ON TWO ALGEBRAIC PARAMETRIZATIONS FOR RATIONAL SOLUTIONS OF THE CUBOID EQUATIONS.

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Abstract. A rational perfect cuboid is a rectangular parallelepiped whose edges and face diagonals are given by rational numbers and whose space diagonal is equal to unity. Its existence is equivalent to the existence of a perfect cuboid with all integer edges and diagonals. Finding such a cuboid or proving its non-existence is an old unsolved problem. Recently, based on a symmetry approach, the equations of a perfect cuboid were transformed to factor equations. The factor equations turned out to be solvable and, being solved, have led to a pair of inverse problems. Our efforts in the present paper are toward solving these inverse problems. Algebraic parametrizations for their solutions using algebraic functions of two rational arguments are found.

1. Introduction.

Referring the reader to [1–44] for the history of cuboid studies, we proceed to the following two cubic equations:

$$x^3 - E_{10}\,x^2 + E_{20}\,x - E_{30} = 0, \tag{1.1}$$

$$d^3 - E_{01}\,d^2 + E_{02}\,d - E_{03} = 0. \tag{1.2}$$

The equations (1.1) and (1.2) were derived as a result of the series of papers [45–50] along with three auxiliary equations

$$\begin{aligned} &x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2 = E_{21},\\ &x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1 = E_{11},\\ &x_1\,d_2\,d_3 + x_2\,d_3\,d_1 + x_3\,d_1\,d_2 = E_{12}. \end{aligned} \tag{1.3}$$

The numbers x_1, x_2, x_3 and d_1, d_2, d_3 in (1.3) are edges and face diagonals of a rational perfect cuboid. The first three of them are roots of the first cubic equation (1.2), the others are roots of the second cubic equation (1.2).

The numbers E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12} in the equations (1.1), (1.2) and (1.3) are rational functions of two arbitrary rational parameters b and c. Here is the formula for the number E_{11} in (1.3):

$$E_{11} = -\frac{b\,(c^2 + 2 - 4\,c)}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2\, + 2\,b}. \tag{1.4}$$

2000 *Mathematics Subject Classification.* 11D25, 11D72, 12E05, 14G05.

The formulas for E_{10}, E_{01} are similar to the formula (1.4) for E_{11}:

$$E_{10} = -\frac{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c\, - c}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}, \tag{1.5}$$

$$E_{01} = -\frac{b\,(c^2 + 2 - 2\,c)}{b^2\,c^2 + 2\,b^2 - 3\,b^2\,c + c - b\,c^2 + 2\,b}. \tag{1.6}$$

Below are the formulas for E_{20}, E_{02}, E_{30}, E_{03}, E_{21}, E_{12} in (1.1), (1.2), and (1.3):

$$\begin{gathered} E_{20} = \frac{b}{2}\,(b\,c^2 - 2\,c - 2\,b)\,(2\,b\,c^2 - c^2 - 6\,b\,c + 2 + 4\,b)\times\\ \times(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \end{gathered} \tag{1.7}$$

$$\begin{gathered} E_{02} = \frac{1}{2}\,(28\,b^2\,c^2 - 16\,b^2\,c - 2\,c^2 - 4\,b^2 - b^2\,c^4 + 4\,b^3\,c^4 - 12\,b^3\,c^3 +\\ +\,4\,b\,c^3 + 24\,b^3\,c - 8\,b\,c - 2\,b^4\,c^4 + 12\,b^4\,c^3 - 26\,b^4\,c^2 - 8\,b^2\,c^3 +\\ +24\,b^4\,c - 16\,b^3 - 8\,b^4)\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \end{gathered} \tag{1.8}$$

$$\begin{gathered} E_{30} = c\,b^2\,(1 - c)\,(c - 2)\,(b\,c^2 - 4\,b\,c + 2 + 4\,b)\,(2\,b\,c^2 - c^2 - 4\,b\,c +\\ +\,2\,b)\,(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-1}\times\\ \times(b\,c - 1 - b)^{-2}\,(-c + b\,c - 2\,b)^{-2}, \end{gathered} \tag{1.9}$$

$$\begin{gathered} E_{03} = \frac{b}{2}\,(b^2\,c^4 - 5\,b^2\,c^3 + 10\,b^2\,c^2 - 10\,b^2\,c + 4\,b^2 + 2\,b\,c + 2\,c^2 -\\ -\,b\,c^3)\,(2\,b^2\,c^4 - 12\,b^2\,c^3 + 26\,b^2\,c^2 - 24\,b^2\,c + 8\,b^2 - c^4\,b + 3\,b\,c^3 -\\ -\,6\,b\,c + 4\,b + c^3 - 2\,c^2 + 2\,c)\,(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 -\\ -12\,b^2\,c + 4\,b^2 + c^2)^{-1}\,(b\,c - 1 - b)^{-2}\,(-c + b\,c - 2\,b)^{-2}, \end{gathered} \tag{1.10}$$

$$\begin{gathered} E_{21} = \frac{b}{2}\,(5\,c^6\,b - 2\,c^6\,b^2 + 52\,c^5\,b^2 - 16\,c^5\,b - 2\,c^7\,b^2 + 2\,b^4\,c^8 -\\ -\,26\,b^4\,c^7 - 426\,b^4\,c^5 - 61\,b^3\,c^6 + 100\,b^3\,c^5 + 14\,c^7\,b^3 - c^8\,b^3 - 20\,b\,c^2 -\\ -\,8\,b^2\,c^2 - 16\,b^2\,c - 128\,b^2\,c^4 - 200\,b^3\,c^3 + 244\,b^3\,c^2 + 32\,b\,c^3 +\\ +\,768\,b^4\,c^4 - 852\,b^4\,c^3 + 568\,b^4\,c^2 + 104\,b^2\,c^3 - 208\,b^4\,c + 8\,c^4 +\\ +16\,b^3 - 112\,b^3\,c + 142\,b^4\,c^6 + 32\,b^4 - 2\,c^5)\,(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 -\\ -12\,b^2\,c - 4\,c^3 + 4\,b^2 + c^2)^{-1}\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}, \end{gathered} \tag{1.11}$$

$$\begin{gathered} E_{12} = (16\,b^6 + 32\,b^5 - 6\,c^5\,b^2 + 2\,c^5\,b - 62\,b^5\,c^6 + 62\,b^6\,c^6 + 16\,b^4 -\\ -\,180\,b^6\,c^5 - c^7\,b^3 + 18\,b^5\,c^7 - 12\,b^6\,c^7 - 2\,b^5\,c^8 + b^6\,c^8 + 248\,b^5\,c^2 +\\ +\,248\,b^6\,c^2 - 96\,b^6\,c + 321\,b^6\,c^4 - 180\,b^5\,c^3 - 144\,b^5\,c - 360\,b^6\,c^3 +\\ +\,b^4\,c^8 + 8\,b^4\,c^6 - 6\,b^4\,c^7 + 18\,b^4\,c^5 + 7\,b^3\,c^6 + 90\,b^5\,c^5 - 14\,b^3\,c^5 +\\ +\,17\,b^2\,c^4 + 32\,b^4\,c^2 + 28\,b^3\,c^3 - 28\,b^3\,c^2 - 4\,b\,c^3 + 8\,b^3\,c - 57\,b^4\,c^4 +\\ +\,36\,b^4\,c^3 - 12\,b^2\,c^3 - 48\,b^4\,c - c^4)\,(b^2\,c^4 - 6\,b^2\,c^3 + 13\,b^2\,c^2 -\\ -\,12\,b^2\,c + 4\,b^2 + c^2)^{-1}\,(b\,c - 1 - b)^{-2}\,(b\,c - c - 2\,b)^{-2}. \end{gathered} \tag{1.12}$$

Once the quantities E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12} are known through the formulas (1.4), (1.5), (1.6), (1.7), (1.8), (1.9), (1.10), (1.11), (1.12), the next step is to find x_1, x_2, x_3 and d_1, d_2, d_3 by solving the equations (1.1), (1.2), (1.3). This step is formulated in the following inverse problems.

Problem 1.1. *Find all pairs of rational numbers b and c for which the cubic equations* (1.1) *and* (1.2) *with the coefficients given by the formulas* (1.5)*,* (1.7)*,* (1.9) *and* (1.6)*,* (1.8)*,* (1.10) *possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations* (1.3) *whose right hand sides are given by the formulas* (1.11)*,* (1.4)*,* (1.12)*.*

Problem 1.2. *Find at least one pair of rational numbers b and c for which the cubic equations* (1.1) *and* (1.2) *with the coefficients given by the formulas* (1.5)*,* (1.7)*,* (1.9) *and* (1.6)*,* (1.8)*,* (1.10) *possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations* (1.3) *whose right hand sides are given by the formulas* (1.11)*,* (1.4)*,* (1.12)*.*

The term "inverse" here means that E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12} are produced from x_1, x_2, x_3, and d_1, d_2, d_3 as the values of elementary multisymmetric polynomials (see [51–71]), which is treated as a direct transform, then recovering x_1, x_2, x_3, d_1, d_2, d_3 through E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12} is an inverse transform. Since E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12} are functions of b and c, expressing x_1, x_2, x_3, d_1, d_2, d_3 through them means expressing x_1, x_2, x_3, d_1, d_2, d_3 through b and c so that the equations (1.1), (1.2), (1.3) are fulfilled. Below in the present paper we find such expressions for x_1, x_2, x_3 and d_1, d_2, d_3 in two ways using two algebraic function $w_1(b,c)$ and $w_2(b,c)$.

2. Cubics with three rational roots.

Lemma 2.1. *A reduced cubic equation $y^3+y^2+D=0$ has three rational roots if and only if there is a rational number w satisfying the sextic equation*

$$D\,(w^2+3)^3+4\,(w-1)^2\,(1+w)^2=0. \tag{2.1}$$

In this case the roots of the cubic equation $y^3+y^2+D=0$ are given by the formulas

$$y_1=-\frac{2\,(w+1)}{w^2+3}, \qquad y_2=\frac{2\,(w-1)}{w^2+3}, \qquad y_3=\frac{1-w^2}{w^2+3}. \tag{2.2}$$

Proof. Sufficiency. Assume that w is a root of the sextic equation (2.1). Note that the denominator w^2+3, which is common for all of the three fractions (2.2), cannot vanish for any rational number w. Therefore (2.2) yields three rational numbers y_1, y_2, y_3. The rest is to substitute them into the product $(y-y_1)\,(y-y_2)\,(y-y_3)$:

$$(y-y_1)\,(y-y_2)\,(y-y_3)=y^3+y^2-\frac{4\,(w-1)^2\,(1+w)^2}{(w^2+3)^3}. \tag{2.3}$$

Since $w^2+3\neq 0$, the equation (2.1) can be resolved with respect to D:

$$D=-\frac{4\,(w-1)^2\,(1+w)^2}{(w^2+3)^3}. \tag{2.4}$$

Comparing (2.4) and (2.3), we find that the sufficiency is proved.

Necessity. Assume that the equation $y^3 + y^2 + D = 0$ has three rational roots y_1, y_2, y_3. Then it can be written as

$$(y - y_2)\,(y^2 + A\,y + B) = y^3 + (A - y_2)\,y^2 + (B - y_2\,A)\,y - y_2\,B = 0, \tag{2.5}$$

where y_1 and y_3 are roots of the quadratic equation $y^2 + A\,y + B = 0$. Comparing (2.5) with the initial cubic equation $y^3 + y^2 + D = 0$, we find

$$A = y_2 + 1, \qquad B = y_2\,(y_2 + 1), \qquad D = -y_2^3 - y_2^2. \tag{2.6}$$

Note that the quadratic equation $y^2 + A\,y + B = 0$ with rational coefficients has two rational roots if and only if its discriminant is a square of some rational number z:

$$A^2 - 4\,B = z^2. \tag{2.7}$$

Applying (2.6) to (2.7), we derive the equation relating y_2 and z:

$$-3\,y_2^2 - 2\,y_2 + 1 = z^2. \tag{2.8}$$

The equation (2.8) is similar to the equation (4.5) in [72]. It is solved similarly with the use of the lemma 2.2 in [49]. Its general solution in rational numbers is

$$y_2 = \frac{2\,t}{(t + 1)^2 + 3}, \qquad z = \frac{t^2 - 4}{(t + 1)^2 + 3}, \tag{2.9}$$

where t is an arbitrary rational number. The roots y_1 and y_3 of the quadratic equation $y^2 + A\,y + B = 0$ are given by the standard formula:

$$y_{1,3} = -\frac{A}{2} \pm \frac{\sqrt{A^2 - 4\,B}}{2}. \tag{2.10}$$

Applying (2.6) and (2.7) to (2.10), we derive

$$y_1 = -\frac{y_2 + 1}{2} + \frac{z}{2}, \qquad y_3 = -\frac{y_2 + 1}{2} - \frac{z}{2}, \tag{2.11}$$

Then we apply (2.9) to (2.11). As a result we obtain

$$y_1 = \frac{-2\,(2 + t)}{(t + 1)^2 + 3}, \qquad y_2 = \frac{2\,t}{(t + 1)^2 + 3}, \qquad y_3 = \frac{-t\,(2 + t)}{(t + 1)^2 + 3}. \tag{2.12}$$

In order to derive the required formulas (2.2) it is sufficient to substitute $t = w - 1$ into (2.12). Thus, we have found that if the cubic equation $y^3 + y^2 + D = 0$ has three rational roots, these roots are expressed through some rational number w by means of the formulas (2.2). The rest is to substitute any one of the roots (2.2) into the initial cubic equation $y^3 + y^2 + D = 0$. Since $w^2 + 3 \neq 0$, this yields the required equation (2.1). The lemma 2.1 is proved. $\square$

Now let's consider a general cubic equation with the coefficients A_0, A_1, A_2, A_3:

$$A_3\, x^3 + A_2\, x^2 + A_1\, x + A_0 = 0. \tag{2.13}$$

Assuming that $A_3 \neq 0$ in (2.13), we substitute

$$x = \tilde{x} - \frac{A_2}{3\, A_3} \tag{2.14}$$

into the cubic equation (2.13). As a result it is transformed to

$$\tilde{x}^3 + \left(\frac{A_1}{A_3} - \frac{A_2^2}{3\, A_3^2}\right) \tilde{x} + \frac{A_0}{A_3} - \frac{A_1\, A_2}{3\, A_3^2} + \frac{2\, A_2^3}{27\, A_3^3} = 0. \tag{2.15}$$

The following notations are for the sake of convenience:

$$B_1 = \frac{A_1}{A_3} - \frac{A_2^2}{3\, A_3^2}, \qquad\qquad B_9 = \frac{A_0}{A_3} - \frac{A_1\, A_2}{3\, A_3^2} + \frac{2\, A_2^3}{27\, A_3^3}. \tag{2.16}$$

In terms of the notations (2.16) the cubic equation (2.15) simplifies to

$$\tilde{x}^3 + B_1\, \tilde{x} + B_0 = 0. \tag{2.17}$$

Assume that $B_0 \neq 0$ and $B_1 \neq 0$. Then $\tilde{x} = 0$ is not a root of the cubic equation (2.17). Therefore the following transformation is applicable to it:

$$\tilde{x} = \frac{B_0}{B_1\, y}. \tag{2.18}$$

Applying (2.18) to the equation (2.17), we transform it to

$$y^3 + y^2 + D = 0, \ \text{ where } \ D = \frac{B_0^2}{B_1^3} \neq 0. \tag{2.19}$$

Now, substituting (2.16) into (2.19), we derive the formula for the parameter D in the case of a general cubic equation (2.13):

$$D = -\frac{(9\, A_1\, A_2\, A_3 - 27\, A_0\, A_3^2 - 2\, A_2^3)^2}{27\, (A_2^2 - 3\, A_1\, A_3)^3}. \tag{2.20}$$

The next step is to transform the formulas (2.2) for the roots of (2.19) backward to the roots of a general cubic equation (2.13) using (2.18) and (2.14):

$$x_1 = \frac{B_0}{B_1\, y_1} - \frac{A_2}{3\, A_3}, \qquad x_2 = \frac{B_0}{B_1\, y_2} - \frac{A_2}{3\, A_3}, \qquad x_3 = \frac{B_0}{B_1\, y_3} - \frac{A_2}{3\, A_3}. \tag{2.21}$$

Substituting (2.16) into (2.21) we derive the following three formulas:

$$\begin{aligned} x_1 = \frac{1}{18}\, &((2\, A_2^3 - 9\, A_1\, A_2\, A_3 + 27\, A_0\, A_3^2)\, w^2 + (18\, A_2\, A_1\, A_3 - 6\, A_2^3)\, w \,- \\ &- 9\, A_1\, A_2\, A_3 + 81\, A_0\, A_3^2)\, A_3^{-1}\, (A_2^2 - 3\, A_1\, A_3)^{-1}\, (1 + w)^{-1}, \end{aligned} \tag{2.22}$$

$$
\begin{aligned}
x_2 = \frac{1}{18}\,((2\,A_2^3 - 9\,A_1\,A_2\,A_3 + 27\,A_0\,A_3^2)\,w^2 - (18\,A_2\,A_1\,A_3 - 6\,A_2^3)\,w -\\
-\,9\,A_1\,A_2\,A_3 + 81\,A_0\,A_3^2)\,A_3^{-1}\,(A_2^2 - 3\,A_1\,A_3)^{-1}\,(1-w)^{-1},
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
x_3 = \frac{1}{9}\,((A_2^3 - 27\,A_0\,A_3^2)\,w^2 + 36\,A_1\,A_2\,A_3 - 81\,A_0\,A_3^2 - 9\,A_2^3)\times\\
\times A_3^{-1}\,(A_2^2 - 3\,A_1\,A_3)^{-1}\,(1-w)^{-1}\,(1+w)^{-1}.
\end{aligned}
\tag{2.24}
$$

Based on the formulas (2.22), (2.23), (2.24), we can formulate the next lemma.

Lemma 2.2. *Assume that the numbers A_0, A_1, A_2, A_3 obey the inequalities*

$$
A_3 \neq 0, \qquad\qquad \frac{A_1}{A_3} - \frac{A_2^2}{3\,A_3^2} \neq 0, \qquad\qquad \frac{A_0}{A_3} - \frac{A_1\,A_2}{3\,A_3^2} + \frac{2\,A_2^3}{27\,A_3^3} \neq 0.
$$

Then the general cubic polynomial (2.13) with the rational coefficients A_0, A_1, A_2, A_3 has three rational roots if and only if there is a rational number w satisfying the sextic equation (2.1) where D is given by the formula (2.20). In this case the roots of the cubic equation (2.13) are given by the formulas (2.22), (2.23), (2.24).

3. The first algebraic parametrization.

The first algebraic parametrization for solutions of the equations (1.1), (1.2), (1.3) is produced from the the first cubic equation (1.1) with the use of the lemma 2.2. Applying this lemma, we get the sextic equation

$$
D_1\,(w^2+3)^3 + 4\,(w-1)^2\,(1+w)^2 = 0 \tag{3.1}
$$

of the form (2.1). Its parameter $D = D_1$ is calculated using the formula (2.20) and the formulas (1.5), (1.7), (1.9) for the coefficients of the equation (1.1):

$$
\begin{aligned}
D_1 = -\frac{2}{27}\,(7812\,b^4\,c^4 - 216\,b^2\,c^4 - 52\,b^2\,c^3 + 1764\,b^3\,c^4 - 1200\,b^4\,c^3 -\\
-\,1848\,b^4\,c^2 + 720\,b^4\,c - 36\,c^4\,b - 1512\,b^3\,c^3 - 36\,c^8\,b^3 + 288\,b^3\,c^2 -\\
-\,108\,c^6\,b^2 + 380\,c^5\,b^2 + 378\,c^7\,b^3 - 231\,c^8\,b^4 - 300\,c^7\,b^4 + 3906\,c^6\,b^4 -\\
-13\,c^7\,b^2 - 8904\,c^5\,b^4 - 882\,c^6\,b^3 + 18\,c^6\,b - 1319\,b^6\,c^8 + 20952\,b^5\,c^3 -\\
-\,11952\,b^5\,c^2 + 2592\,b^5\,c - 48372\,b^6\,c^4 + 31620\,b^6\,c^3 - 10552\,b^6\,c^2 +\\
+\,816\,b^6\,c + 1494\,b^5\,c^8 - 5238\,b^5\,c^7 - 4\,c^5 + 7905\,b^6\,c^7 - 24186\,b^6\,c^6 +\\
+\,288\,b^6 + 43740\,b^6\,c^5 + 7686\,b^5\,c^6 + 576\,b^7 + 128\,b^8 - 15372\,b^5\,c^4 -\\
-\,1080\,b^7\,c^8 - 3546\,b^7\,c^6 + 51\,c^9\,b^6 + 400\,b^8\,c^8 - 162\,c^9\,b^5 + 8640\,b^7\,c^2 -\\
-\,3456\,b^7\,c + 2808\,b^7\,c^7 - 1560\,b^8\,c^7 + 3940\,b^8\,c^6 + 216\,c^9\,b^7 - 960\,b^8\,c -\\
-\,6240\,b^8\,c^3 + 9\,c^{10}\,b^6 + 7880\,b^8\,c^4 + 4\,c^{10}\,b^8 - 6732\,b^8\,c^5 + 45\,c^9\,b^4 +\\
+\,3200\,b^8\,c^2 - 11232\,b^7\,c^3 + 7092\,b^7\,c^4 - 18\,c^{10}\,b^7 - 60\,c^9\,b^8)^2\,(2\,c^2 +\\
+\,2\,b^4\,c^4 - 12\,b^4\,c^3 + 26\,b^4\,c^2 - 24\,b^4\,c + 8\,b^4 - 6\,b^3\,c^4 + 18\,b^3\,c^3 -\\
-\,36\,b^3\,c + 24\,b^3 + 3\,b^2\,c^4 + 8\,b^2\,c^3 - 36\,b^2\,c^2 + 16\,b^2\,c + 12\,b^2 - 6\,b\,c^3 +\\
+\,12\,b\,c)^3\,(b^2\,c^4 - 6\,b^2\,c^{-3} + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-2}.
\end{aligned}
\tag{3.2}
$$

The fraction $2/27$ in (3.2) can be expressed as $2^3/(2^2\,3^3)$. So the structure of (3.2) as a ratio of some square and some cube is the same as the structure of D in (2.20). The formulas (3.1) and (3.2) define an algebraic function $w = w_1(b, c)$. This function is used below as w without showing its arguments.

Now we proceed to the formulas (2.22), (2.23), (2.24). Using these formulas, we get explicit expressions for x_1, x_2, x_3 through b, c, and w, where $w = w_1(b, c)$:

$$x_1 = x_1(b, c, w), \qquad x_2 = x_2(b, c, w), \qquad x_3 = x_3(b, c, w). \tag{3.3}$$

However, the expressions for the functions (3.3) are very huge. They comprise more than 100 terms in each. Therefore these expressions are given in Appendix 1 in a machine readable form.

The functions (3.3) are roots of the cubic equation (1.1). This fact follows from the lemma 2.2. Moreover, this fact has been tested computationally with the use of the explicit formulas for them in Appendix 1.

Apart from being roots of the cubic equation (1.1), the functions (3.3) obey the following cuboid equation saying that its space diagonal is equal to unity:

$$x_1^2 + x_2^2 + x_3^2 = 1. \tag{3.4}$$

This fact follows from the theory of the cuboid factor equations in [46], [47]. The equality (3.4) has also been tested computationally.

The nest step is to derive the formulas for the face diagonals d_1, d_2, d_3 of a cuboid. For this purpose we use the following equations:

$$\begin{aligned}
&x_1\,x_2\,d_3 + x_2\,x_3\,d_1 + x_3\,x_1\,d_2 = E_{21},\\
&x_1\,d_2 + d_1\,x_2 + x_2\,d_3 + d_2\,x_3 + x_3\,d_1 + d_3\,x_1 = E_{11},\\
&d_1 + d_2 + d_3 = E_{01}.
\end{aligned} \tag{3.5}$$

The first two of them are taken from (1.3). The last equation (3.5) follows from (1.2) since d_1, d_2, d_3 should be the roots of this cubic equation.

It is easy to see that the equations (3.5) are linear with respect to d_1, d_2, d_3, while x_1, x_2, x_3 are already known from (3.3). Resolving the system of linear equations (3.5), we get three functions similar to x_1, x_2, x_3 in (3.3):

$$d_1 = x_1(b, c, w), \qquad d_2 = d_2(b, c, w), \qquad d_3 = d_3(b, c, w). \tag{3.6}$$

The explicit formulas for the functions (3.6) are extremely huge. They comprise more than 800 terms in each. For this reason we do not provide these formulas.

The functions (3.6) should be the roots of the second cubic equation (1.2). Moreover, they should satisfy the last equation (1.3) which is not used in (3.5) for determining them. These two facts follow from the theory of the cuboid factor equations in [46], [47]. However, they cannot be verified even with the use of symbolic computations since the formulas for the functions are extremely huge. These facts have been tested numerically for a series of random pairs of rational numbers b and c.

4. The second algebraic parametrization.

The second algebraic parametrization for solutions of the equations (1.1), (1.2), (1.3) is similar to the first one. However, in tis case we start with the second cubic equation (1.2). Applying the lemma 2.2 to it, we get the sextic equation

$$D_2\,(w^2+3)^3+4\,(w-1)^2\,(1+w)^2=0 \tag{4.1}$$

of the form (2.1). Its parameter $D=D_2$ is calculated using the formula (2.20) and the formulas (1.6), (1.8), (1.10) for the coefficients of the equation (1.2):

$$\begin{gathered}
D_2=-\frac{2\,b^2}{27}\,(832\,b^2\,c^2-1440\,b^2\,c^4-840\,b^2\,c^3+4788\,b^3\,c^4+396\,b\,c^3+\\
+720\,b^3\,c+808\,b^4\,c^4+3032\,b^4\,c^3-2576\,b^4\,c^2-96\,b^4\,c+448\,b^4-\\
-504\,c^4\,b-4176\,b^3\,c^3-9\,c^8\,b^3+72\,b^3\,c^2-720\,c^6\,b^2+2288\,c^5\,b^2+\\
+1044\,c^7\,b^3-322\,c^8\,b^4+758\,c^7\,b^4+404\,c^6\,b^4-210\,c^7\,b^2-2464\,c^5\,b^4-\\
-2394\,c^6\,b^3+72\,c^4+252\,c^6\,b+3168\,b^6\,c^8+441\,c^9\,b^5-7056\,b^5\,c+\\
+57960\,b^6\,c^4-47232\,b^6\,c^3+25344\,b^6\,c^2-8064\,b^6\,c-1809\,b^5\,c^8+\\
+14472\,b^5\,c^2+3951\,b^5\,c^7-72\,c^5+36\,c^6-11808\,b^6\,c^7+1440\,b^5+\\
+28980\,b^6\,c^6-49032\,b^6\,c^5-4410\,b^5\,c^6+8820\,b^5\,c^4-15804\,b^5\,c^3+\\
+1152\,b^6-504\,c^9\,b^6-45\,c^9\,b^3-6\,c^9\,b^4+104\,c^8\,b^2+36\,c^{10}\,b^6+\\
+14\,c^{10}\,b^4-45\,c^{10}\,b^5-99\,c^7\,b)^2/(6\,b^4\,c^4-36\,b^4\,c^3+78\,b^4\,c^2-72\,b^4\,c+\\
+24\,b^4-12\,b^3\,c^4+36\,b^3\,c^3-72\,b^3\,c+48\,b^3+5\,b^2\,c^4+16\,b^2\,c^3-\\
-68\,b^2\,c^2+32\,b^2\,c+20\,b^2-12\,b\,c^3+24\,b\,c+6\,c^2)^{-3}\,(b^2\,c^4-6\,b^2\,c^3+\\
+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-2}.
\end{gathered} \tag{4.2}$$

The formulas (4.1) and (4.2) define an algebraic function $w=w_2(b,c)$. This function is used below as w without showing its arguments.

Now we proceed to the formulas (2.22), (2.23), (2.24). Using these formulas, we get explicit expressions for d_1, d_2, d_3 through b, c, and w, where $w=w_2(b,c)$:

$$d_1=d_1(b,c,w), \qquad d_2=d_2(b,c,w), \qquad d_3=d_3(b,c,w). \tag{4.3}$$

Again, the expressions for the functions (4.3) are very huge. They comprise more than 100 terms in each. Therefore these expressions are given in Appendix 2 in a machine readable form.

The functions (4.3) are roots of the cubic equation (1.2). This fact follows from the lemma 2.2. Moreover, this fact has been tested computationally with the use of the explicit formulas for them in Appendix 2.

Apart from being roots of the cubic equation (1.1), the functions (4.3) obey the following equation that can be derived from the original cuboid equations:

$$d_1^2+d_2^2+d_3^2=2. \tag{4.4}$$

However, which is more important, the equation (4.4) follows from the cuboid factor equations (see (6.10) in [46] or (1.12) and (1.19) in [47] and recall (3.4)). Despite the theoretical background from [46] and [47], the equality (4.4) has also been tested computationally using the explicit expressions for (4.3).

The nest step is to derive the formulas for x_1, x_2, x_3 through the formulas for d_1, d_2, d_3. For this purpose we use the following equations:

$$\begin{aligned}&x_1+x_2+x_3=E_{10},\\&x_1\,d_2+d_1\,x_2+x_2\,d_3+d_2\,x_3+x_3\,d_1+d_3\,x_1=E_{11},\\&x_1\,d_2\,d_3+x_2\,d_3\,d_1+x_3\,d_1\,d_2=E_{12}.\end{aligned}\tag{4.5}$$

The last two of them are taken from (1.3). The first equation (4.5) follows from (1.1) since x_1, x_2, x_3 should be the roots of this cubic equation.

It is easy to see that the equations (4.5) are linear with respect to x_1, x_2, x_3, while d_1, d_2, d_3 are already known from (4.3). Resolving the system of linear equations (4.5), we get three functions similar to d_1, d_2, d_3 in (4.3):

$$x_1=x_1(b,c,w),\qquad\qquad x_2=x_2(b,c,w),\qquad\qquad x_3=x_3(b,c,w).\tag{4.6}$$

The explicit formulas for the functions (4.6) are extremely huge. They comprise more than 800 terms in each. For this reason we do not provide these formulas.

The functions (4.6) should be the roots of the first cubic equation (1.1). Moreover, they should satisfy the first equation (1.3) which is not used in (4.5) for determining them. These two facts follow from the theory of the cuboid factor equations in [46], [47]. However, they cannot be verified even with the use of symbolic computations since the formulas for the functions are extremely huge. These facts have been tested numerically for a series of random pairs of rational numbers b and c.

5. Concluding remarks.

Thus, two sets explicit formulas for possible solutions of the inverse cuboid problems 1.1 and 1.2 are obtained. However, in order to produce an actual solution one should solve at least one of the two sextic equations (3.1) or (4.1) in rational numbers. The equations (3.1) or (4.1) are similar to the twelfth order equation derived in [40]. They are sextic with respect to w. However their total degrees with respect to b, c and w are 42 and 40 respectively.

The equations (3.1) or (4.1) produce two algebraic functions $w=w_1(b,c)$ and $w=w_2(b,c)$ which are different since the parameters D_1 and D_2 in (3.2) and (4.2) are different. These two functions probably are related to each other as $w_1=p_1(w_2,b,c)$ and $w_2=p_2(w_1,b,c)$, where p_1 and p_2 are polynomials in w_2 and w_1 respectively. But the relation can be more complicated, i. e. $P(w_1,w_2,b,c)=0$, where P is a single polynomial of four variables. Which of these two options is valid? This question is to be studied in a separate paper.

The formulas (3.1) and (4.1) as well as the formulas in Appendix 1 and Appendix 2 and those huge formulas which are not presented explicitly have denominators. Some of them correspond to singularities studied in [73]. Others are new. These new singularities are also to be studied in a separate paper.

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Appendix 1.

Here are the formulas for $x_1 = x_1(b, c, w)$, $x_2 = x_2(b, c, w)$, $x_3 = x_3(b, c, w)$ from (3.3). They are written in a machine readable form convenient for to copy-paste into some symbolic computations package:

```
x1=1/18*(7686*w^2*b^5*c^6+2592*b^5*w^2*c-36*w^2*c^8*b^3+2808*w^2*b^7
*c^7+128*b^8*w^2-720*b^2*c^4-36*b^2*c^3+6084*b^3*c^4+26748*b^4*c^4-4
176*b^4*c^3-6264*b^4*c^2+2448*b^4*c+4*w^2*c^10*b^8-36*c^4*b-5400*b^3
*c^3-1560*w^2*b^8*c^7-144*c^8*b^3+1152*b^3*c^2-360*c^6*b^2+1116*c^5*
b^2+1350*c^7*b^3-783*c^8*b^4-1044*c^7*b^4+13374*c^6*b^4-9*c^7*b^2-30
456*c^5*b^4-3042*c^6*b^3+18*c^6*b-4059*b^6*c^8+63720*b^5*c^3-36144*b
^5*c^2+7776*b^5*c-130716*b^6*c^4+88308*b^6*c^3-32472*b^6*c^2+4464*b^
6*c+4518*b^5*c^8-15930*b^5*c^7+9*w^2*c^10*b^6+22077*b^6*c^7+288*b^6-
65358*b^6*c^6+117180*b^6*c^5+23454*b^5*c^6+31620*c^3*b^6*w^2-6732*w^
2*b^8*c^5+1494*w^2*b^5*c^8+576*b^7-46908*b^5*c^4-1080*b^7*c^8+2808*b
^7*c^7-3546*b^7*c^6+7092*b^7*c^4-11232*b^7*c^3+8640*b^7*c^2-3456*b^7
*c-486*c^9*b^5+279*c^9*b^6+216*c^9*b^7+153*c^9*b^4+9*c^10*b^6-18*c^1
0*b^7-1319*w^2*b^6*c^8-60*w^2*c^9*b^8-1080*w^2*b^7*c^8-162*w^2*c^9*b
^5+43740*w^2*b^6*c^5+400*w^2*b^8*c^8+51*w^2*c^9*b^6-11232*w^2*b^7*c^
3-6240*w^2*b^8*c^3-216*b^2*w^2*c^4-11952*c^2*b^5*w^2-3546*w^2*b^7*c^
6+3200*w^2*b^8*c^2+3940*w^2*b^8*c^6-300*w^2*c^7*b^4+7905*w^2*b^6*c^7
+3906*w^2*c^6*b^4+380*w^2*c^5*b^2-1848*b^4*w^2*c^2-5238*w^2*b^5*c^7+
7880*w^2*b^8*c^4+816*c*b^6*w^2+576*b^7*w^2-231*w^2*c^8*b^4-24186*w^2
*b^6*c^6+7092*w^2*b^7*c^4+288*b^6*w^2-1152*b^7*w-52*b^2*w^2*c^3-4837
2*w^2*b^6*c^4+378*w^2*c^7*b^3+216*w^2*c^9*b^7+45*w^2*c^9*b^4+1764*c^
4*b^3*w^2-720*b^4*w*c^2-72*b^2*w*c^4-864*b^3*w*c^3+720*b^4*w^2*c+288
*b^3*w^2*c^2+288*b^4*w*c+288*b^3*w*c^2+120*b^2*w*c^3-4*w^2*c^5+72*b*
w*c^4-36*b*w^2*c^4-10552*c^2*b^6*w^2-1512*b^3*w^2*c^3+18*w^2*c^6*b-1
8*c^10*b^7*w^2-960*c*b^8*w^2-108*c^6*b^2*w^2-3456*c*b^7*w^2-13*c^7*b
^2*w^2+8640*c^2*b^7*w^2-882*c^6*b^3*w^2+20952*c^3*b^5*w^2-15372*c^4*
b^5*w^2+7812*c^4*b^4*w^2-8904*c^5*b^4*w^2-1200*c^3*b^4*w^2-576*b^6*w
```

```
+3312*b^4*w*c^4-792*b^5*w*c^4+864*b^5*w*c^3-288*b^5*w*c^2-576*b^4*w*
c^3-90*b^4*w*c^8-396*b^3*w*c^6-144*b^4*w*c^7-3744*b^4*w*c^5-24*b^2*w
*c^5+36*b^5*w*c^8+396*b^5*w*c^6+1656*b^4*w*c^6-36*b^2*w*c^6+2160*b^7
|*w*c^8-6552*b^6*w*c^3-816*b^6*w*c^2+2016*b^6*w*c-14184*b^7*w*c^4+22
464*b^7*w*c^3-17280*b^7*w*c^2-102*b^6*w*c^8-1638*b^6*w*c^7-14040*b^6
*w*c^5-36*b*w*c^6+7092*b^7*w*c^6+7200*b^6*w*c^6+14400*b^6*w*c^4-1200
*b^8*w*c^8+4680*b^8*w*c^7-11820*b^8*w*c^6+20196*b^8*w*c^5+792*b^3*w*
c^4-23640*b^8*w*c^4+18720*b^8*w*c^3-9600*b^8*w*c^2+126*b^6*w*c^9-432
*b^7*w*c^9+180*b^8*w*c^9+30*b^2*w*c^7+18*b^4*w*c^9-36*b^3*w*c^8+36*b
^7*w*c^10-12*b^8*w*c^10-18*b^6*w*c^10-384*b^8*w+2880*w*b^8*c+12*w*c^
5-5616*b^7*w*c^7+6912*b^7*w*c-216*b^5*w*c^7+216*b^3*w*c^7)/((1+w)(20
*b^2*c^2+16*b*c^2+10*b^2*c^4+36*b^2*c-66*b^2*c^3+78*b^3*c^4+80*b^3*c
-78*b^4*c^4+238*b^4*c^3-156*b^4*c^2-68*b^4*c+24*b^3+72*b^4+2*c^3-8*c
^4*b-156*b^3*c^2+9*c^5*b^2-20*c^5*b^3+9*c^6*b^4-17*c^5*b^4-3*c^6*b^3
+176*b^5*c^2-192*b^5*c+66*b^6*c^4-126*b^6*c^3+132*b^6*c^2-72*b^6*c+4
8*b^5*c^5+64*b^5+16*b^6+2*b^6*c^6-18*b^6*c^5-8*b^5*c^6-88*b^5*c^4)(b
^2*c^4-6*b^2*c^3+13*b^2*c^2-12*b^2*c+4*b^2+c^2));
```

```
x2=-1/18*(7686*w^2*b^5*c^6+2592*b^5*w^2*c-36*w^2*c^8*b^3+2808*w^2*b^
7*c^7+128*b^8*w^2-720*b^2*c^4-36*b^2*c^3+6084*b^3*c^4+26748*b^4*c^4-
4176*b^4*c^3-6264*b^4*c^2+2448*b^4*c+4*w^2*c^10*b^8-36*c^4*b-5400*b^
3*c^3-1560*w^2*b^8*c^7-144*c^8*b^3+1152*b^3*c^2-360*c^6*b^2+1116*c^5
*b^2+1350*c^7*b^3-783*c^8*b^4-1044*c^7*b^4+13374*c^6*b^4-9*c^7*b^2-3
0456*c^5*b^4-3042*c^6*b^3+18*c^6*b-4059*b^6*c^8+63720*b^5*c^3-36144*
b^5*c^2+7776*b^5*c-130716*b^6*c^4+88308*b^6*c^3-32472*b^6*c^2+4464*b
^6*c+4518*b^5*c^8-15930*b^5*c^7+9*w^2*c^10*b^6+22077*b^6*c^7+288*b^6
-65358*b^6*c^6+117180*b^6*c^5+23454*b^5*c^6+31620*c^3*b^6*w^2-6732*w
^2*b^8*c^5+1494*w^2*b^5*c^8+576*b^7-46908*b^5*c^4-1080*b^7*c^8+2808*
b^7*c^7-3546*b^7*c^6+7092*b^7*c^4-11232*b^7*c^3+8640*b^7*c^2-3456*b^
7*c-486*c^9*b^5+279*c^9*b^6+216*c^9*b^7+153*c^9*b^4+9*c^10*b^6-18*c^
10*b^7-1319*w^2*b^6*c^8-60*w^2*c^9*b^8-1080*w^2*b^7*c^8-162*w^2*c^9*
b^5+43740*w^2*b^6*c^5+400*w^2*b^8*c^8+51*w^2*c^9*b^6-11232*w^2*b^7*c
^3-6240*w^2*b^8*c^3-216*b^2*w^2*c^4-11952*c^2*b^5*w^2-3546*w^2*b^7*c
^6+3200*w^2*b^8*c^2+3940*w^2*b^8*c^6-300*w^2*c^7*b^4+7905*w^2*b^6*c^
7+3906*w^2*c^6*b^4+380*w^2*c^5*b^2-1848*b^4*w^2*c^2-5238*w^2*b^5*c^7
+7880*w^2*b^8*c^4+816*c*b^6*w^2+576*b^7*w^2-231*w^2*c^8*b^4-24186*w^
2*b^6*c^6+7092*w^2*b^7*c^4+288*b^6*w^2+1152*b^7*w-52*b^2*w^2*c^3-483
72*w^2*b^6*c^4+378*w^2*c^7*b^3+216*w^2*c^9*b^7+45*w^2*c^9*b^4+1764*c
^4*b^3*w^2+720*b^4*w*c^2+72*b^2*w*c^4+864*b^3*w*c^3+720*b^4*w^2*c+28
8*b^3*w^2*c^2-288*b^4*w*c-288*b^3*w*c^2-120*b^2*w*c^3-4*w^2*c^5-72*b
*w*c^4-36*b*w^2*c^4-10552*c^2*b^6*w^2-1512*b^3*w^2*c^3+18*w^2*c^6*b-
18*c^10*b^7*w^2-960*c*b^8*w^2-108*c^6*b^2*w^2-3456*c*b^7*w^2-13*c^7*
b^2*w^2+8640*c^2*b^7*w^2-882*c^6*b^3*w^2+20952*c^3*b^5*w^2-15372*c^4
*b^5*w^2+7812*c^4*b^4*w^2-8904*c^5*b^4*w^2-1200*c^3*b^4*w^2+576*b^6*
w-3312*b^4*w*c^4+792*b^5*w*c^4-864*b^5*w*c^3+288*b^5*w*c^2+576*b^4*w
*c^3+90*b^4*w*c^8+396*b^3*w*c^6+144*b^4*w*c^7+3744*b^4*w*c^5+24*b^2*
w*c^5-36*b^5*w*c^8-396*b^5*w*c^6-1656*b^4*w*c^6+36*b^2*w*c^6-2160*b^
7*w*c^8+6552*b^6*w*c^3+816*b^6*w*c^2-2016*b^6*w*c+14184*b^7*w*c^4-22
464*b^7*w*c^3+17280*b^7*w*c^2+102*b^6*w*c^8+1638*b^6*w*c^7+14040*b^6
*w*c^5+36*b*w*c^6-7092*b^7*w*c^6-7200*b^6*w*c^6-14400*b^6*w*c^4+1200
```

```
*b^8*w*c^8-4680*b^8*w*c^7+11820*b^8*w*c^6-20196*b^8*w*c^5-792*b^3*w*
c^4+23640*b^8*w*c^4-18720*b^8*w*c^3+9600*b^8*w*c^2-126*b^6*w*c^9+432
*b^7*w*c^9-180*b^8*w*c^9-30*b^2*w*c^7-18*b^4*w*c^9+36*b^3*w*c^8-36*b
^7*w*c^10+12*b^8*w*c^10+18*b^6*w*c^10+384*b^8*w-2880*w*b^8*c-12*w*c^
5+5616*b^7*w*c^7-6912*b^7*w*c+216*b^5*w*c^7-216*b^3*w*c^7)/((20*b^2*
c^2+16*b*c^2+10*b^2*c^4+36*b^2*c-66*b^2*c^3+78*b^3*c^4+80*b^3*c-78*b
^4*c^4+238*b^4*c^3-156*b^4*c^2-68*b^4*c+24*b^3+72*b^4+2*c^3-8*c^4*b-
156*b^3*c^2+9*c^5*b^2-20*c^5*b^3+9*c^6*b^4-17*c^5*b^4-3*c^6*b^3+176*
b^5*c^2-192*b^5*c+66*b^6*c^4-126*b^6*c^3+132*b^6*c^2-72*b^6*c+48*b^5
*c^5+64*b^5+16*b^6+2*b^6*c^6-18*b^6*c^5-8*b^5*c^6-88*b^5*c^4)(w-1)(b
^2*c^4-6*b^2*c^3+13*b^2*c^2-12*b^2*c+4*b^2+c^2));
```

```
x3=-2/9*(-3942*w^2*b^5*c^6-1296*b^5*w^2*c+27*w^2*c^8*b^3+32*b^8*w^2+
306*b^2*c^4+108*b^2*c^3-2448*b^3*c^4-10890*b^4*c^4+1656*b^4*c^3+2592
*b^4*c^2-1008*b^4*c+w^2*c^10*b^8+72*c^4*b+2052*b^3*c^3-390*w^2*b^8*c
^7+45*c^8*b^3-360*b^3*c^2+153*c^6*b^2-576*c^5*b^2-513*c^7*b^3+324*c^
8*b^4+414*c^7*b^4-5445*c^6*b^4+27*c^7*b^2+12420*c^5*b^4+1224*c^6*b^3
-36*c^6*b+1953*b^6*c^8-31212*b^5*c^3+17856*b^5*c^2-3888*b^5*c+76158*
b^6*c^4-49068*b^6*c^3+15624*b^6*c^2-720*b^6*c-2232*b^5*c^8+7803*b^5*
c^7+9*c^5-12267*b^6*c^7-576*b^6+38079*b^6*c^6-69120*b^6*c^5-11430*b^
5*c^6-14172*c^3*b^6*w^2-1683*w^2*b^8*c^5-756*w^2*b^5*c^8-1152*b^7-28
8*b^8+22860*b^5*c^4+2160*b^7*c^8-5616*b^7*c^7+7092*b^7*c^6-14184*b^7
*c^4+22464*b^7*c^3-17280*b^7*c^2+6912*b^7*c-900*b^8*c^8+3510*b^8*c^7
-8865*b^8*c^6+15147*b^8*c^5-17730*b^8*c^4+14040*b^8*c^3-7200*b^8*c^2
+2160*b^8*c+243*c^9*b^5-45*c^9*b^6-432*c^9*b^7-63*c^9*b^4-18*c^10*b^
6+36*c^10*b^7-9*c^10*b^8+135*c^9*b^8+685*w^2*b^6*c^8-15*w^2*c^9*b^8+
81*w^2*c^9*b^5-18360*w^2*b^6*c^5+100*w^2*b^8*c^8-57*w^2*c^9*b^6-1560
*w^2*b^8*c^3+126*b^2*w^2*c^4+6048*c^2*b^5*w^2+800*w^2*b^8*c^2+985*w^
2*b^8*c^6+186*w^2*c^7*b^4-3543*w^2*b^6*c^7-2367*w^2*c^6*b^4-184*w^2*
c^5*b^2+1104*b^4*w^2*c^2+2673*w^2*b^5*c^7+1970*w^2*b^8*c^4-912*c*b^6
*w^2+138*w^2*c^8*b^4+10293*w^2*b^6*c^6-4*b^2*w^2*c^3+20586*w^2*b^6*c
^4-243*w^2*c^7*b^3-27*w^2*c^9*b^4-1080*c^4*b^3*w^2-432*b^4*w^2*c-216
*b^3*w^2*c^2-w^2*c^5+5480*c^2*b^6*w^2+972*b^3*w^2*c^3-240*c*b^8*w^2+
63*c^6*b^2*w^2-c^7*b^2*w^2+540*c^6*b^3*w^2-10692*c^3*b^5*w^2+7884*c^
4*b^5*w^2-4734*c^4*b^4*w^2+5388*c^5*b^4*w^2+744*c^3*b^4*w^2)/((1+w)*
(w-1)*(20*b^2*c^2+16*b*c^2+10*b^2*c^4+36*b^2*c-66*b^2*c^3+78*b^3*c^4
+80*b^3*c-78*b^4*c^4+238*b^4*c^3-156*b^4*c^2-68*b^4*c+24*b^3+72*b^4+
2*c^3-8*c^4*b-156*b^3*c^2+9*c^5*b^2-20*c^5*b^3+9*c^6*b^4-17*c^5*b^4-
3*c^6*b^3+176*b^5*c^2-192*b^5*c+66*b^6*c^4-126*b^6*c^3+132*b^6*c^2-7
2*b^6*c+48*b^5*c^5+64*b^5+16*b^6+2*b^6*c^6-18*b^6*c^5-8*b^5*c^6-88*b
^5*c^4)*(b^2*c^4-6*b^2*c^3+13*b^2*c^2-12*b^2*c+4*b^2+c^2));
```

Appendix 2.

Here are the formulas for $d_1 = d_1(b, c, w)$, $d_2 = d_2(b, c, w)$, $d_3 = d_3(b, c, w)$ from (4.3). They are written in a machine readable form convenient for to copy-paste into some symbolic computations package:

```
d1=-1/18*(-4410*w^2*b^5*c^6+30*c^10*b^4*w+792*c^9*b^5*w-45*w^2*c^9*b
^3-7056*b^5*w^2*c-9*w^2*c^8*b^3+3024*b^2*c^2-3456*b^2*c^4-3528*b^2*c
^3+10620*b^3*c^4+1476*b*c^3+3312*b^3*c-26856*b^4*c^4+26136*b^4*c^3-1
```

```
0368*b^4*c^2-2592*b^4*c+2304*b^4-1800*c^4*b-6768*b^3*c^3+477*c^8*b^3
-3816*b^3*c^2-1728*c^6*b^2+6264*c^5*b^2+1692*c^7*b^3-1296*c^8*b^4+65
34*c^7*b^4-13428*c^6*b^4-882*c^7*b^2+19656*c^5*b^4-5310*c^6*b^3+900*
c^6*b+12672*b^6*c^8-83412*b^5*c^3+72792*b^5*c^2-33840*b^5*c+231840*b
^6*c^4-188928*b^6*c^3+101376*b^6*c^2-32256*b^6*c-9099*b^5*c^8+20853*
b^5*c^7+288*c^4-288*c^5+144*c^6+36*w^2*c^10*b^6-47232*b^6*c^7+6624*b
^5+4608*b^6+115920*b^6*c^6-196128*b^6*c^5-24174*b^5*c^6-47232*c^3*b^
6*w^2-1809*w^2*b^5*c^8+104*w^2*c^8*b^2+48348*b^5*c^4+2115*c^9*b^5-20
16*c^9*b^6-207*c^9*b^3-162*c^9*b^4+378*c^8*b^2+144*c^10*b^6+72*c^10*
b^4+3168*w^2*b^6*c^8+441*w^2*c^9*b^5-49032*w^2*b^6*c^5+72*w^2*c^4-72
*c^9*b^3*w-504*w^2*c^9*b^6-1440*b^2*w^2*c^4+14472*c^2*b^5*w^2+758*w^
2*c^7*b^4+36*w*c^6-11808*w^2*b^6*c^7+404*w^2*c^6*b^4+2288*w^2*c^5*b^
2-2576*b^4*w^2*c^2+3951*w^2*b^5*c^7+72*w*c^4-8064*c*b^6*w^2-322*w^2*
c^8*b^4+28980*w^2*b^6*c^6+1152*b^6*w^2+36*w^2*c^6-840*b^2*w^2*c^3+14
*w^2*c^10*b^4+57960*w^2*b^6*c^4+1044*w^2*c^7*b^3-6*w^2*c^9*b^4+4788*
c^4*b^3*w^2-2640*b^4*w*c^2+864*b^2*w*c^4+5760*b^3*w*c^3-96*b^4*w^2*c
-12672*b^5*c*w+72*b^3*w^2*c^2+1440*b^5*w^2-2304*b^4*w*c-72*c^10*b^5*
w+2304*b^5*w-4032*b^3*w*c^2-1008*b^2*w*c^3-45*b^5*w^2*c^10-72*w^2*c^
5-288*b*w*c^4-504*b*w^2*c^4+66*c^8*b^2*w-207*c^10*b^5+25344*c^2*b^6*
w^2-4176*b^3*w^2*c^3+252*w^2*c^6*b-720*c^6*b^2*w^2-210*c^7*b^2*w^2-2
394*c^6*b^3*w^2-15804*c^3*b^5*w^2+8820*c^4*b^5*w^2+808*c^4*b^4*w^2-2
464*c^5*b^4*w^2+3032*c^3*b^4*w^2+1152*b^6*w-29280*b^4*w*c^4+21888*b^
5*w*c^4-36000*b^5*w*c^3+29376*b^5*w*c^2+17040*b^4*w*c^3-330*b^4*w*c^
8+1872*b^3*w*c^6+4260*b^4*w*c^7+27048*b^4*w*c^5-600*b^2*w*c^5-3672*b
^5*w*c^8-10944*b^5*w*c^6-14640*b^4*w*c^6+432*b^2*w*c^6-47232*b^6*w*c
^3+25344*b^6*w*c^2-8064*b^6*w*c+3168*b^6*w*c^8-11808*b^6*w*c^7-49032
*b^6*w*c^5+144*b*w*c^6+28980*b^6*w*c^6+57960*b^6*w*c^4-3744*b^3*w*c^
4-504*b^6*w*c^9-252*b^2*w*c^7-144*b^4*w*c^9+504*b^3*w*c^8+36*b^6*w*c
^10-72*w*c^5+9000*b^5*w*c^7-1440*b^3*w*c^7+960*b^4*w+448*b^4*w^2-99*
b*w^2*c^7-72*b*w*c^7-369*c^7*b+720*b^3*w^2*c+1152*b^3*c*w+528*b^2*w*
c^2+832*b^2*w^2*c^2+396*b*w^2*c^3+288*b*w*c^3)*b/((1+w)(44*b^2*c^2+3
6*b*c^2+22*b^2*c^4+68*b^2*c-134*b^2*c^3+150*b^3*c^4+160*b^3*c-166*b^
4*c^4+490*b^4*c^3-332*b^4*c^2-116*b^4*c+40*b^3+136*b^4+6*c^3-18*c^4*
b-300*b^3*c^2+17*c^5*b^2-40*c^5*b^3+17*c^6*b^4-29*c^5*b^4-5*c^6*b^3+
396*b^5*c^2-432*b^5*c+198*b^6*c^4-378*b^6*c^3+396*b^6*c^2-216*b^6*c+
108*b^5*c^5+144*b^5+48*b^6+6*b^6*c^6-54*b^6*c^5-18*b^5*c^6-198*b^5*c
^4)(b^2*c^4-6*b^2*c^3+13*b^2*c^2-12*b^2*c+4*b^2+c^2));
```

```
d2=1/18*(-4410*w^2*b^5*c^6-30*c^10*b^4*w-792*c^9*b^5*w-45*w^2*c^9*b^
3-7056*b^5*w^2*c-9*w^2*c^8*b^3+3024*b^2*c^2-3456*b^2*c^4-3528*b^2*c^
3+10620*b^3*c^4+1476*b*c^3+3312*b^3*c-26856*b^4*c^4+26136*b^4*c^3-10
368*b^4*c^2-2592*b^4*c+2304*b^4-1800*c^4*b-6768*b^3*c^3+477*c^8*b^3-
3816*b^3*c^2-1728*c^6*b^2+6264*c^5*b^2+1692*c^7*b^3-1296*c^8*b^4+653
4*c^7*b^4-13428*c^6*b^4-882*c^7*b^2+19656*c^5*b^4-5310*c^6*b^3+900*c
^6*b+12672*b^6*c^8-83412*b^5*c^3+72792*b^5*c^2-33840*b^5*c+231840*b^
6*c^4-188928*b^6*c^3+101376*b^6*c^2-32256*b^6*c-9099*b^5*c^8+20853*b
^5*c^7+288*c^4-288*c^5+144*c^6+36*w^2*c^10*b^6-47232*b^6*c^7+6624*b^
5+4608*b^6+115920*b^6*c^6-196128*b^6*c^5-24174*b^5*c^6-47232*c^3*b^6
*w^2-1809*w^2*b^5*c^8+104*w^2*c^8*b^2+48348*b^5*c^4+2115*c^9*b^5-201
6*c^9*b^6-207*c^9*b^3-162*c^9*b^4+378*c^8*b^2+144*c^10*b^6+72*c^10*b
```

```
^4+3168*w^2*b^6*c^8+441*w^2*c^9*b^5-49032*w^2*b^6*c^5+72*w^2*c^4+72*
c^9*b^3*w-504*w^2*c^9*b^6-1440*b^2*w^2*c^4+14472*c^2*b^5*w^2+758*w^2
*c^7*b^4-36*w*c^6-11808*w^2*b^6*c^7+404*w^2*c^6*b^4+2288*w^2*c^5*b^2
-2576*b^4*w^2*c^2+3951*w^2*b^5*c^7-72*w*c^4-8064*c*b^6*w^2-322*w^2*c
^8*b^4+28980*w^2*b^6*c^6+1152*b^6*w^2+36*w^2*c^6-840*b^2*w^2*c^3+14*
w^2*c^10*b^4+57960*w^2*b^6*c^4+1044*w^2*c^7*b^3-6*w^2*c^9*b^4+4788*c
^4*b^3*w^2+2640*b^4*w*c^2-864*b^2*w*c^4-5760*b^3*w*c^3-96*b^4*w^2*c+
12672*b^5*c*w+72*b^3*w^2*c^2+1440*b^5*w^2+2304*b^4*w*c+72*c^10*b^5*w
-2304*b^5*w+4032*b^3*w*c^2+1008*b^2*w*c^3-45*b^5*w^2*c^10-72*w^2*c^5
+288*b*w*c^4-504*b*w^2*c^4-66*c^8*b^2*w-207*c^10*b^5+25344*c^2*b^6*w
^2-4176*b^3*w^2*c^3+252*w^2*c^6*b-720*c^6*b^2*w^2-210*c^7*b^2*w^2-23
94*c^6*b^3*w^2-15804*c^3*b^5*w^2+8820*c^4*b^5*w^2+808*c^4*b^4*w^2-24
64*c^5*b^4*w^2+3032*c^3*b^4*w^2-1152*b^6*w+29280*b^4*w*c^4-21888*b^5
*w*c^4+36000*b^5*w*c^3-29376*b^5*w*c^2-17040*b^4*w*c^3+330*b^4*w*c^8
-1872*b^3*w*c^6-4260*b^4*w*c^7-27048*b^4*w*c^5+600*b^2*w*c^5+3672*b^
5*w*c^8+10944*b^5*w*c^6+14640*b^4*w*c^6-432*b^2*w*c^6+47232*b^6*w*c^
3-25344*b^6*w*c^2+8064*b^6*w*c-3168*b^6*w*c^8+11808*b^6*w*c^7+49032*
b^6*w*c^5-144*b*w*c^6-28980*b^6*w*c^6-57960*b^6*w*c^4+3744*b^3*w*c^4
+504*b^6*w*c^9+252*b^2*w*c^7+144*b^4*w*c^9-504*b^3*w*c^8-36*b^6*w*c^
10+72*w*c^5-9000*b^5*w*c^7+1440*b^3*w*c^7-960*b^4*w+448*b^4*w^2-99*b
*w^2*c^7+72*b*w*c^7-369*c^7*b+720*b^3*w^2*c-1152*b^3*c*w-528*b^2*w*c
^2+832*b^2*w^2*c^2+396*b*w^2*c^3-288*b*w*c^3)*b/((w-1)(44*b^2*c^2+36
*b*c^2+22*b^2*c^4+68*b^2*c-134*b^2*c^3+150*b^3*c^4+160*b^3*c-166*b^4
*c^4+490*b^4*c^3-332*b^4*c^2-116*b^4*c+40*b^3+136*b^4+6*c^3-18*c^4*b
-300*b^3*c^2+17*c^5*b^2-40*c^5*b^3+17*c^6*b^4-29*c^5*b^4-5*c^6*b^3+3
96*b^5*c^2-432*b^5*c+198*b^6*c^4-378*b^6*c^3+396*b^6*c^2-216*b^6*c+1
08*b^5*c^5+144*b^5+48*b^6+6*b^6*c^6-54*b^6*c^5-18*b^5*c^6-198*b^5*c^
4)(b^2*c^4-6*b^2*c^3+13*b^2*c^2-12*b^2*c+4*b^2+c^2));
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d3=-1/9*(-9882*w^2*b^5*c^6-81*w^2*c^9*b^3-13392*b^5*w^2*c+243*w^2*c^
8*b^3+2232*b^2*c^2-4752*b^2*c^4-2016*b^2*c^3+16236*b^3*c^4+1044*b*c^
3+1584*b^3*c+17064*b^4*c^4+576*b^4*c^3-6408*b^4*c^2+864*b^4*c+864*b^
4-1368*c^4*b-15408*b^3*c^3-279*c^8*b^3+2232*b^3*c^2-2376*c^6*b^2+716
4*c^5*b^2+3852*c^7*b^3-801*c^8*b^4+144*c^7*b^4+8532*c^6*b^4-504*c^7*
b^2-20916*c^5*b^4-8118*c^6*b^3+684*c^6*b+7920*b^6*c^8-29412*b^5*c^3+
28728*b^5*c^2-14832*b^5*c+144900*b^6*c^4-118080*b^6*c^3+63360*b^6*c^
2-20160*b^6*c-3591*b^5*c^8+7353*b^5*c^7+180*c^4-180*c^5+90*c^6+54*w^
2*c^10*b^6-29520*b^6*c^7+3168*b^5+2880*b^6+72450*b^6*c^6-122580*b^6*
c^5-7758*b^5*c^6-70848*c^3*b^6*w^2-3645*w^2*b^5*c^8+137*w^2*c^8*b^2+
15516*b^5*c^4+927*c^9*b^5-1260*c^9*b^6-99*c^9*b^3+54*c^9*b^4+279*c^8
*b^2+90*c^10*b^6+27*c^10*b^4+4752*w^2*b^6*c^8+837*w^2*c^9*b^5-73548*
w^2*b^6*c^5+108*w^2*c^4-756*w^2*c^9*b^6-1008*b^2*w^2*c^4+29160*c^2*b
^5*w^2+2888*w^2*c^7*b^4-17712*w^2*b^6*c^7-6916*w^2*c^6*b^4+1988*w^2*
c^5*b^2-3896*b^4*w^2*c^2+8451*w^2*b^5*c^7-12096*c*b^6*w^2-487*w^2*c^
8*b^4+43470*w^2*b^6*c^6+1728*b^6*w^2+54*w^2*c^6-1344*b^2*w^2*c^3+29*
w^2*c^10*b^4+86940*w^2*b^6*c^4+324*w^2*c^7*b^3-78*w^2*c^9*b^4+2916*c
^4*b^3*w^2-1248*b^4*w^2*c-1944*b^3*w^2*c^2+2592*b^5*w^2-81*b^5*w^2*c
^10-108*w^2*c^5-648*b*w^2*c^4-99*c^10*b^5+38016*c^2*b^6*w^2-1296*b^3
*w^2*c^3+324*w^2*c^6*b-504*c^6*b^2*w^2-336*c^7*b^2*w^2-1458*c^6*b^3*
w^2-33804*c^3*b^5*w^2+19764*c^4*b^5*w^2-13832*c^4*b^4*w^2+11060*c^5*
```

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b^4*w^2+11552*c^3*b^4*w^2+928*b^4*w^2-135*b*w^2*c^7-261*c^7*b+1296*b
^3*w^2*c+1096*b^2*w^2*c^2+540*b*w^2*c^3)*b/((1+w)(w-1)(44*b^2*c^2+36
*b*c^2+22*b^2*c^4+68*b^2*c-134*b^2*c^3+150*b^3*c^4+160*b^3*c-166*b^4
*c^4+490*b^4*c^3-332*b^4*c^2-116*b^4*c+40*b^3+136*b^4+6*c^3-18*c^4*b
-300*b^3*c^2+17*c^5*b^2-40*c^5*b^3+17*c^6*b^4-29*c^5*b^4-5*c^6*b^3+3
96*b^5*c^2-432*b^5*c+198*b^6*c^4-378*b^6*c^3+396*b^6*c^2-216*b^6*c+1
08*b^5*c^5+144*b^5+48*b^6+6*b^6*c^6-54*b^6*c^5-18*b^5*c^6-198*b^5*c^
4)(b^2*c^4-6*b^2*c^3+13*b^2*c^2-12*b^2*c+4*b^2+c^2)).
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