the first cubic equation (1.2) reduces to the following one:

$$x((c^2-2c+2)x+2(c-1))((c^2-2c+2)x-c(c-2)) = 0. \quad (8.2)$$

It is easy to see that the equation (8.2) has three rational roots

$$x_1 = 0, \quad x_2 = \frac{-2(c-1)}{(c-1)^2+1}, \quad x_3 = \frac{c(c-2)}{(c-1)^2+1}. \quad (8.3)$$

Now let's substitute (8.1) into the formulas (1.13), (1.16), and (1.19). As a result we find that the second cubic equation (1.3) reduces to the following one:

$$(d-1)((c^2-2c+2)d-2(c-1))((c^2-2c+2)d-c(c-2))=0. \quad (8.4)$$

Again, it is easy to see that the equation (8.4) has three rational roots

$$d_1 = 1, \quad d_2 = \frac{c(c-2)}{(c-1)^2+1}, \quad d_3 = \frac{2(c-1)}{(c-1)^2+1}. \quad (8.5)$$

The formulas (8.3) and (8.5) are almost identical to the formulas (7.3) and (7.5). The only difference is the sign of d_1 .

Now let's substitute (7.1) into the formulas (1.17), (1.18), and (1.12). This yields

$$\begin{aligned} E_{21} &= -\frac{2(c-2)(c-1)c}{((c-1)^2+1)^2}, \\ E_{11} &= \frac{2c((c-2)^2-2)(c-1)}{((c-1)^2+1)^2}, \\ E_{12} &= \frac{((c-2)^2-2)(c^2-2)}{((c-1)^2+1)^2}. \end{aligned} \quad (8.6)$$

If we substitute the formulas (8.3), (8.5), and (8.6) into (1.5), we find that the auxiliary equations (1.5) are fulfilled. Similarly, substituting (8.3) and (8.5) along with $L = 1$ into the equations (1.1), we find that the cuboid equations are also fulfilled. But like in all previous cases, the formulas (8.3) and (8.5) do not provide a perfect cuboid since x_1 is zero.

Note that (8.1) provides the restriction $c \neq 1$. Substituting (7.1) into (2.1), we get the condition $c((c-2)^2+1) \neq 0$, which reduces to $c \neq 0$ since $(c-2)^2+1$ is always positive. Summarizing the results of this section, we can formulate the following theorem.

Theorem 8.1. *If $c \neq 0$, $c \neq 1$, and $2b(c-1)^2 = c^2$, then the cubic polynomials in (1.2) and (1.3) are reducible over the field of rational numbers \mathbb{Q} . Moreover, each of the cubic equations (1.2) and (1.3) has three rational roots satisfying the auxiliary equations (1.5) but not resolving the problem 1.2.*

9. CONCLUDING REMARKS.

Six special cases of reducibility considered in the above sections do not exhaust all of the cases where the cubic equations (1.2) and (1.3) are reducible over \mathbb{Q} . There is one very special case with the following values of b and c :

$$b = \frac{14}{5}, \quad c = -\frac{7}{2}.$$

In this very special case the equations (1.2) and (1.3) are factored as follows:

$$\begin{aligned}(17x + 15)(9248x^2 + 3128x - 495) &= 0, \\ (17d + 8)(9248d^2 - 952d - 8175) &= 0.\end{aligned}\tag{9.1}$$

The search for such special cases of reducibility like (9.1) is planned for the future.

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