A NOTE ON SOLUTIONS OF THE CUBOID FACTOR EQUATIONS.

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Abstract. A rational perfect cuboid is a rectangular parallelepiped whose edges and face diagonals are given by rational numbers and whose space diagonal is equal to unity. It is described by a system of four quadratic equations with respect to six variables. The cuboid factor equations were derived from these four equations by symmetrization procedure. They constitute a system of eight polynomial equations. Recently two sets of formulas were derived providing two solutions for the cuboid factor equations. These two solutions are studied in the present paper. They are proved to coincide with each other up to a change of parameters in them.

1. Introduction.

Finding a rational perfect cuboid is equivalent to finding a perfect cuboid with all integer edges and diagonals, which is an old unsolved problem known since 1719. The history of cuboid studies can be followed through the references [1–44]. Here are the equations describing perfect cuboids:

$$\begin{aligned} &x_1^2+x_2^2+x_3^2-L^2=0, && x_2^2+x_3^2-d_1^2=0,\\ &x_3^2+x_1^2-d_2^2=0, && x_1^2+x_2^2-d_3^2=0. \end{aligned} \tag{1.1}$$

The variables x_1, x_2, x_3 in (1.1) represent edges of a cuboid, the variables d_1, d_2, d_3 are its face diagonals, and L is its space diagonal. In the case of a rational perfect cuboid we set $L=1$.

Let's denote through p_0, p_1, p_2, p_3 the left hand sides of the cuboid equations (1.1), i. e. let's introduce the following notations:

$$\begin{aligned} &p_0=x_1^2+x_2^2+x_3^2-L^2, && p_1=x_2^2+x_3^2-d_1^2,\\ &p_2=x_3^2+x_1^2-d_2^2, && p_3=x_1^2+x_2^2-d_3^2. \end{aligned} \tag{1.2}$$

Using the polynomials (1.2), the following eight equations are written:

$$\begin{aligned} &p_0=0, && \sum_{i=1}^{3} p_i=0,\\ &\sum_{i=1}^{3} d_i\,p_i=0, && \sum_{i=1}^{3} x_i\,p_i=0, \end{aligned} \tag{1.3}$$

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$$\begin{aligned}
&\sum_{i=1}^{3} d_i^2\,p_i=0, &&\sum_{i=1}^{3} x_i^2\,p_i=0,\\
&\sum_{i=1}^{3} x_i\,d_i\,p_i=0, &&\sum_{i=1}^{3} x_i^2\,d_i^2\,p_i=0.
\end{aligned}\tag{1.4}$$

The equations (1.3) and (1.4) are called the cuboid factor equations. They were derived as a result of a symmetry approach to the original cuboid equations (1.1) initiated in [45] (see also [46–48]).

It is easy to see that each solution of the original cuboid equations (1.1) is a solution for the factor equations (1.3) and (1.4). Generally speaking, the converse is not true. However, in [47] the following theorem was proved.

Theorem 1.1. *Each integer or rational solution of the factor equations* (1.3) *and* (1.4) *such that* $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, $d_1 > 0$, $d_2 > 0$, *and* $d_3 > 0$ *is an integer or rational solution for the equations* (1.1).

Due to the theorem 1.1 the factor equations (1.3) and (1.4) are equivalent to the equations (1.1) in studying perfect cuboids. But in this paper, saying a solution of the factor equations we assume any integer or rational solution, i. e. even such that some of the inequalities $x_1 > 0$, $x_2 > 0$, $x_3 > 0$, $d_1 > 0$, $d_2 > 0$, $d_3 > 0$ or all of them are not fulfilled.

Note that the left hand sides of the factor equations are multisimmetric polynomials in x_1, x_2, x_3 and d_1, d_2, d_3, i. e. they are invariant with respect to the S_3 permutation group acting upon x_1, x_2, x_3, d_1, d_2, d_3, and L as follows:

$$\sigma(x_i)=x_{\sigma i},\qquad \sigma(d_i)=d_{\sigma i},\qquad \sigma(L)=L.$$

For the theory of multisymmetric polynomials the reader is referred to [49–69]. According to this theory, each multisymmetric polynomial is expressed through the following nine elementary multisymmetric polynomials:

$$\begin{aligned}
&x_1+x_2+x_3=E_{10},\\
&x_1\,x_2+x_2\,x_3+x_3\,x_1=E_{20},\\
&x_1\,x_2\,x_3=E_{30},
\end{aligned}\tag{1.5}$$

$$\begin{aligned}
&d_1+d_2+d_3=E_{01},\\
&d_1\,d_2+d_2\,d_3+d_3\,d_1=E_{02},\\
&d_1\,d_2\,d_3=E_{03},
\end{aligned}\tag{1.6}$$

$$\begin{aligned}
&x_1\,x_2\,d_3+x_2\,x_3\,d_1+x_3\,x_1\,d_2=E_{21},\\
&x_1\,d_2+d_1\,x_2+x_2\,d_3+d_2\,x_3+x_3\,d_1+d_3\,x_1=E_{11},\\
&x_1\,d_2\,d_3+x_2\,d_3\,d_1+x_3\,d_1\,d_2=E_{12}.
\end{aligned}\tag{1.7}$$

Expressing the left hand sides of the factor equations (1.3) and (1.4) through the polynomials (1.5), (1.6) and (1.7), one gets polynomial equations with respect to the variables E_{10}, E_{20}, E_{30}, E_{01}, E_{02}, E_{03}, E_{21}, E_{11}, E_{12}, and L (see (3.1) through (3.7) in [48]). These equations were complemented with fourteen identities expressing the algebraic dependence of the elementary multisymmetric polynomials (1.5), (1.6), and (1.7) (see (3.8) in [48]). As a result a system of twenty two polynomial equations was obtained. In [48] this huge system of twenty two polynomial equations was reduced to the following single polynomial equation for E_{10}, E_{01}, E_{11}, and L:

$$(2\,E_{11})^2+(E_{01}^2+L^2-E_{10}^2)^2-8\,E_{01}^2\,L^2=0. \tag{1.8}$$

The other variables E_{20}, E_{30}, E_{02}, E_{03}, E_{21}, E_{12} are expressed as rational functions of E_{10}, E_{01}, E_{11}, and L (see formulas (4.1), (4.3), (5.1), (5.2), (4.6), (4.7) in [48]).

The equation (1.8) was solved by John Ramsden in [70]. In the case of a rational perfect cuboid, where $L=1$, omitting some inessential special cases, the general solution of the equation (1.8) is given by the formulas

$$E_{11}=-\frac{b\,(c^2+2-4\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}, \tag{1.9}$$

$$E_{10}=-\frac{b^2\,c^2+2\,b^2-3\,b^2\,c-c}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}, \tag{1.10}$$

$$E_{01}=-\frac{b\,(c^2+2-2\,c)}{b^2\,c^2+2\,b^2-3\,b^2\,c+c-b\,c^2+2\,b}. \tag{1.11}$$

Below are the formulas for E_{12}, E_{21}, E_{03}, E_{30}, E_{02}, E_{20} in (1.5), (1.6), and (1.7):

$$\begin{gathered}
E_{12}=(16\,b^6+32\,b^5-6\,c^5\,b^2+2\,c^5\,b-62\,b^5\,c^6+62\,b^6\,c^6+16\,b^4-\\
-180\,b^6\,c^5-c^7\,b^3+18\,b^5\,c^7-12\,b^6\,c^7-2\,b^5\,c^8+b^6\,c^8+248\,b^5\,c^2+\\
+248\,b^6\,c^2-96\,b^6\,c+321\,b^6\,c^4-180\,b^5\,c^3-144\,b^5\,c-360\,b^6\,c^3+\\
+b^4\,c^8+8\,b^4\,c^6-6\,b^4\,c^7+18\,b^4\,c^5+7\,b^3\,c^6+90\,b^5\,c^5-14\,b^3\,c^5+\\
+17\,b^2\,c^4+32\,b^4\,c^2+28\,b^3\,c^3-28\,b^3\,c^2-4\,b\,c^3+8\,b^3\,c-57\,b^4\,c^4+\\
+36\,b^4\,c^3-12\,b^2\,c^3-48\,b^4\,c-c^4)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\
-12\,b^2\,c+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},
\end{gathered} \tag{1.12}$$

$$\begin{gathered}
E_{21}=\frac{b}{2}\,(5\,c^6\,b-2\,c^6\,b^2+52\,c^5\,b^2-16\,c^5\,b-2\,c^7\,b^2+2\,b^4\,c^8-\\
-26\,b^4\,c^7-426\,b^4\,c^5-61\,b^3\,c^6+100\,b^3\,c^5+14\,c^7\,b^3-c^8\,b^3-20\,b\,c^2-\\
-8\,b^2\,c^2-16\,b^2\,c-128\,b^2\,c^4-200\,b^3\,c^3+244\,b^3\,c^2+32\,b\,c^3+\\
+768\,b^4\,c^4-852\,b^4\,c^3+568\,b^4\,c^2+104\,b^2\,c^3-208\,b^4\,c+8\,c^4+\\
+16\,b^3-112\,b^3\,c+142\,b^4\,c^6+32\,b^4-2\,c^5)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\
-12\,b^2\,c-4\,c^3+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2},
\end{gathered} \tag{1.13}$$

$$\begin{gathered}
E_{03}=\frac{b}{2}\,(b^2\,c^4-5\,b^2\,c^3+10\,b^2\,c^2-10\,b^2\,c+4\,b^2+2\,b\,c+2\,c^2-\\
-b\,c^3)\,(2\,b^2\,c^4-12\,b^2\,c^3+26\,b^2\,c^2-24\,b^2\,c+8\,b^2-c^4\,b+3\,b\,c^3-\\
-6\,b\,c+4\,b+c^3-2\,c^2+2\,c)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-\\
-12\,b^2\,c+4\,b^2+c^2)^{-1}\,(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2},
\end{gathered} \tag{1.14}$$

$$E_{30} = c\,b^2\,(1-c)\,(c-2)\,(b\,c^2-4\,b\,c+2+4\,b)\,(2\,b\,c^2-c^2-4\,b\,c+ \\ +2\,b)\,(b^2\,c^4-6\,b^2\,c^3+13\,b^2\,c^2-12\,b^2\,c+4\,b^2+c^2)^{-1}\times \\ \times(b\,c-1-b)^{-2}\,(-c+b\,c-2\,b)^{-2}, \tag{1.15}$$

$$E_{02} = \frac{1}{2}\,(28\,b^2\,c^2-16\,b^2\,c-2\,c^2-4\,b^2-b^2\,c^4+4\,b^3\,c^4-12\,b^3\,c^3+ \\ +4\,b\,c^3+24\,b^3\,c-8\,b\,c-2\,b^4\,c^4+12\,b^4\,c^3-26\,b^4\,c^2-8\,b^2\,c^3+ \\ +24\,b^4\,c-16\,b^3-8\,b^4)\,(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2}, \tag{1.16}$$

$$E_{20} = \frac{b}{2}\,(b\,c^2-2\,c-2\,b)\,(2\,b\,c^2-c^2-6\,b\,c+2+4\,b)\times \\ \times(b\,c-1-b)^{-2}\,(b\,c-c-2\,b)^{-2}. \tag{1.17}$$

The formulas (1.12), (1.13), (1.14), (1.15), (1.16), (1.17) were derived in [71] by substituting (1.9), (1.10), and (1.11) along with $L=1$ into the corresponding formulas from [48].

Thus, the right hand sides of the equalities (1.5), (1.6), and (1.7) turned out to be expressed through two arbitrary rational parameters b and c. The next step was to resolve these equalities with respect to x_1, x_2, x_3, d_1, d_2, d_3. For this purpose in [71] the following two cubic equations were written:

$$x^3 - E_{10}\,x^2 + E_{20}\,x - E_{30} = 0, \tag{1.18}$$

$$d^3 - E_{01}\,d^2 + E_{02}\,d - E_{03} = 0. \tag{1.19}$$

Note that the left hand sides of the equalities (1.5) are regular symmetric polynomials of the variables x_1, x_2, x_3 (see [72]). Similarly, the left hand sides of the equalities (1.6) are regular symmetric polynomials of the variables d_1, d_2, d_3. For this reason x_1, x_2, x_3 can be found as roots of the cubic equation (1.18). Similarly, d_1, d_2, d_3 are roots of the second cubic equation (1.19). Relying on these facts, in [71] the following two inverse problems were formulated.

Problem 1.1. *Find all pairs of rational numbers b and c for which the cubic equations* (1.18) *and* (1.19) *with the coefficients given by the formulas* (1.10), (1.11), (1.14), (1.15), (1.16), (1.17) *possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations* (1.7) *whose right hand sides are given by the formulas* (1.9), (1.12), *and* (1.13).

Problem 1.2. *Find at least one pair of rational numbers b and c for which the cubic equations* (1.18) *and* (1.19) *with the coefficients given by the formulas* (1.10), (1.11), (1.14), (1.15), (1.16), (1.17) *possess positive rational roots x_1, x_2, x_3, d_1, d_2, d_3 obeying the auxiliary polynomial equations* (1.7) *whose right hand sides are given by the formulas* (1.9), (1.12), *and* (1.13).

Due to the theorem 1.1 the inverse problems 1.1 and 1.2 are equivalent to finding all rational perfect cuboids and to finding at least one rational perfect cuboid respectively. Singularities of the inverse problems 1.1 and 1.2 due to the denominators in the formulas (1.9) through (1.17) were studied in [73]. Some special cases where the equations (1.5), (1.6), (1.7) are solvable with respect to the cuboid variables x_1, x_2, x_3 and d_1, d_2, d_3 were found in [74]. However, none of these special cases have produced a perfect cuboid since the inequalities

$$\begin{aligned} &x_1 > 0, && x_2 > 0, && x_3 > 0, \\ &d_1 > 0, && d_2 > 0, && d_3 > 0 \end{aligned} \tag{1.20}$$

required for solving the problems 1.1 and 1.2 are not fulfilled in these special cases.

Again, neglecting the inequalities (1.20), an approach to solving the equations (1.5), (1.6), (1.7) was found in [75]. It exploits the following lemma.

Lemma 1.1. *A reduced cubic equation $y^3 + y^2 + D = 0$ has three rational roots if and only if there is a rational number w satisfying the sextic equation*

$$D\,(w^2 + 3)^3 + 4\,(w - 1)^2\,(1 + w)^2 = 0. \tag{1.21}$$

In this case the roots of the cubic equation $y^3 + y^2 + D = 0$ are given by the formulas

$$y_1 = -\frac{2\,(w + 1)}{w^2 + 3}, \qquad y_2 = \frac{2\,(w - 1)}{w^2 + 3}, \qquad y_3 = \frac{1 - w^2}{w^2 + 3}. \tag{1.22}$$

Based on the lemma 1.1 and on the cubic equations (1.18) and (1.19), in [75] two sextic equations of the form (1.21) were derived:

$$D_1\,(w_1^2 + 3)^3 + 4\,(w_1 - 1)^2\,(1 + w_1)^2 = 0, \tag{1.23}$$

$$D_2\,(w_2^2 + 3)^3 + 4\,(w_2 - 1)^2\,(1 + w_2)^2 = 0. \tag{1.24}$$

The D-parameters D_1 and D_2 of the sextic equations (1.23) and (1.24) depend on the same two rational numbers b and c as E_{11}, E_{10}, E_{01}, E_{12}, E_{21}, E_{03}, E_{30}, E_{02}, E_{20} in the formulas (1.9) through (1.17). They are given by the formulas

$$\begin{aligned} D_1 = -\frac{2}{27}\,&(7812\,b^4\,c^4 - 216\,b^2\,c^4 - 52\,b^2\,c^3 + 1764\,b^3\,c^4 - 1200\,b^4\,c^3 - \\ &- 1848\,b^4\,c^2 + 720\,b^4\,c - 36\,c^4\,b - 1512\,b^3\,c^3 - 36\,c^8\,b^3 + 288\,b^3\,c^2 - \\ &- 108\,c^6\,b^2 + 380\,c^5\,b^2 + 378\,c^7\,b^3 - 231\,c^8\,b^4 - 300\,c^7\,b^4 + 3906\,c^6\,b^4 - \\ &- 13\,c^7\,b^2 - 8904\,c^5\,b^4 - 882\,c^6\,b^3 + 18\,c^6\,b - 1319\,b^6\,c^8 + 20952\,b^5\,c^3 - \\ &- 11952\,b^5\,c^2 + 2592\,b^5\,c - 48372\,b^6\,c^4 + 31620\,b^6\,c^3 - 10552\,b^6\,c^2 + \\ &+ 816\,b^6\,c + 1494\,b^5\,c^8 - 5238\,b^5\,c^7 - 4\,c^5 + 7905\,b^6\,c^7 - 24186\,b^6\,c^6 + \\ &+ 288\,b^6 + 43740\,b^6\,c^5 + 7686\,b^5\,c^6 + 576\,b^7 + 128\,b^8 - 15372\,b^5\,c^4 - \\ &- 1080\,b^7\,c^8 - 3546\,b^7\,c^6 + 51\,c^9\,b^6 + 400\,b^8\,c^8 - 162\,c^9\,b^5 + 8640\,b^7\,c^2 - \\ &- 3456\,b^7\,c + 2808\,b^7\,c^7 - 1560\,b^8\,c^7 + 3940\,b^8\,c^6 + 216\,c^9\,b^7 - 960\,b^8\,c - \\ &- 6240\,b^8\,c^3 + 9\,c^{10}\,b^6 + 7880\,b^8\,c^4 + 4\,c^{10}\,b^8 - 6732\,b^8\,c^5 + 45\,c^9\,b^4 + \\ &+ 3200\,b^8\,c^2 - 11232\,b^7\,c^3 + 7092\,b^7\,c^4 - 18\,c^{10}\,b^7 - 60\,c^9\,b^8)^2\,(2\,c^2 + \\ &+ 2\,b^4\,c^4 - 12\,b^4\,c^3 + 26\,b^4\,c^2 - 24\,b^4\,c + 8\,b^4 - 6\,b^3\,c^4 + 18\,b^3\,c^3 - \\ &- 36\,b^3\,c + 24\,b^3 + 3\,b^2\,c^4 + 8\,b^2\,c^3 - 36\,b^2\,c^2 + 16\,b^2\,c + 12\,b^2 - 6\,b\,c^3 + \\ &+ 12\,b\,c)^{-3}\,(b^2\,c^4 - 6\,b^2\,c^{-3} + 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-2}, \end{aligned} \tag{1.25}$$

$$
\begin{aligned}
D_2 = -\frac{2\,b^2}{27}\,&(832\,b^2\,c^2 - 1440\,b^2\,c^4 - 840\,b^2\,c^3 + 4788\,b^3\,c^4 + 396\,b\,c^3 + \\
&+ 720\,b^3\,c + 808\,b^4\,c^4 + 3032\,b^4\,c^3 - 2576\,b^4\,c^2 - 96\,b^4\,c + 448\,b^4 - \\
&- 504\,c^4\,b - 4176\,b^3\,c^3 - 9\,c^8\,b^3 + 72\,b^3\,c^2 - 720\,c^6\,b^2 + 2288\,c^5\,b^2 + \\
&+ 1044\,c^7\,b^3 - 322\,c^8\,b^4 + 758\,c^7\,b^4 + 404\,c^6\,b^4 - 210\,c^7\,b^2 - 2464\,c^5\,b^4 - \\
&- 2394\,c^6\,b^3 + 72\,c^4 + 252\,c^6\,b + 3168\,b^6\,c^8 + 441\,c^9\,b^5 - 7056\,b^5\,c + \\
&+ 57960\,b^6\,c^4 - 47232\,b^6\,c^3 + 25344\,b^6\,c^2 - 8064\,b^6\,c - 1809\,b^5\,c^8 + \\
&+ 14472\,b^5\,c^2 + 3951\,b^5\,c^7 - 72\,c^5 + 36\,c^6 - 11808\,b^6\,c^7 + 1440\,b^5 + \\
&+ 28980\,b^6\,c^6 - 49032\,b^6\,c^5 - 4410\,b^5\,c^6 + 8820\,b^5\,c^4 - 15804\,b^5\,c^3 + \\
&+ 1152\,b^6 - 504\,c^9\,b^6 - 45\,c^9\,b^3 - 6\,c^9\,b^4 + 104\,c^8\,b^2 + 36\,c^{10}\,b^6 + \\
&+ 14\,c^{10}\,b^4 - 45\,c^{10}\,b^5 - 99\,c^7\,b)^2\,(6\,b^4\,c^4 - 36\,b^4\,c^3 + 78\,b^4\,c^2 - 72\,b^4\,c + \\
&+ 24\,b^4 - 12\,b^3\,c^4 + 36\,b^3\,c^3 - 72\,b^3\,c + 48\,b^3 + 5\,b^2\,c^4 + 16\,b^2\,c^3 - \\
&- 68\,b^2\,c^2 + 32\,b^2\,c + 20\,b^2 - 12\,b\,c^3 + 24\,b\,c + 6\,c^2)^{-3}\,(b^2\,c^4 - 6\,b^2\,c^3 + \\
&+ 13\,b^2\,c^2 - 12\,b^2\,c + 4\,b^2 + c^2)^{-2}.
\end{aligned}
\tag{1.26}
$$

Along with (1.25) and (1.26), in [75] twelve rational functions were derived:

$$
\begin{aligned}
&x_1 = x_1(b,c,w_1), &&x_2 = x_2(b,c,w_1), &&x_3 = x_3(b,c,w_1),\\
&d_1 = d_1(b,c,w_1), &&d_2 = d_2(b,c,w_1), &&d_3 = d_3(b,c,w_1),
\end{aligned}
\tag{1.27}
$$

$$
\begin{aligned}
&x_1 = x_1(b,c,w_2), &&x_2 = x_2(b,c,w_2), &&x_3 = x_3(b,c,w_2),\\
&d_1 = d_1(b,c,w_2), &&d_2 = d_2(b,c,w_2), &&d_3 = d_3(b,c,w_2).
\end{aligned}
\tag{1.28}
$$

The explicit formulas for (1.27) and (1.28) are very huge. Therefore we provide them in the ancillary files **Solutions_1.txt** and **Solutions_2.txt** attached to this arXiv submission. The main result of [75] is formulated in the following theorems.

Theorem 1.2. *If (b, c, w_1) is a triple of rational numbers solving the equation (1.23), where D_1 is given by (1.25), and belonging to the domain of the rational functions (1.27), then the values of these functions provide a rational solution for the equations (1.5), (1.6), (1.7) and for the cuboid factor equations (1.3), (1.4).*

Theorem 1.3. *If (b, c, w_2) is a triple of rational numbers solving the equation (1.24), where D_2 is given by (1.26), and belonging to the domain of the rational functions (1.28), then the values of these functions provide a rational solution for the equations (1.5), (1.6), (1.7) and for the cuboid factor equations (1.3), (1.4).*

The main goal of the present paper is to prove that the sets of solutions to the equations (1.5), (1.6), (1.7) and to the cuboid factor equations (1.3) and (1.4) provided by the theorems 1.2 and 1.3 do essentially coincide.

2. Some prerequisites.

Let's consider a general cubic equation with the coefficients A_0, A_1, A_2, A_3:

$$A_3\,x^3 + A_2\,x^2 + A_1\,x + A_0 = 0. \tag{2.1}$$

Under some certain restrictions for its coefficients, the cubic equation (2.1) can be transformed to its reduced form $y^3 + y^2 + D = 0$, where D is given by the formula

$$D = -\frac{(9\,A_1\,A_2\,A_3 - 27\,A_0\,A_3^2 - 2\,A_2^3)^2}{27\,(A_2^2 - 3\,A_1\,A_3)^3}. \tag{2.2}$$

Applying the lemma 1.1 to the reduced form of the equation (2.1), one gets the formulas (1.22) for y_1, y_2, y_3. Then the backward transformation of y_1, y_2, y_3 to the roots of the equation (2.1) yields the formulas

$$\begin{aligned} x_1 = \frac{1}{18}\,&((2\,A_2^3 - 9\,A_1\,A_2\,A_3 + 27\,A_0\,A_3^2)\,w^2 + (18\,A_2\,A_1\,A_3 - 6\,A_2^3)\,w\,- \\ &-9\,A_1\,A_2\,A_3 + 81\,A_0\,A_3^2)\,A_3^{-1}\,(A_2^2 - 3\,A_1\,A_3)^{-1}\,(1+w)^{-1}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} x_2 = \frac{1}{18}\,&((2\,A_2^3 - 9\,A_1\,A_2\,A_3 + 27\,A_0\,A_3^2)\,w^2 - (18\,A_2\,A_1\,A_3 - 6\,A_2^3)\,w\,- \\ &-9\,A_1\,A_2\,A_3 + 81\,A_0\,A_3^2)\,A_3^{-1}\,(A_2^2 - 3\,A_1\,A_3)^{-1}\,(1-w)^{-1}, \end{aligned} \tag{2.4}$$

$$\begin{aligned} x_3 = \frac{1}{9}\,&((A_2^3 - 27\,A_0\,A_3^2)\,w^2 + 36\,A_1\,A_2\,A_3 - 81\,A_0\,A_3^2 - 9\,A_2^3)\,\times \\ &\times A_3^{-1}\,(A_2^2 - 3\,A_1\,A_3)^{-1}\,(1-w)^{-1}\,(1+w)^{-1}. \end{aligned} \tag{2.5}$$

As a result one can formulate the following lemma for the equation (2.1).

Lemma 2.1. *Assume that the numbers A_0, A_1, A_2, A_3 obey the inequalities*

$$A_3 \neq 0, \qquad \frac{A_1}{A_3} - \frac{A_2^2}{3\,A_3^2} \neq 0, \qquad \frac{A_0}{A_3} - \frac{A_1\,A_2}{3\,A_3^2} + \frac{2\,A_2^3}{27\,A_3^3} \neq 0. \tag{2.6 }$$

Then the general cubic polynomial (2.1) with the rational coefficients A_0, A_1, A_2, A_3 has three rational roots if and only if there is a rational number w satisfying the sextic equation (2.1) where D is given by the formula (2.2). In this case the roots of the cubic equation (2.1) are given by the formulas (2.3), (2.4), (2.5).

The detailed proofs of the lemmas 1.1 and 2.1 can be found in [75].

Now, assume that we have a cubic equation with three rational roots x_1, x_2, x_3. Then this cubic equation can be written as follows:

$$(x - x_1)(x - x_2)(x - x_3) = 0 \tag{2.7}$$

Expanding the left hand side of the equation (2.7), we find

$$\begin{aligned} &A_3 = 1, && A_1 = x_1\,x_2 + x_2\,x_3 + x_3\,x_1, \\ &A_0 = -x_1\,x_2\,x_3, && A_2 = -(x_1 + x_2 + x_3). \end{aligned} \tag{2.8}$$

The condition $A_3 \neq 0$ from (2.6) is fulfilled for the polynomial (2.7) since $A_3 = 1$ in (2.8). The second condition (2.6) for the polynomial (2.7) is written as

$$x_1^2 + x_2^2 + x_3^2 - x_2\,x_3 - x_1\,x_3 - x_1\,x_2 \neq 0. \tag{2.9}$$

The third condition (2.6) is the most interesting of all three. Applying (2.8) to it, we find that for the polynomial (2.7) this condition is written as follows:

$$(2\,x_1-x_2-x_3)\,(2\,x_2-x_3-x_1)\,(2\,x_3-x_1-x_2)\neq 0. \tag{2.10}$$

The condition (2.10) can be written as three conditions

$$\begin{aligned} &u_1=2\,x_1-x_2-x_3\neq 0,\\ &u_2=2\,x_2-x_3-x_1\neq 0,\\ &u_3=2\,x_3-x_1-x_2\neq 0. \end{aligned} \tag{2.11}$$

It is not obvious, but the condition (2.9) can be written as

$$(2\,x_1-x_2-x_3)^2+(2\,x_2-x_3-x_1)^2+(2\,x_3-x_1-x_2)^2\neq 0.$$

Therefore it is clear that the conditions (2.11) imply both (2.9) and (2.10).

Now let's substitute (2.8) into (2.2). Then we find that the D-parameter of the sextic equation (1.21) corresponding to the cubic equation (2.7) is written as

$$D=-\frac{8\,(u_1\,u_2\,u_3)^2}{(u_1^2+u_2^2+u_3^2)^3} \tag{2.12}$$

The denominator of (2.12) is nonzero due to (2.11). Substituting (2.12) into the sextic equation, we find that it factors explicitly

$$D\,(w-\tilde{w}_1)\,(w-\tilde{w}_2)\,(w-\tilde{w}_3)\,(w-\tilde{w}_4)\,(w-\tilde{w}_5)\,(w-\tilde{w}_6)=0, \tag{2.13}$$

where D is given by (2.12) and $\tilde{w}_1$, $\tilde{w}_2$, $\tilde{w}_3$ $\tilde{w}_4$ $\tilde{w}_5$, $\tilde{w}_6$ are given by the formulas

$$\begin{aligned} &\tilde{w}_1=\frac{u_1-u_2}{u_3}, &&\tilde{w}_2=-\frac{u_1-u_2}{u_3},\\ &\tilde{w}_3=\frac{u_2-u_3}{u_1}, &&\tilde{w}_4=-\frac{u_2-u_3}{u_1},\\ &\tilde{w}_5=\frac{u_3-u_1}{u_2}, &&\tilde{w}_6=-\frac{u_3-u_1}{u_2}. \end{aligned} \tag{2.14}$$

The numbers u_1, u_2, u_3 in (2.12) and (2.14) are determined by the formulas (2.11).

The quantities $\tilde{w}_1$, $\tilde{w}_2$, $\tilde{w}_3$, $\tilde{w}_4$, $\tilde{w}_5$, $\tilde{w}_6$ in (2.13) are roots of the sextic equation (1.21). Therefore, substituting (2.8) into (2.3), (2.4), (2.5) and substituting one of the quantities (2.14) for w into these formulas, we express x_1, x_2, x_3 through x_1, x_2, x_3, but up to some permutation of them. Here are the permutations associated with the quantities $\tilde{w}_1$, $\tilde{w}_2$, $\tilde{w}_3$, $\tilde{w}_4$, $\tilde{w}_5$, $\tilde{w}_6$ from (2.14):

$$\begin{aligned} &\tilde{w}_1\colon\ (x_1,x_2,x_3)\mapsto(x_1,x_2,x_3), &&\tilde{w}_2\colon\ (x_1,x_2,x_3)\mapsto(x_2,x_1,x_3),\\ &\tilde{w}_3\colon\ (x_1,x_2,x_3)\mapsto(x_2,x_3,x_1), &&\tilde{w}_4\colon\ (x_1,x_2,x_3)\mapsto(x_3,x_2,x_1),\\ &\tilde{w}_5\colon\ (x_1,x_2,x_3)\mapsto(x_3,x_1,x_2), &&\tilde{w}_6\colon\ (x_1,x_2,x_3)\mapsto(x_1,x_2,x_3). \end{aligned} \tag{2.15}$$

As we see in (2.15), the first quantity $\tilde{w}_1$ from (2.14) plays the role of the identical permutation belonging to the permutation group S_3 and being its unit element.

3. Conversion formulas.

Let's choose the first formula (2.14). It expresses a root of the sextic equation (1.21) through the roots of the associated cubic equation (2.1) given by the formulas (2.3), (2.4), (2.5). We write this formula as follows:

$$w=\frac{3\,(x_1-x_2)}{2\,x_3-x_1-x_2}. \tag{3.1}$$

The formula (3.1) is inverse to the formulas (2.3), (2.4), and (2.5). Indeed, applying the formulas (2.8) to (2.3), (2.4), and (2.5), we obtain

$$\begin{aligned}x_1=\frac{1}{18}\bigl(&(x_1+x_2-2\,x_3)\,(x_2+x_3-2\,x_1)\,(x_3+x_1-2\,x_2)\,w^2+\\&+6\,(x_3+x_2+x_1)\,(x_1^2+x_2^2+x_3^2-x_2\,x_3-x_1\,x_3-x_1\,x_2)\,w+\\&+9\,x_1\,x_2^2+9\,x_1\,x_3^2+9\,x_2\,x_3^2+9\,x_2\,x_1^2+9\,x_3\,x_1^2+9\,x_3\,x_2^2-\\&-54\,x_1\,x_2\,x_3\bigr)\,(x_1^2+x_2^2+x_3^2-x_1\,x_2-x_2\,x_3-x_3\,x_1)^{-1}\,(1+w)^{-1},\end{aligned} \tag{3.2}$$

$$\begin{aligned}x_2=\frac{1}{18}\bigl(&(x_1+x_2-2\,x_3)\,(x_2+x_3-2\,x_1)\,(x_3+x_1-2\,x_2)\,w^2-\\&-6\,(x_3+x_2+x_1)\,(x_1^2+x_2^2+x_3^2-x_2\,x_3-x_1\,x_3-x_1\,x_2)\,w+\\&+9\,x_1\,x_2^2+9\,x_1\,x_3^2+9\,x_2\,x_3^2+9\,x_2\,x_1^2+9\,x_3\,x_1^2+9\,x_3\,x_2^2-\\&-54\,x_1\,x_2\,x_3\bigr)\,(x_1^2+x_2^2+x_3^2-x_1\,x_2-x_2\,x_3-x_3\,x_1)^{-1}\,(1-w)^{-1},\end{aligned} \tag{3.3}$$

$$\begin{aligned}x_3=\frac{1}{9}\bigl(&(21\,x_1\,x_2\,x_3+3\,x_1\,x_2^2+3\,x_1\,x_3^2+3\,x_2\,x_3^2+3\,x_2\,x_1^2+3\,x_3\,x_1^2+\\&+3\,x_3\,x_2^2-x_1^3-x_2^3-x_3^3)\,w^2-9\,x_1\,x_2^2-9\,x_1\,x_3^2-9\,x_2\,x_3^2-9\,x_2\,x_1^2-\\&-9\,x_3\,x_1^2-9\,x_3\,x_2^2+27\,x_1\,x_2\,x_3+9\,x_1^3+9\,x_2^3+9\,x_3^3\bigr)\times\\&\times(x_1^2+x_2^2+x_3^2-x_1\,x_2-x_2\,x_3-x_3\,x_1)^{-1}\,(1-w)^{-1}\,(1+w)^{-1}.\end{aligned} \tag{3.4}$$

Substituting (3.2), (3.3), (3.4) into (3.1), we get the identity $w=w$. Conversely, substituting (3.1) into (3.2), (3.3), (3.4), we get three identities $x_1=x_1$, $x_2=x_2$, and $x_3=x_3$. This result proves the following theorem.

Theorem 3.1. *Let x_1, x_2, x_3 be three roots of a general cubic equation (2.1) such that the conditions (2.6) are fulfilled. Then the formula (3.1) yields a solution w of the associated sextic equation (1.21) whose D-parameter is given by the formula (2.2). In this case the roots x_1, x_2, x_3 are backward expressed through w by means of the formulas (2.3), (2.4), and (2.5).*

Let's return to the formulas (1.27), (1.28) and let's recall that the functions $x_1(b,c,w_1)$, $x_2(b,c,w_1)$, $x_3(b,c,w_1)$ from (1.27) were produced in [75] by applying the formulas (2.3), (2.4), (2.5) to the cubic equation (1.18). Therefore, setting $w=w_1$ and substituting these functions for x_1, x_2, x_3 into (3.1), we get the identity $w_1=w_1$. However, we have the other three functions $x_1(b,c,w_2)$, $x_2(b,c,w_2)$, $x_3(b,c,w_2)$ in (1.28). They represent the same three roots x_1, x_2, x_3 of the same cubic equation (1.18). Substituting them into (3.1), we get the same quantity w_1.

But now w_1 turns out to be expressed through w_2, i. e. we get the formula

$$w_1=\frac{3\,x_1(b,c,w_2)-3\,x_2(b,c,w_2)}{2\,x_3(b,c,w_2)-x_1(b,c,w_2)-x_2(b,c,w_2)},\tag{3.5}$$

which is not an identity. The formula (3.5) is the first conversion formula. It expresses w_1 through b, c, and w_2, i. e. (3.5) yields a function $w_1=w_1(b,c,w_2)$. If we substitute this function into the argument w_1 of the functions $x_1(b,c,w_1)$, $x_2(b,c,w_1)$, $x_3(b,c,w_1)$ from (1.27), then, according to the theorem 3.1, we get back three roots $x_1(b,c,w_2)$, $x_2(b,c,w_2)$, $x_3(b,c,w_2)$ used in (3.5). This means that we have the following relationships based on the formula (3.5):

$$\begin{aligned}x_1(b,c,w_1(b,c,w_2))&=x_1(b,c,w_2),\\ x_2(b,c,w_1(b,c,w_2))&=x_2(b,c,w_2),\\ x_3(b,c,w_1(b,c,w_2))&=x_3(b,c,w_2).\end{aligned}\tag{3.6}$$

Note that (1.19) is another cubic equation. It is associated with the other sextic equation (1.24) and it has its own formula like (3.1):

$$w=\frac{3\,(d_1-d_2)}{2\,d_3-d_1-d_2}.\tag{3.7}$$

The functions $d_1(b,c,w_2)$, $d_2(b,c,w_2)$, $d_3(b,c,w_2)$ were produced in [75] by applying the formulas (2.3), (2.4), (2.5) to the cubic equation (1.19). Therefore, setting $w=w_2$ and substituting these functions for d_1, d_2, d_3 into (3.7), we get the identity $w_2=w_2$. However, there are the other three functions $d_1(b,c,w_1)$, $d_2(b,c,w_1)$, $d_3(b,c,w_1)$ in (1.27). They represent the same three roots d_1, d_2, d_3 of the same cubic equation (1.19). Substituting them into (3.7), we get the same quantity w_2. But now w_2 turns out to be expressed through w_1, i. e. we get the formula

$$w_2=\frac{3\,d_1(b,c,w_1)-3\,d_2(b,c,w_1)}{2\,d_3(b,c,w_1)-d_1(b,c,w_1)-d_2(b,c,w_1)},\tag{3.8}$$

which is not an identity. The formula (3.8) is the second conversion formula. It expresses w_2 through b, c, and w_1, i. e. (3.8) yields a function $w_2=w_1(b,c,w_1)$. If we substitute this function into the argument w_2 of the functions $d_1(b,c,w_2)$, $x_2(b,c,w_2)$, $x_3(b,c,w_2)$ from (1.28), then, according to the theorem 3.1, we get back three roots $d_1(b,c,w_1)$, $d_2(b,c,w_1)$, $d_3(b,c,w_1)$ used in (3.8). This means that we have the following relationships based on the formula (3.8):

$$\begin{aligned}d_1(b,c,w_2(b,c,w_1))&=d_1(b,c,w_1),\\ d_2(b,c,w_2(b,c,w_1))&=d_2(b,c,w_1),\\ d_3(b,c,w_2(b,c,w_1))&=d_3(b,c,w_1).\end{aligned}\tag{3.9}$$

The formulas (3.5) and (3.8) provide two transformations $w_1=w_1(b,c,w_2)$ and $w_2=w_2(b,c,w_1)$. Our next step is to prove that these transformations are inverse to each other. For this purpose let's recall that the functions (1.27) obey the relationships (1.5), (1.6), (1.7) whose right hand sides are given by the formulas (1.9) through (1.17). The same is true for the functions (1.28). In particular, we have the following three relationships for the functions (1.28):

$$\begin{aligned}
&x_1(b,c,w_2)\,x_2(b,c,w_2)\,d_3(b,c,w_2)+x_2(b,c,w_2)\,x_3(b,c,w_2)\,\times\\
&\quad\times\; d_1(b,c,w_2)+x_3(b,c,w_2)\,x_1(b,c,w_2)\,d_2(b,c,w_2)=E_{21}(b,c),\\
&x_1(b,c,w_2)\,d_2(b,c,w_2)+d_1(b,c,w_2)\,x_2(b,c,w_2)+x_2(b,c,w_2)\,\times\\
&\quad\times\; d_3(b,c,w_2)+d_2(b,c,w_2)\,x_3(b,c,w_2)+x_3(b,c,w_2)\,\times\\
&\quad\times\, d_1(b,c,w_2)+d_3(b,c,w_2)\,x_1(b,c,w_2)=E_{11}(b,c),\\
&d_1(b,c,w_2)+d_2(b,c,w_2)+d_3(b,c,w_2)=E_{01}(b,c).
\end{aligned}\tag{3.10}$$

Let's apply the formulas (3.6) to (3.10). This yields

$$\begin{aligned}
&x_1(b,c,w_1)\,x_2(b,c,w_1)\,d_3+x_2(b,c,w_1)\,x_3(b,c,w_1)\,d_1\,+\\
&\qquad\qquad+\;x_3(b,c,w_1)\,x_1(b,c,w_1)\,d_2=E_{21}(b,c),\\
&x_1(b,c,w_1)\,d_2+d_1\,x_2(b,c,w_1)+x_2(b,c,w_1)\,d_3\,+\\
&\quad+\;d_2\,x_3(b,c,w_1)+x_3(b,c,w_1)\,d_1+d_3\,x_1(b,c,w_1)=E_{11}(b,c),\\
&d_1+d_2+d_3=E_{01}(b,c),
\end{aligned}\tag{3.11}$$

where $w_1=w_1(b,c,w_2)$ and $d_i=d_i(b,c,w_2)$. The equalities (3.11) are linear with respect to d_1, d_2, d_3. They constitute that very system of linear equations which was used in deriving the functions $d_i=d_i(b,c,w_1)$ (see (3.5) in [75]). This yields

$$\begin{aligned}
d_1(b,c,w_1(b,c,w_2))&=d_1(b,c,w_2),\\
d_2(b,c,w_1(b,c,w_2))&=d_2(b,c,w_2),\\
d_3(b,c,w_1(b,c,w_2))&=d_3(b,c,w_2).
\end{aligned}\tag{3.12}$$

Apart from (3.10) one can extract other three equations from (1.5), (1.6), (1.7) and write them as equalities for the functions (1.27), i. g. we can write

$$\begin{aligned}
&x_1(b,c,w_1)+x_2(b,c,w_1)+x_3(b,c,w_1)=E_{10}(b,c),\\
&x_1(b,c,w_1)\,d_2(b,c,w_1)+d_1(b,c,w_1)\,x_2(b,c,w_1)+x_2(b,c,w_1)\,\times\\
&\quad\times\; d_3(b,c,w_1)+d_2(b,c,w_1)\,x_3(b,c,w_1)+x_3(b,c,w_1)\,\times\\
&\quad\times\, d_1(b,c,w_1)+d_3(b,c,w_1)\,x_1(b,c,w_1)=E_{11}(b,c),\\
&x_1(b,c,w_1)\,d_2(b,c,w_1)\,d_3(b,c,w_1)+x_2(b,c,w_1)\,d_3(b,c,w_1)\,\times\\
&\quad\times\; d_1(b,c,w_1)+x_3(b,c,w_1)\,d_1(b,c,w_1)\,d_2(b,c,w_1)=E_{12}(b,c).
\end{aligned}\tag{3.13}$$

Like in the case of (3.10), applying (3.9) to (3.13), we get

$$\begin{aligned}
&x_1+x_2+x_3=E_{10}(b,c),\\
&x_1\,d_2(b,c,w_2)+d_1(b,c,w_2)\,x_2+x_2\,d_3(b,c,w_2)\,+\\
&\quad+\;d_2(b,c,w_2)\,x_3+x_3\,d_1(b,c,w_2)+d_3(b,c,w_2)\,x_1=E_{11}(b,c),\\
&x_1\,d_2(b,c,w_2)\,d_3(b,c,w_2)+x_2\,d_3(b,c,w_2)\,d_1(b,c,w_2)\,+\\
&\qquad\qquad+\;x_3\,d_1(b,c,w_2)\,d_2(b,c,w_2)=E_{12}(b,c),
\end{aligned}\tag{3.14}$$

where $w_2 = w_2(b, c, w_1)$ and $x_i = x_i(b, c, w_1)$. The equalities (3.14) are linear with respect to x_1, x_2, x_3. They constitute that very system of linear equations which was used in deriving the functions $x_i = x_i(b, c, w_2)$ (see (4.5) in [75]). This yields

$$\begin{aligned}
&x_1(b, c, w_2(b, c, w_1)) = x_1(b, c, w_1),\\
&x_2(b, c, w_2(b, c, w_1)) = x_2(b, c, w_1),\\
&x_3(b, c, w_2(b, c, w_1)) = x_3(b, c, w_1).
\end{aligned} \tag{3.15}$$

The relationships (3.15) are similar to (3.6) and the relationships (3.12) are similar to (3.9). But these four groups of relationships do not coincide with each other.

Now let's consider the composite function $w_2(b, c, w_1(b, c, w_2))$. Then, applying the formula (3.8) to this function, we derive

$$\begin{aligned}
&w_2(b, c, w_1(b, c, w_2)) = \bigl(3\,d_1(b, c, w_1(b, c, w_2)) - 3\,d_2(b, c, w_1(b, c, w_2))\bigr)\times\\
&\times\bigl(2\,d_3(b, c, w_1(b, c, w_2)) - d_1(b, c, w_1(b, c, w_2)) - d_2(b, c, w_1(b, c, w_2))\bigr)^{-1}.
\end{aligned}$$

If we take into account (3.12), then the above formula can be written as

$$w_2(b, c, w_1(b, c, w_2)) = \frac{3\,d_1(b, c, w_2) - 3\,d_2(b, c, w_2)}{2\,d_3(b, c, w_2) - d_1(b, c, w_2) - d_2(b, c, w_2)}. \tag{3.16}$$

The right hand side of the formula (3.16) can be produced by substituting the functions $d_1(b, c, w_2)$, $d_2(b, c, w_2)$, $d_3(b, c, w_2)$ from (1.28) for d_1, d_2, d_3 into the formula (3.7). The formula (3.7) is a version of the formula (3.1) for $w = w_2$, while $d_1(b, c, w_2)$, $d_2(b, c, w_2)$, $d_3(b, c, w_2)$ are the roots of the cubic equation (1.19) produced by means of the formulas (2.3), (2.4), (2.5) with $w = w_2$ applied to the cubic equation (1.19). Therefore, the theorem 3.1 in this case means that the right hand side of (3.16) is equal to w_2. Thus, we have derived the formula

$$w_2(b, c, w_1(b, c, w_2)) = w_2. \tag{3.17}$$

The formula $w_1(b, c, w_2(b, c, w_1)) = w_1$ is derived similarly. For this purpose we consider the function $w_1(b, c, w_2(b, c, w_1))$ and apply the formula (3.5) to it:

$$\begin{aligned}
&w_1(b, c, w_2(b, c, w_1)) = \bigl(3\,x_1(b, c, w_2(b, c, w_1)) - 3\,x_2(b, c, w_2(b, c, w_1))\bigr)\times\\
&\times\bigl(2\,x_3(b, c, w_2(b, c, w_1)) - x_1(b, c, w_2(b, c, w_1)) - x_2(b, c, w_2(b, c, w_1))\bigr)^{-1}.
\end{aligned}$$

Using the relationships (3.15), the above formula is transformed to

$$w_1(b, c, w_2(b, c, w_1)) = \frac{3\,x_1(b, c, w_1) - 3\,x_2(b, c, w_1)}{2\,x_3(b, c, w_1) - x_1(b, c, w_1) - x_2(b, c, w_1)}. \tag{3.18}$$

The right hand side of the formula (3.18) can be produced by substituting the functions $x_1(b, c, w_1)$, $x_2(b, c, w_1)$, $x_3(b, c, w_1)$ from (1.27) for x_1, x_2, x_3 into the formula (3.1), where $w = w_1$, while $x_1(b, c, w_2)$, $x_2(b, c, w_2)$, $x_3(b, c, w_2)$ are the roots of the cubic equation (1.18) produced by means of the formulas (2.3), (2.4), (2.5) with $w = w_1$ applied to the cubic equation (1.18). Therefore, the theorem 3.1

in this case means that the right hand side of (3.18) is equal to w_1. Thus, we have derived the required formula for the composite function $w_1(b, c, w_2(b, c, w_1))$:

$$w_1(b, c, w_2(b, c, w_1)) = w_1. \tag{3.19}$$

The functions $w_1 = w_1(b, c, w_2)$ and $w_2 = w_2(b, c, w_1)$ are given by the formulas (3.5) and (3.8). Using the explicit formulas for the functions (1.27) and (1.28) given in the ancillary files **Solutions_1.txt** and **Solutions_2.txt**, the formulas (3.5) and (3.8) are converted to explicit formulas for the functions $w_1 = w_1(b, c, w_2)$ and $w_2 = w_2(b, c, w_1)$. These explicit formulas are placed in the ancillary file **Conversion_formulas.txt** attached to this arXiv submission.

Theoretically, we could prove the relationships (3.17) and (3.19) by direct calculations. However, the explicit formulas for the functions $w_1(b, c, w_2)$ and $w_2(b, c, w_1)$ are so huge that they cannot be handled on my personal computer.

4. Conclusions.

The formulas (3.17) and (3.19) mean that the transformations given by the conversion functions (3.5) and (3.8) are inverse to each other. Then the formulas (3.6), (3.9), (3.12), and (3.15) mean that (1.27) and (1.28) are not two different solutions of the cuboid factor equations (1.3) and (1.4), but two presentations of a single solution. This fact is the main result of the present paper.

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