A MODEL WITH TWO QUANTUM PARTICLES
SIMILAR TO THE HYDROGEN ATOM.

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ABSTRACT. The hydrogen atom with the Coulomb interaction is one of the exactly solvable non-relativistic quantum models. Unlike many other exactly solvable models it describes a real physical object providing the formulas for energy levels and stationary state wave functions of a real hydrogen atom. In this paper we modify the model replacing the Coulomb interaction by the interaction of the proton and the electron with the classical electromagnetic field serving as an intermediary transmitting the electromagnetic interaction of these two charged quantum particles.

1. INTRODUCTION.

The standard hydrogen atom is a system composed by the proton p^+ with the mass m_p and the electron e^- with the mass m_e. This system is described by the Shrödinger equation $H\Psi = \mathcal{E}\Psi$ with the following Hamilton operator:

$$H = -\frac{\hbar^2}{2\,m_p}\,\nabla_p^2 - \frac{\hbar^2}{2\,m_e}\,\nabla_e^2 - \frac{e^2}{|\mathbf{r}_p - \mathbf{r}_e|} \tag{1.1}$$

(see [1] or [2]). In order to solve the Shrödinger equation $H\Psi = \mathcal{E}\Psi$ the so called center-of-mass coordinates are used:

$$\mathbf{R} = \frac{m_e\,\mathbf{r}_e + m_p\,\mathbf{r}_p}{m_p + m_e}, \qquad\qquad \mathbf{r} = \mathbf{r}_e - \mathbf{r}_p. \tag{1.2}$$

Applying (1.2) to (1.1), one can easily derive the following expression for H:

$$H = -\frac{\hbar^2}{2\,(m_p + m_e)}\,\nabla_{\mathbf{R}}^2 - \frac{\hbar^2}{2\,\tilde{m}}\,\nabla_{\mathbf{r}}^2 - \frac{e^2}{|\mathbf{r}|}, \text{ where } \tilde{m} = \frac{m_e\,m_p}{m_e + m_p}. \tag{1.3}$$

The formula (1.3) then is used for separating variables in the Shrödinger equation $H\Psi = \mathcal{E}\Psi$ and deriving the standard formulas for $\mathcal{E}$ and Ψ.

In this paper we choose a different approach by omitting the Coulomb interaction term in (1.1) and writing the Hamilton operator in the following form:

$$H = -\frac{\hbar^2}{2\,m_p}\,\nabla_p^2 - \frac{\hbar^2}{2\,m_e}\,\nabla_e^2 + H_{\text{int}}. \tag{1.4}$$

Here H_{int} is the **interaction** term describing the interaction of the classical elec-

tromagnetic field with the proton and the electron. The explicit expression for H including the interaction term H_{int} can be obtained from the formula (3.6) below.

Apart from (1.4), we consider the backward influence of two quantum particles upon the classical electromagnetic field. For this purpose the variational approach is applied and the corresponding Lagrangian is written.

2. A QUANTUM PARTICLE IN THE CLASSICAL ELECTROMAGNETIC FIELD.

Massive particles in quantum mechanics are described by their wave functions. Various wave equations are usually written for these wave functions. The time dependent Shrödinger equation is one of them. It is written as follows:

$$i\,\hbar\,\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2\,m}\nabla^2\psi. \tag{2.1}$$

The equation (2.1) can be derived variationally, using the following action integral:

$$S_\psi = \frac{i\,\hbar}{2}\int\left(\frac{\partial \psi}{\partial t}\,\overline{\psi} - \psi\,\frac{\partial \overline{\psi}}{\partial t}\right)d^3r\,dt - \frac{\hbar^2}{2\,m}\int|\nabla\psi|^2\,d^3r\,dt. \tag{2.2}$$

Though the equation (2.1) is not a relativistic equation, the integrals in (2.2) are taken over the four-dimensional Minkowski space. The electromagnetic field is introduced into the equation (2.1) through the following momentum transformation:

$$\frac{i\,\hbar}{c}\,\frac{\partial}{\partial t} \longrightarrow \frac{i\,\hbar}{c}\,\frac{\partial}{\partial t} - \frac{e}{c}\,\phi, \qquad\qquad -i\,\hbar\,\nabla \longrightarrow -i\,\hbar\,\nabla - \frac{e}{c}\,\mathbf{A}. \tag{2.3}$$

Here ϕ and $\mathbf{A}$ are the scalar potential and the vector potential of the electromagnetic field respectively. The transformation (2.3) is known as the minimal coupling (see [3]). Applying the minimal coupling transformation (2.3) to (2.1), we derive

$$i\,\hbar\,\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2\,m}\nabla^2\psi + e\,\phi\,\psi + \frac{i\,e\,\hbar}{2\,m\,c}\big((\mathbf{A},\nabla) + (\nabla,\mathbf{A})\big)\psi + \frac{e^2\,|\mathbf{A}|^2}{2\,m\,c^2}\,\psi. \tag{2.4}$$

The equation (2.4) describes a non-relativistic quantum particle with the mass m and with the electric charge e in the classical electromagnetic field. Applying (2.3) to (2.2), one can derive the action integral for the equation (2.4):

$$S_\psi = \frac{i\,\hbar}{2}\int\left(\frac{\partial \psi}{\partial t}\,\overline{\psi} - \psi\,\frac{\partial \overline{\psi}}{\partial t}\right)d^3r\,dt - e\int\phi\,|\psi|^2\,d^3r\,dt -$$
$$- \frac{\hbar^2}{2\,m}\int|\nabla\psi|^2\,d^3r\,dt - \frac{i\,e\,\hbar}{2\,m\,c}\int\overline{\psi}\,(\mathbf{A},\nabla\psi)\,d^3r\,dt + \tag{2.5}$$
$$+ \frac{i\,e\,\hbar}{2\,m\,c}\int\psi\,(\mathbf{A},\nabla\overline{\psi})\,d^3r\,dt - \frac{e^2}{2\,m\,c^2}\int|\mathbf{A}|^2\,|\psi|^2\,d^3r\,dt.$$

The action integral (2.5) corresponds to a quantum particle experiencing the influence of the classical electromagnetic field. In order to describe the electromagnetic field itself the action integral of the free electromagnetic field is used (see [4] or [5]):

$$S_{\text{fef}} = -\frac{1}{16\,\pi\,c}\int\sum_{p=0}^{3}\sum_{q=0}^{3}F_{pq}\,F^{pq}\,d^4r = \int\frac{|\mathbf{E}|^2 - |\mathbf{H}|^2}{8\,\pi}\,d^3r\,dt. \tag{2.6}$$

If we need to describe the backward influence of a quantum particle upon the electromagnetic field, we should add the action integrals (2.5) and (2.6):

$$S = S_\psi + S_{\text{fef}} \tag{2.7}$$

The total action integral S implies three Euler-Lagrange equations:

$$\frac{\delta S}{\delta \overline{\psi}} = 0, \qquad\qquad \frac{\delta S}{\delta \phi} = 0, \qquad\qquad \frac{\delta S}{\delta \mathbf{A}} = 0. \tag{2.8}$$

The first equation (2.8) leads to (2.4). The other two equations (2.8) coincide with the Maxwell equations comprising charges and currents:

$$\operatorname{div} \mathbf{E} = 4\pi\rho, \qquad\qquad \operatorname{rot} \mathbf{H} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}. \tag{2.9}$$

Due to (2.7) the charge density ρ and the current density $\mathbf{j}$ in (2.9) are determined by the action integral (2.5) according to the formulas

$$\rho = -\frac{\delta S_\psi}{\delta \phi}, \qquad\qquad \mathbf{j} = c\,\frac{\delta S_\psi}{\delta \mathbf{A}}. \tag{2.10}$$

Substituting the explicit expression (2.5) for S_ψ into (2.10), we derive:

$$\rho = e\,|\psi|^2, \qquad\qquad \mathbf{j} = \frac{i\,e\,\hbar}{2\,m}\left(\psi\,\nabla\overline{\psi} - \overline{\psi}\,\nabla\psi\right) - \frac{e^2\mathbf{A}}{m\,c}\,|\psi|^2. \tag{2.11}$$

The formulas (2.11) are compatible with the probabilistic interpretations of the wave function ψ (see [1] and [2]).

The following formulas are well known (see [4] or [5]):

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \qquad\qquad \mathbf{H} = \operatorname{rot}\mathbf{A}. \tag{2.12}$$

They express the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ through the corresponding vectorial potential $\mathbf{A}$ and scalar potential ϕ.

3. A couple of quantum particles in the classical electromagnetic field.

The hydrogen atom, as well as our present model, is a couple of two quantum particles — the proton and the electron. Such a couple of particles is described by a single wave function Ψ but depending on two spatial variables $\mathbf{r}_p$ and $\mathbf{r}_e$ which are radius-vectors of the proton and the electron respectively. Apart from $\mathbf{r}_p$ and $\mathbf{r}_e$, the joint wave function of the proton and the electron depends on the time variable:

$$\Psi = \Psi(t, \mathbf{r}_p, \mathbf{r}_e). \tag{3.1}$$

The analog of the action integral (2.2) for the wave function (3.1) is written as

$$S_\Psi = \frac{i\,\hbar}{2}\int\left(\frac{\partial\Psi}{\partial t}\,\overline{\Psi} - \Psi\,\frac{\partial\overline{\Psi}}{\partial t}\right)d^3r_p\,d^3r_e\,dt -$$

$$-\int\left(\frac{\hbar^2\,|\nabla_p\Psi|^2}{2\,m_p} + \frac{\hbar^2\,|\nabla_e\Psi|^2}{2\,m_e}\right)d^3r_p\,d^3r_e\,dt.$$

Similarly, the analog of the action integral (2.5) is written as follows:

$$
\begin{aligned}
S_\Psi = {} & \frac{i\,\hbar}{2} \int \left(\frac{\partial\,\Psi}{\partial\,t}\,\overline{\Psi} - \Psi\,\frac{\partial\,\overline{\Psi}}{\partial\,t} \right) d^3 r_p\, d^3 r_e\, dt - \\
& - e \int (\phi_p - \phi_e)\,|\Psi|^2\, d^3 r_p\, d^3 r_e\, dt - \\
& - \int \left(\frac{\hbar^2\,|\nabla_p \Psi|^2}{2\,m_p} + \frac{\hbar^2\,|\nabla_e \Psi|^2}{2\,m_e} \right) d^3 r_p\, d^3 r_e\, dt - \\
& - \frac{i\,e\,\hbar}{2\,c} \int \left(\frac{\overline{\Psi}\,(\mathbf{A}_p, \nabla_p \Psi)}{m_p} - \frac{\overline{\Psi}\,(\mathbf{A}_e, \nabla_e \Psi)}{m_e} \right) d^3 r_p\, d^3 r_e\, dt + \\
& + \frac{i\,e\,\hbar}{2\,c} \int \left(\frac{\Psi\,(\mathbf{A}_p, \nabla_p \overline{\Psi})}{m_p} - \frac{\Psi\,(\mathbf{A}_e, \nabla_e \overline{\Psi})}{m_e} \right) d^3 r_p\, d^3 r_e\, dt - \\
& - \frac{e^2}{2\,c^2} \int \left(\frac{|\mathbf{A}_p|^2}{m_p} + \frac{|\mathbf{A}_e|^2}{m_e} \right) |\Psi|^2\, d^3 r_p\, d^3 r_e\, dt.
\end{aligned}
\tag{3.2}
$$

For the sake of brevity in (3.2) the following notations are used:

$$
\begin{aligned}
\phi_p &= \phi(t, \mathbf{r}_p), & \mathbf{A}_p &= \mathbf{A}(t, \mathbf{r}_p), \\
\phi_e &= \phi(t, \mathbf{r}_e), & \mathbf{A}_e &= \mathbf{A}(t, \mathbf{r}_e).
\end{aligned}
\tag{3.3}
$$

Like in (2.5), in order to describe the compete system including the classical electromagnetic field we add the action integral (2.6) to (3.2):

$$
S = S_\Psi + S_{\text{fef}}
\tag{3.4}
$$

Then, like in (2.8), three Euler-Lagrange equations for S in (3.4) are written:

$$
\frac{\delta S}{\delta \overline{\Psi}} = 0, \qquad\qquad \frac{\delta S}{\delta \phi} = 0, \qquad\qquad \frac{\delta S}{\delta \mathbf{A}} = 0.
\tag{3.5}
$$

The first equation (3.5) leads to the time dependent Shrödinger equation for the proton and the electron in the classical electromagnetic field. This equation is similar to the equation (2.4), but a little bit more complicated than it:

$$
\begin{aligned}
i\,\hbar\,\frac{\partial\Psi}{\partial t} = {} & -\frac{\hbar^2\,\nabla_p^2 \Psi}{2\,m_p} - \frac{\hbar^2\,\nabla_e^2 \Psi}{2\,m_e} + e\,\phi_p\,\Psi - e\,\phi_e\,\Psi + \\
& + \frac{i\,e\,\hbar}{2\,m_p\,c} \big((\mathbf{A}_p, \nabla_p) + (\nabla_p, \mathbf{A}_p)\big)\Psi - \frac{i\,e\,\hbar}{2\,m_e\,c} \big((\mathbf{A}_e, \nabla_e) + (\nabla_e, \mathbf{A}_e)\big)\Psi + \\
& + \frac{e^2\,|\mathbf{A}_p|^2}{2\,m_p\,c^2}\,\Psi + \frac{e^2\,|\mathbf{A}_e|^2}{2\,m_e\,c^2}\,\Psi.
\end{aligned}
\tag{3.6}
$$

In the equation (3.6) the same notations (3.3) are used as in (3.2).

The second and the third equations (3.5) lead to the Maxwell equations (2.9). The formulas for ρ and $\mathbf{j}$ are similar to (2.10):

$$
\rho = -\frac{\delta S_\Psi}{\delta\,\phi}, \qquad\qquad \mathbf{j} = c\,\frac{\delta S_\Psi}{\delta\,\mathbf{A}}.
$$

However, the ultimate expressions for them are different from (2.11):

$$\rho = e \int |\Psi_p|^2 \, d^3 r_e - e \int |\Psi_e|^2 \, d^3 r_p,$$

$$\mathbf{j} = \frac{i\,e\,\hbar}{2\,m_p} \int \left(\Psi_p \, \nabla \overline{\Psi}_p - \overline{\Psi}_p \, \nabla \Psi_p \right) d^3 r_e - \frac{e^2 \mathbf{A}}{m_p\,c} \int |\Psi_p|^2 \, d^3 r_e - \qquad (3.7)$$

$$- \frac{i\,e\,\hbar}{2\,m_e} \int \left(\Psi_e \, \nabla \overline{\Psi}_e - \overline{\Psi}_e \, \nabla \Psi_e \right) d^3 r_p - \frac{e^2 \mathbf{A}}{m_e\,c} \int |\Psi_e|^2 \, d^3 r_p.$$

In the above formulas (3.7) the following notations are used:

$$\Psi_p = \Psi(t, \mathbf{r}, \mathbf{r}_e), \qquad\qquad \Psi_e = \Psi(t, \mathbf{r}_p, \mathbf{r}). \qquad (3.8)$$

The notations (3.8) are similar to the notations (3.3). For the sake of completeness, in addition to (3.6) and (2.9), one should write the equations (2.12) expressing $\mathbf{H}$ and $\mathbf{E}$ through ϕ and $\mathbf{A}$.

4. Spherically symmetric eigenstates of the model.

It is known that the ground state of the hydrogen atom is spherically symmetric. By analogy we look for eigenstates of our model in the class of spherically symmetric wave functions. Each eigenstate is a stationary state. Its time dependence is given by an oscillating exponential factor. As a result we get

$$\Psi(t, \mathbf{r}_p, \mathbf{r}_e) = \psi(r_p, r_e) \, \exp(-i\,\mathcal{E}\,t/\hbar). \qquad (4.1)$$

where $r_p = |\mathbf{r}_p|$ and $r_e = |\mathbf{r}_e|$. Apart from (4.1), we assume the electric and magnetic fields $\mathbf{E}$ and $\mathbf{H}$ associated with the eigenstate to be stationary and spherically symmetric. As a result, denoting $r = |\mathbf{r}|$, we get

$$\mathbf{E}(t, \mathbf{r}) = E(r) \, \frac{\mathbf{r}}{r}, \qquad\qquad \mathbf{H}(t, \mathbf{r}) = H(r) \, \frac{\mathbf{r}}{r}. \qquad (4.2)$$

Lemma 4.1. *Each spherically symmetric stationary magnetic field given by the second formula* (4.2) *is identically zero.*

Proof. Note that the second equality (2.12) is equivalent to the Maxwell equation $\mathrm{div}\,\mathbf{H} = 0$. The integral presentation of $\mathrm{div}\,\mathbf{H} = 0$ is

$$\oint_S (\mathbf{H}, \, \mathbf{n}) \, dS = 0, \qquad (4.3)$$

where S is an arbitrary closed surface and $\mathbf{n}$ is the unit normal vector of S. Substituting the second formula (4.2) into (4.3) and applying (4.3) to the sphere with the radius r whose center is at the origin, we derive

$$\oint_S (\mathbf{H}, \, \mathbf{n}) \, dS = 4\,\pi\,r^2 \, H(r) = 0. \qquad (4.4)$$

The equality (4.4) yields $H(r) = 0$. Thus Lemma 4.1 is proved. □

Since $\mathbf{H} = 0$, we can choose $\mathbf{A} = 0$ for the corresponding vector potential $\mathbf{A}$. The spherically symmetric stationary electric field given by the first formula (4.2) is associated with a spherically symmetric stationary scalar potential. Thus we have

$$\phi(t, \mathbf{r}) = \phi(r), \qquad\qquad \mathbf{A}(t, \mathbf{r}) = 0. \qquad (4.5)$$

The next step is to apply (2.12) and (2.9) to (4.5). This yields

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = -4\pi\rho, \qquad\qquad \mathbf{j} = 0. \qquad (4.6)$$

The first equation (4.6) is a version of the Poisson equation $\triangle\phi = -4\pi\rho$ (see [6]).

Now let's substitute (4.6) along with (4.1) into the time-dependent Shrödinger equation (3.6). As a result this equation transforms to

$$-\frac{\hbar^2\,\nabla_p^2\psi}{2\,m_p} - \frac{\hbar^2\,\nabla_e^2\psi}{2\,m_e} + e\,\phi_p\,\psi - e\,\phi_e\,\psi = \mathcal{E}\,\psi. \qquad (4.7)$$

The notations (3.3) now yield $\phi_p = \phi(r_p)$ and $\phi_e = \phi(r_e)$, where $r_p = |\mathbf{r}_p|$ and $r_e = |\mathbf{r}_e|$. Keeping in mind these notations, we can separate variables and break (4.7) into two equations if we substitute $\psi(r_p, r_e) = \psi_p(r_p)\,\psi_e(r_e)$:

$$-\frac{\hbar^2\,\nabla_p^2\psi_p}{2\,m_p} + e\,\phi_p\,\psi_p = \mathcal{E}_p\,\psi_p,$$
$$\qquad\qquad (4.8)$$
$$-\frac{\hbar^2\,\nabla_e^2\psi_e}{2\,m_e} - e\,\phi_e\,\psi_e = \mathcal{E}_e\,\psi_e.$$

The total eigenvalue $\mathcal{E}$ is the sum of two eigenvalues $\mathcal{E}_p$ and $\mathcal{E}_e$ in (4.8):

$$\mathcal{E} = \mathcal{E}_p + \mathcal{E}_e. \qquad (4.9)$$

Having subdivided the equation (4.7) into two equations (4.8), we can omit the particle indices in the arguments of the functions ψ_p and ψ_e, i.e. we can write $\psi_p = \psi_p(r)$ and $\psi_e = \psi_e(r)$, where $r = |\mathbf{r}|$. Then the equations (4.8) are written as

$$-\frac{\hbar^2\,\nabla^2\psi_p}{2\,m_p} + e\,\phi(r)\,\psi_p = \mathcal{E}_p\,\psi_p,$$
$$\qquad\qquad (4.10)$$
$$-\frac{\hbar^2\,\nabla^2\psi_e}{2\,m_e} - e\,\phi(r)\,\psi_e = \mathcal{E}_e\,\psi_e.$$

The functions $\psi_p = \psi_p(r)$ and $\psi_e = \psi_e(r)$ are spherically symmetric. Therefore the equations (4.10) are written as ordinary differential equations

$$-\frac{\hbar^2}{2\,m_p}\left(\frac{d^2\psi_p}{dr^2} + \frac{2}{r}\frac{d\psi_p}{dr}\right) + e\,\phi(r)\,\psi_p = \mathcal{E}_p\,\psi_p,$$
$$\qquad\qquad (4.11)$$
$$-\frac{\hbar^2}{2\,m_e}\left(\frac{d^2\psi_e}{dr^2} + \frac{2}{r}\frac{d\psi_e}{dr}\right) - e\,\phi(r)\,\psi_e = \mathcal{E}_e\,\psi_e.$$

Though the equations (4.11) look like two independent equations, they are not actually independent since $\phi(r)$ is not a given function in them. It is defined by the first equation (4.6). In order to specify the charge density ρ in the right hand side of this equation we use the first formula (3.7). Upon substituting the product $\psi(r_p, r_e) = \psi_p(r_p)\,\psi_e(r_e)$ and (4.9) into (4.1) we get

$$\Psi(t, \mathbf{r}_p, \mathbf{r}_e) = \psi_p(r_p)\,\psi_e(r_e)\,\exp(-i\,\mathcal{E}_p\,t/\hbar)\,\exp(-i\,\mathcal{E}_e\,t/\hbar). \tag{4.12}$$

The wave functions $\psi_p = \psi_p(r)$ and $\psi_e = \psi_e(r)$ in (4.12) are assumed to be normalized to unity according to their probabilistic interpretations:

$$\int |\psi_p|^2\,d^3r = 4\,\pi \int\limits_0^\infty |\psi_p(r)|^2\,r^2\,dr = 1,$$
$$\int |\psi_e|^2\,d^3r = 4\,\pi \int\limits_0^\infty |\psi_e(r)|^2\,r^2\,dr = 1. \tag{4.13}$$

Substituting (4.12) into the first formula (3.7) and taking into account (4.13), we derive the following expression for the charge density ρ:

$$\rho = \rho(r) = e\,|\psi_p(r)|^2 - e\,|\psi_e(r)|^2. \tag{4.14}$$

Combining (4.14) with the first equation (4.6), we get the differential equation

$$\frac{d^2\phi}{dr^2} + \frac{2}{r}\frac{d\phi}{dr} = 4\,\pi\,e\,|\psi_e(r)|^2 - 4\,\pi\,e\,|\psi_p(r)|^2. \tag{4.15}$$

The equations (4.11) complemented with the equation (4.15) constitute a closed system of three ordinary differential equations for two complex-valued functions $\psi_p = \psi_p(r)$ and $\psi_e = \psi_e(r)$ and for one real-valued function $\phi = \phi(r)$.

Definition 4.1. A solution of the system of differential equations (4.11) and (4.15) on the half-line $0 < r < +\infty$ is called an *eigenstate* of the model if the functions $\psi_p = \psi_p(r)$ and $\psi_e = \psi_e(r)$ are square-integrable in the sense of the formulas (4.13) along with their derivatives so that

$$-\int \overline{\psi_p}\,\nabla^2\psi_p\,d^3r = \int |\nabla\psi_p|^2\,d^3r < \infty,$$
$$-\int \overline{\psi_e}\,\nabla^2\psi_e\,d^3r = \int |\nabla\psi_e|^2\,d^3r < \infty. \tag{4.16}$$

Lemma 4.2. *For any eigenstate of the model given by a solution of the equations* (4.11) *and* (4.15) *not only the total eigenvalue $\mathcal{E}$ in* (4.9) *is a real number, but the separate eigenvalues $\mathcal{E}_p$ and $\mathcal{E}_e$ are also real numbers.*

Lemma 4.2 is easily proved by applying (4.13) and (4.16) to the equations (4.11) written in the form of (4.10).

Note that if the functions $\psi_p = \psi_p(r)$ and $\psi_e = \psi_e(r)$ are given, the equation (4.15) is a linear non-homogeneous ordinary differential equation of the second

order. Its general solution is defined by the formula with two arbitrary constants

$$\phi = \phi_0 + C_1 + \frac{C_2}{r^2}, \tag{4.17}$$

provided some particular solution $\phi = \phi_0(r)$ is known.

Lemma 4.3. *For any eigenstate of the model given by a solution of the equations* (4.11) *and* (4.15) *there is a particular solution of the equation* (4.15) *such that*

$$\phi_0(r) = \int\limits_r^\infty \frac{1}{r^2} \left(\int\limits_0^r 4\,\pi\,e \left(|\psi_p(r)|^2 - |\psi_e(r)|^2 \right) r^2\,dr \right) dr. \tag{4.18}$$

The integrals in (4.18) do exist and are finite since ψ_p and ψ_e are square-integrable functions in the sense of (4.13). Therefore Lemma 4.3 is easily proved by direct calculations upon substituting (4.18) into (4.15). Note that the function (4.18) obeys the condition $\phi_0 \longrightarrow 0$ as $r \longrightarrow +\infty$. Combining this fact with the formula (4.17), we derive the following lemma.

Lemma 4.4. *For any eigenstate of the model given by a solution of the equations* (4.11) *and* (4.15) *there is a finite limit* $C = \lim\limits_{r\to+\infty} \phi(r) \neq \infty$.

Note that the equations (4.11) and (4.15) are invariant under the transformations

$$\phi \to \phi - C, \qquad\qquad \mathcal{E}_p \to \mathcal{E}_p - e\,C, \qquad\qquad \mathcal{E}_e \to \mathcal{E}_e + e\,C, \tag{4.19}$$

where C is some constant. The transformations (4.19) do not change the total eigenvalue $\mathcal{E}$ in (4.9).

Combining Lemma 4.4 with (4.19), we can formulate the next lemma.

Lemma 4.5. *For any eigenstate of the model given by a solution of the equations* (4.11) *and* (4.15) *there is an associated eigenstate such that* $\lim\limits_{r\to+\infty} \phi(r) = 0$. *The partial eigenvalues* $\mathcal{E}_p$ *and* $\mathcal{E}_e$ *of this associated eigenstate are called energy levels of the proton and the electron respectively.*

If the function $\phi = \phi(r)$ is fixed, then the equations (4.11) are linear homogeneous ordinary differential equations of the second order with respect to the functions ψ_p and ψ_e. These equations are very similar to each other. Therefore we can write them in a unified way omitting the particle indices:

$$-\frac{\hbar^2}{2\,m} \left(\psi'' + \frac{2}{r}\,\psi' \right) + u(r)\,\psi = \varepsilon\,\psi. \tag{4.20}$$

The square-integrability conditions (4.13) and (4.16) can also be written in a unified way as the following two conditions for solutions of the equation (4.20):

$$\int\limits_0^\infty |\psi|^2\,r^2\,dr < \infty, \qquad\qquad \int\limits_0^\infty |\psi'|^2\,r^2\,dr < \infty. \tag{4.21}$$

For any two solutions ψ_1 and ψ_2 of the differential equation (4.20) we construct their Wronskian: $W = W(\psi_1, \psi_2) = \psi_1\,\psi_2' - \psi_1'\,\psi_2$ (see [7]). One can easily prove the following facts concerning the Wronskian W:

1) for any two solutions ψ_1 and ψ_2 of the differential equation (4.20) their Wronskian $W = W(\psi_1, \psi_2)$ obeys the first order ordinary differential equation $W' + 2\,W/r = 0$ whose general solution is

$$W = \frac{C}{r^2}, \quad \text{where} \quad C = \text{const}; \tag{4.22}$$

2) two solutions ψ_1 and ψ_2 of the differential equation (4.20) are linearly dependent if and only if their Wronskian $W = W(\psi_1, \psi_2)$ is zero.

Lemma 4.6. *Any two solutions ψ_1 and ψ_2 of the differential equation (4.20) obeying both square-integrability conditions (4.21) are linearly dependent.*

Proof. It is known that the product of any two square-integrable functions is integrable (see [8]). Using this fact, from (4.21) we derive

$$\left| \int\limits_0^\infty W\,r^2\,dr \right| \leqslant \int\limits_0^\infty |\psi_1\,\psi_2' - \psi_1'\,\psi_2|\,r^2\,dr \leqslant \int\limits_0^\infty (|\psi_1\,\psi_2'| + |\psi_1'\,\psi_2|)\,r^2\,dr < \infty. \tag{4.23}$$

On the other hand, substituting (4.22) into the left hand side of (4.23), we get

$$\left| \int\limits_0^\infty W\,r^2\,dr \right| = \left| \int\limits_0^\infty C\,dr \right| = \begin{cases} 0 & \text{if } C = 0, \\ \infty & \text{if } C \neq 0. \end{cases} \tag{4.24}$$

Comparing (4.24) and (4.23), we find that $C = 0$ and $W = W(\psi_1, \psi_2) = 0$. The solutions ψ_1 and ψ_2 are linearly dependent since their Wronskian is zero. $\square$

Lemma 4.7. *Any nonzero complex-valued solution $\psi_{\mathbb{c}}$ of the differential equation (4.20) obeying both square-integrability conditions (4.21) is produced from some real-valued solution $\psi_{\mathbb{R}}$ of the same differential equation obeying both conditions (4.21) by multiplying it by some complex constant: $\psi_{\mathbb{c}} = C\,\psi_{\mathbb{R}}$.*

Proof. The equation (4.20) is a linear ordinary differential equation with real coefficients. Therefore, if $\psi_{\mathbb{c}}$ is its nonzero solution, then the complex conjugate function $\overline{\psi_{\mathbb{c}}}$ is also a solution of the equation (4.20). It is clear that if $\psi_{\mathbb{c}}$ obeys both conditions (4.21), then so does the complex conjugate function $\overline{\psi_{\mathbb{c}}}$. Applying Lemma 4.6 to the functions $\psi_{\mathbb{c}}$ and $\overline{\psi_{\mathbb{c}}}$, we find that they are linearly dependent, i.e. $\overline{\psi_{\mathbb{c}}} = K\,\psi_{\mathbb{c}}$, where K is some complex constant. Now we define two real-valued functions

$$\psi_1 = \frac{\psi_{\mathbb{c}} + \overline{\psi_{\mathbb{c}}}}{2} = \frac{1 + K}{2}\,\psi_{\mathbb{c}}, \qquad\qquad \psi_2 = \frac{\psi_{\mathbb{c}} - \overline{\psi_{\mathbb{c}}}}{2\,i} = \frac{1 - K}{2\,i}\,\psi_{\mathbb{c}}. \tag{4.25}$$

It is easy to see that the functions (4.25) are real-valued solutions of the equation (4.20) obeying both square-integrability conditions (4.21). At least one of them is nonzero. Therefore at least one of the following two formulas

$$\psi_{\mathbb{c}} = \frac{2}{1 + K}\,\psi_1, \qquad\qquad \psi_{\mathbb{c}} = \frac{2\,i}{1 - K}\,\psi_2$$

yields a valid expression $\psi_{\mathbb{C}} = C \, \psi_{\mathbb{R}}$ proving Lemma 4.7.   □

Now we return to the equations (4.11) and apply Lemma 4.7 to them. As a result we easily prove the following lemma strengthening Lemma 4.5.

Lemma 4.8. *For any eigenstate of the model given by a solution of the equations* (4.11) *and* (4.15) *there is an associated eigenstate with purely real-valued eigenfunctions* $\psi_p(r)$ *and* $\psi_e(r)$ *such that* $\lim\limits_{r \to +\infty} \phi(r) = 0$.

Let's recall that apart from the equations (4.11) and (4.15) we have the equality $\mathbf{j} = 0$ in (4.6). Applying the second formula (3.7) to it and taking into account the equality $\mathbf{A} = 0$ from (4.5), we derive the equation

$$\frac{i \, e \, \hbar}{2 \, m_p} \left(\psi_p \, \nabla \overline{\psi_p} - \overline{\psi_p} \, \nabla \psi_p \right) = \frac{i \, e \, \hbar}{2 \, m_e} \left(\psi_e \, \nabla \overline{\psi_e} - \overline{\psi_e} \, \nabla \psi_e \right). \tag{4.26}$$

The equation (4.26) is fulfilled identically if we apply Lemma 4.8 and choose purely real-valued eigenfunctions ψ_p and ψ_e. But even if they are not purely real-valued, according to Lemma 4.7, they differ from purely real ones by some constant factors. In this case the equation (4.26) is also fulfilled identically.

Going on, one can use more complicated methods from [9] for studying the equations (4.11). Conjecturing some integrability conditions for $\phi(r)$ at infinity, one can prove that the energy levels $\mathcal{E}_p$ and $\mathcal{E}_e$ are non-positive, i.e. $\mathcal{E}_p \leqslant 0$ and $\mathcal{E}_e \leqslant 0$. The Volterra-type integral equations from [9] can be applied for calculating $\psi_p(r)$, $\psi_e(r)$, and $\phi(r)$ numerically along with the associated eigenvalues $\mathcal{E}_p$ and $\mathcal{E}_e$. However these steps require much more efforts such as choosing a proper software and writing computer code. They are left for a separate paper.

5. Comparison with the standard model of the hydrogen atom.

As the reader can see above the standard model of the hydrogen atom is much more simple than the present model. Its spectrum is calculated explicitly, while our model require numeric computations. But there is also a conceptual difference. In the standard model the proton and the electron behave as classical point charges when producing the electromagnetic field in the form of Coulomb potential. Then they interact with the produced Coulomb field quantum mechanically as distributed charges according to their wave function in the Shrödinger equation with the Hamilton operator (1.1).

Our present model is more logically coherent. The proton and the electron in this model behave quantum mechanically as distributed sources in creating the electromagnetic field (see formulas (3.7)) and then they behave again quantum mechanically when interacting with the created field (see Shrödinger equation (3.6)).

Which of the two models is more congruent to the nature? I hope to answer this question upon computing the spectrum of the new model in forthcoming papers.

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