CLUSTERS OF EXPONENTIAL FUNCTIONS IN THE SPACE OF SQUARE INTEGRABLE FUNCTIONS.

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Abstract. Finite dimensional subspaces spanned by exponential functions in the space of square integrable functions on a finite interval of the real line are considered. Their limiting positions are studied and described in terms of expo-polynomials.

1. Introduction.

Exponential functions of the form $e^{\lambda\,x}$ naturally arise in Fourier series. More general series of exponential functions were studied by A. F. Leontiev (see [1]) and his school. In this paper we consider finite sequences of exponential functions

$$e^{\lambda_1 x},\,\ldots\,,e^{\lambda_n x},\tag{1.1}$$

where $\lambda_1,\,\ldots\,,\lambda_n$ are distinct complex numbers, i.e. $\lambda_i\neq\lambda_j$. These numbers are called the spectrum of the sequence (1.1).

The exponential functions (1.1) are treated as elements of the space of square integrable functions $L^2([a,b])$, where $-\infty<a<b<+\infty$. Without loss of generality we can take $a=-\pi$ and $b=+\pi$ like in Fourier analysis. The functions (1.1) span an n-dimensional subspace in $L^2([-\pi,+\pi])$:

$$L=\operatorname{Span}(e^{\lambda_1 x},\,\ldots\,,e^{\lambda_n x}).\tag{1.2}$$

The goal of this paper is to describe the behavior of the subspace (1.2) when the numbers $\lambda_1,\,\ldots\,,\lambda_n$ subdivide into m clusters and tend to some distinct limit values $\Lambda_1,\,\ldots\,,\Lambda_n$ common within each cluster. Therefore below we use the following double index notation for the numbers $\lambda_1,\,\ldots\,,\lambda_n$:

$$\lambda_{ij},\ \text{ where }\ i=1,\,\ldots\,,m\ \text{ and }\ j=1,\,\ldots\,,k_i.\tag{1.3}$$

The numbers $k_1,\ldots\,,k_m$ in (1.3) are called multiplicities of clusters. Their sum is equal to the total number of lambdas in (1.2):

$$k_1+\ldots+k_m=n.\tag{1.4}$$

Due to (1.3) and (1.4) we write (1.2) as

$$L=\operatorname{Span}(\{e^{\lambda_{ij}x}\}_{j=1,\,\ldots\,,\,k_i}^{i=1,\,\ldots\,,\,m})$$

2000 *Mathematics Subject Classification.* 42C15, 46C07.

or as $L = \operatorname{Span}(\{e^{\lambda_{ij} x}\})$ for short. As it was said above, we assume that

$$\lambda_{ij} \to \Lambda_i \tag{1.5}$$

in a sequence of samplings or in a continuous process. It is convenient to denote

$$\varepsilon = \max\{|\lambda_{ij} - \Lambda_i|\}^{i=1,\,\ldots\,,\,m}_{j=1,\,\ldots\,,\,k_i}. \tag{1.6}$$

Then we can write the formula (1.5) in the following way:

$$\lambda_{ij} \to \Lambda_i \quad \text{as} \quad \varepsilon \to 0. \tag{1.7}$$

2. Convergence of subspaces in a Hilbert space.

Let $\mathcal{H}$ be a Hilbert space (see [2]). The space of square integrable functions $L^2([-\pi, +\pi])$ is an example of a Hilbert space.

Definition 2.1. A sequence L_q of n-dimensional subspaces of a Hilbert space $\mathcal{H}$ is said to converge to an n-dimensional subspace M if there are some bases $\mathbf{e}_{1q}, \ldots, \mathbf{e}_{nq}$ in L_q and there is some basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in M such that

$$\mathbf{e}_{iq} \to \mathbf{e}_i \quad \text{as} \quad q \to \infty$$

in the sense of the norm of the Hilbert space $\mathcal{H}$.

The definition 2.1 can be reformulated in order to apply to the case of continuous parametric sets of subspaces.

Definition 2.2. Let L_ε be a parametric set of n-dimensional subspaces of a Hilbert space $\mathcal{H}$. It is said to converge to an n-dimensional subspace M as $\varepsilon \to 0$ if there are some bases $\mathbf{e}_{1\varepsilon}, \ldots, \mathbf{e}_{n\varepsilon}$ in L_ε and there is a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in M such that

$$\mathbf{e}_{i\varepsilon} \to \mathbf{e}_i \quad \text{as} \quad \varepsilon \to 0$$

in the sense of the norm of the Hilbert space $\mathcal{H}$.

3. Taylor expansions of exponential functions.

This section is a preliminary one. Assume for a while that we have only one cluster (i. e. $m = 1$) with $\Lambda_1 = 0$. Then we can use the initial notations $\lambda_1, \ldots, \lambda_n$ for lambdas and write the formula (1.5) as $\lambda_i \to 0$. The exponential functions (1.2) have the following Taylor expansions:

$$\begin{aligned} &e^{\lambda_1 x} = 1 + \lambda_1 x + \ldots + \frac{\lambda_1^{n-1}\, x^{n-1}}{(n-1)!} + \ldots = \sum_{q=0}^{\infty} \frac{\lambda_1^q\, x^q}{q!}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &e^{\lambda_n x} = 1 + \lambda_n x + \ldots + \frac{\lambda_n^{n-1} x^{n-1}}{(n-1)!} + \ldots = \sum_{q=0}^{\infty} \frac{\lambda_n^q x^q}{q!}. \end{aligned} \tag{3.1}$$

Initial parts of the power series (3.1) define the polynomials

$$p_i(x) = 1 + \lambda_i x + \ldots + \frac{\lambda_i^{n-1}\, x^{n-1}}{(n-1)!}, \quad \text{where} \quad i = 1, \ldots, n. \tag{3.2}$$

Using the polynomials (3.2), we define the following equation for $a_1, \ldots, a_n$:

$$a_1\,p_1(x)+\ldots+a_n\,p_n(x)=\frac{x^{n-1}}{(n-1)!}. \tag{3.3}$$

The polynomial equation (3.3) is equivalent to a matrix equation for $a_1, \ldots, a_n$:

$$\begin{Vmatrix} 1 & 1 & \ldots & 1\\ \lambda_1 & \lambda_2 & \ldots & \lambda_n\\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_n^2\\ \vdots & \vdots & \ddots & \vdots\\ \lambda_1^{n-1} & \lambda_2^{n-1} & \ldots & \lambda_n^{n-1} \end{Vmatrix} \cdot \begin{Vmatrix} a_1\\ a_2\\ a_3\\ \vdots\\ a_n \end{Vmatrix} = \begin{Vmatrix} 0\\ 0\\ 0\\ \vdots\\ 1 \end{Vmatrix}. \tag{3.4}$$

The matrix in (3.4) is the transpose of the Vandermonde matrix (see [3]):

$$W=\begin{Vmatrix} 1 & \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^{n-1}\\ 1 & \lambda_2 & \lambda_2^2 & \ldots & \lambda_2^{n-1}\\ 1 & \lambda_3 & \lambda_3^2 & \ldots & \lambda_3^{n-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & \lambda_n & \lambda_n^2 & \ldots & \lambda_n^{n-1} \end{Vmatrix}. \tag{3.5}$$

The Vandermonde matrix (3.5) is non-degenerate for distinct lambdas, i. e. if $\lambda_i\neq\lambda_j$. In this case it has the inverse matrix $U=W^{-1}$ (see [4]). In order to write the elements of the inverse matrix $U=W^{-1}$ explicitly we use the polynomials

$$P_q(\lambda)=\frac{\displaystyle\prod_{s\neq q}^{n}(\lambda-\lambda_s)}{\displaystyle\prod_{s\neq q}^{n}(\lambda_q-\lambda_s)},\quad\text{where}\quad q=1,\,\ldots,\,n. \tag{3.6}$$

It is easy to see that the polynomials (3.6) obey the equality

$$P_q(\lambda_i)=\begin{cases} 1 & \text{for}\quad q=i,\\ 0 & \text{for}\quad q\neq i. \end{cases} \tag{3.7}$$

If we present the polynomials (3.6) as the power expansions

$$P_q(\lambda)=\sum_{r=1}^{n}U_{rq}\,\lambda^{r-1}=U_{1q}+U_{2q}\,\lambda+\ldots+U_{nq}\,\lambda^{n-1}, \tag{3.8}$$

then the equality (3.7) can be rewritten as

$$\sum_{r=1}^{n}\lambda_i^{r-1}\,U_{rq}=\begin{cases} 1 & \text{for}\quad q=i,\\ 0 & \text{for}\quad q\neq i. \end{cases} \tag{3.9}$$

Looking at the matrix (3.5), we see that the equality (3.9) is equivalent to the matrix equality $W \cdot U = 1$, where U is the matrix whose components coincide with the coefficients in power expansions (3.8):

$$U = \begin{Vmatrix} U_{11} & U_{12} & U_{13} & \ldots & U_{1n} \\ U_{21} & U_{22} & U_{23} & \ldots & U_{2n} \\ U_{31} & U_{32} & U_{33} & \ldots & U_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ U_{n1} & U_{n2} & U_{n3} & \ldots & U_{nn} \end{Vmatrix}. \tag{3.10}$$

Explicit expressions for the components of the matrix (3.10) are derived from (3.6):

$$U_{rq} = \frac{1}{(r-1)!} \left.\frac{d^{r-1} P_q(\lambda)}{d\lambda^{r-1}}\right|_{\lambda=0}. \tag{3.11}$$

The equality $W \cdot U = 1$ derived from (3.9) means that the matrix (3.10) with the components (3.11) is inverse to the Vandermonde matrix (3.5).

Since $(W^{\top})^{-1} = U^{\top}$, we can apply the transpose of the matrix U in order to solve the matrix equation (3.4). Its solution is written as

$$a_q = U_{nq} = \frac{1}{(n-1)!} \left.\frac{d^{n-1} P_q(\lambda)}{d\lambda^{n-1}}\right|_{\lambda=0} = \frac{1}{\displaystyle\prod_{s \neq q}^{n} (\lambda_q - \lambda_s)}. \tag{3.12}$$

Along with solving the matrix equation (3.4), the quantities (3.12) solve the polynomial equation (3.3) as well.

Now, using the quantities (3.12) as coefficients, we define the following linear combination of the exponential functions (1.1):

$$f_n(x) = a_1\, e^{\lambda_1 x} + \ldots + a_n\, e^{\lambda_n x}. \tag{3.13}$$

Taking into account (3.1), (3.2) and (3.3), for $f_n(x)$ we derive the power expansion

$$f_n(x) = \frac{x^{n-1}}{(n-1)!} + \sum_{q=1}^{\infty} \frac{B_{n\,q}\, x^{n-1+q}}{(n-1+q)!}. \tag{3.14}$$

The coefficients $B_{n\,q}$ in (3.14) are given by the formula

$$B_{n\,q} = \sum_{i=1}^{n} a_i\, \lambda_i^{n-1+q}. \tag{3.15}$$

Lemma 3.1. *For mutually distinct numbers $\lambda_i \neq \lambda_j$ and for $q \geqslant 1$ the quantities $B_{n\,q}$ from (3.15) obey the recurrent relationship*

$$B_{n+1\,q} = B_{n\,q} + \lambda_{n+1}\, B_{n+1\,q-1}. \tag{3.16}$$

Lemma 3.1 is easily proved by means of direct calculations using (3.12) and (3.15). Substituting $q = 0$ into (3.15) and taking into account the matrix equality

(3.4) for the quantities $a_1, \ldots, a_n$, we find that

$$B_{n\,0} = 1 \quad \text{for all} \quad n \geqslant 1. \tag{3.17}$$

If $n = 1$, the formula (3.12) turns to $a_q = 1$. Then (3.15) yields

$$B_{1\,q} = \lambda_1^q \quad \text{for all} \quad q \geqslant 0. \tag{3.18}$$

The formula (3.18) can be derived directly from the first Tailor expansion (3.1).

The formulas (3.17) and (3.18) along with the recurrent relationship (3.16) are sufficient to determine all of $B_{n\,q}$ inductively on two parameters n and q.

Lemma 3.2. *The quantities $B_{n\,q}$ defined for mutually distinct lambdas $\lambda_i \neq \lambda_j$ through the formulas (3.12) and (3.15) are given by the explicit formula*

$$B_{n\,q} \;= \sum_{1\leqslant i_1\leqslant\,\ldots\,\leqslant i_q\leqslant n} \!\!\cdots\!\! \sum \lambda_{i_1}\cdot\ldots\cdot\lambda_{i_q}, \quad \textit{where} \quad n \geqslant 1 \quad \textit{and} \quad q \geqslant 1. \tag{3.19}$$

In order to prove Lemma 3.2 it is sufficient to verify the formulas (3.16) and (3.18) upon substituting (3.19) into them. The formula (3.19), along with the formula (3.17), determines all of the quantities $B_{n\,q}$.

Let's denote through $N_{n\,q}$ the number of summands in the formula (3.19). This number is estimated in the following way:

$$N_{n\,q} \leqslant n^q. \tag{3.20}$$

Let's recall that in present section we consider the special case where the number of clusters $m = 1$, $k_1 = n$, and $\Lambda_1 = 0$. Therefore let's denote

$$\varepsilon = \max(|\lambda_1|, \ldots, |\lambda_n|). \tag{3.21}$$

The notation (3.21) is a version of (1.6) adapted to our present case. Applying (3.20) and (3.21) to the summands in (3.14), for $q \geqslant 1$ we get

$$\left|\frac{B_{n\,q}\,x^{n-1+q}}{(n-1+q)!}\right| \leqslant \frac{n^q\,\varepsilon^q\,|x|^{n-1+q}}{(n-1)!\,n\,(n+1)\cdot\ldots\cdot(h+q-1)}. \tag{3.22}$$

A weaker estimate is sufficient for our purposes. Therefore from (3.22) we derive

$$\left|\frac{B_{n\,q}\,x^{n-1+q}}{(n-1+q)!}\right| \leqslant \frac{\varepsilon^q\,|x|^{n-1+q}}{(n-1)!}, \quad \text{where} \quad q \geqslant 1. \tag{3.23}$$

From (3.23) one can easily derive the norm estimate

$$\left\|\frac{B_{n\,q}\,x^{n-1+q}}{(n-1+q)!}\right\| \leqslant \sqrt{\frac{2\,\pi}{2\,n+2\,q-1}}\,\frac{\pi^{n-1}}{(n-1)!}\,(\pi\,\varepsilon)^q. \tag{3.24}$$

in term of the L^2-norm of the Hilbert space $\mathcal{H} = L^2([-\pi, +\pi])$. Since $n = \text{const}$ in (3.1) and (3.24), the estimate (3.24) can be simplified. For this purpose we introduce the following constant that does not depend on $q\geqslant 1$:

$$C_n=\sqrt{\frac{2\,\pi}{2\,n+1}}\,\frac{\pi^{n-1}}{(n-1)!}. \tag{3.25}$$

Using the constant (3.25), the estimate (3.24) is simplified as

$$\left\|\frac{B_{n\,q}\,x^{n-1+q}}{(n-1+q)!}\right\|\leqslant C_n\,(\pi\,\varepsilon)^q,\ \text{ where }\ q\geqslant 1. \tag{3.26}$$

If $\pi\,\varepsilon<1/2$, then the estimate (3.26) produces an estimate for the function $f_n(x)$ from (3.13) and (3.14). Here is this estimate:

$$\left\|f_n(x)-\frac{x^{n-1}}{(n-1)!}\right\|\leqslant\frac{C_n\,\pi\,\varepsilon}{1-\pi\,\varepsilon}\leqslant 2\,C_n\,\pi\,\varepsilon. \tag{3.27}$$

Due to (3.13) the function $f_n(x)$ is a linear combination of the exponential functions $e^{\lambda_1 x},\,\ldots\,,\,e^{\lambda_n x}$, i.e. $f_n(x)\in\operatorname{Span}(e^{\lambda_1 x},\,\ldots\,,e^{\lambda_n x})$. Therefore we can formulate the following theorem.

Theorem 3.1. *For any n mutually distinct complex quantities $\lambda_1,\,\ldots\,,\,\lambda_n$ tending to zero there is a function $f(x)$ belonging to the subspace*

$$L=\operatorname{Span}(e^{\lambda_1 x},\,\ldots\,,e^{\lambda_n x}).$$

of the Hilbert space of square integrable functions $\mathcal{H}=L^2([-\pi,+\pi])$ and such that

$$\|f(x)-x^{n-1}\|\to 0 \tag{3.28}$$

as $\lambda_1,\,\ldots\,,\,\lambda_n$ tend to zero.

Theorem 3.1 is immediate from the inequality (3.27). It is important to note that the norm convergence in (3.28) is irrespective of any mutual relations of $\lambda_1,\,\ldots\,,\,\lambda_n$ and is irrespective of the individual convergence rates of $\lambda_i\to 0$.

Having n mutually distinct complex quantities $\lambda_1,\,\ldots\,,\,\lambda_n$ converging to zero, we can take a part of them $\lambda_1,\,\ldots\,,\,\lambda_s$, where $1\leqslant s\leqslant n$. Applying the theorem 3.1 to all of such parts, we derive the following result.

Theorem 3.2. *For any n mutually distinct complex quantities $\lambda_1,\,\ldots\,,\,\lambda_n$ tending to zero there are n function $f_1(x),\,\ldots\,,\,f_n(x)$ belonging to the subspace*

$$L=\operatorname{Span}(e^{\lambda_1 x},\,\ldots\,,e^{\lambda_n x}).$$

of the Hilbert space of square integrable functions $\mathcal{H}=L^2([-\pi,+\pi])$ and such that

$$\|f_s(x)-x^{s-1}\|\to 0 \tag{3.29}$$

as $\lambda_1,\,\ldots\,,\,\lambda_n$ tend to zero for all $s=1,\,\ldots\,,\,n$.

Again, it is important to note that the norm convergences in (3.29) are irrespective of any mutual relations of $\lambda_1, \ldots, \lambda_n$ and are irrespective of the individual convergence rates of $\lambda_i \to 0$.

The quantities $\lambda_1, \ldots, \lambda_n$ tending to zero can go through some discrete sets of values or their convergence can be a continuous process. In both cases, relying on the definitions 2.1 and 2.2, we can derive the following theorem.

Theorem 3.3. *For any n mutually distinct complex quantities $\lambda_1, \ldots, \lambda_n$ tending to zero the span of exponential functions*

$$L = \mathrm{Span}(e^{\lambda_1 x}, \ldots, e^{\lambda_n x})$$

converges to the span of polynomials

$$M = \mathrm{Span}(1, x, \ldots, x^{n-1})$$

in the Hilbert space of square integrable functions $\mathcal{H} = L^2([-\pi, +\pi])$.

Theorem 3.3 is immediate from the previous theorem 3.2.

4. The case of multiple clusters.

Now we proceed to the general case where lambdas are subdivided into m clusters with $k_1, \ldots, k_m$ being the multiplicities of clusters (see (1.3)). They tend to m mutually distinct complex numbers $\Lambda_1, \ldots, \Lambda_m$ according to (1.5) or (1.7). Therefore in this case we introduce the deflection numbers

$$\delta_{ij} = \lambda_{ij} - \Lambda_i$$

tending to zero and, instead of (3.1), we write

$$e^{\lambda_{i1} x} = e^{\Lambda_i x}\left(1 + \delta_{i1}\, x + \ldots + \frac{\delta_{i1}^{n-1}\, x^{n-1}}{(n-1)!}\right) + \ldots = \sum_{q=1}^{\infty} \frac{\delta_{i1}^q\, x^q\, e^{\Lambda_i x}}{q!},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$e^{\lambda_{ik_i} x} = e^{\Lambda_i x}\left(1 + \delta_{ik_i}\, x + \ldots + \frac{\delta_{ik_i}^{n-1} x^{n-1}}{(n-1)!}\right) + \ldots = \sum_{q=1}^{\infty} \frac{\delta_{ik_i}^q\, x^q\, e^{\Lambda_i x}}{q!}.$$

As a result, instead of Theorem 3.1, here we get the following theorem.

Theorem 4.1. *For a set of mutually distinct complex quantities λ_{ij}, where $i = 1, \ldots, m$ and $j = 1, \ldots, k_i$, tending to m mutually distinct complex numbers $\Lambda_1, \ldots, \Lambda_m$ so that $\lambda_{ij} \to \Lambda_i$ there is a function $f_s(x)$ belonging to the subspace*

$$L = \mathrm{Span}(\{e^{\lambda_{ij} x}\}_{j=1,\ldots,k_i}^{i=1,\ldots,m})$$

of the Hilbert space of square integrable functions $\mathcal{H} = L^2([-\pi, +\pi])$ and such that

$$\|f_s(x) - x^{k_s - 1}\, e^{\Lambda_s x}\| \to 0 \quad \text{as} \quad \lambda_{ij} \to \Lambda_i. \tag{4.1}$$

The proof of Theorem 4.1 is basically the same as the proof of Theorem 3.1. The main difference is that, instead of the polynomial x^{n-1}, here in Theorem 4.1 we have the expo-polynomial $x^{k_i-1}\,e^{\Lambda_i x}$.

The expo-polynomial $x^{k_i-1}\,e^{\Lambda_i x}$ is not unique. Repeating the reasons used in deriving Theorem 3.2 from Theorem 3.1, we can write the following theorem that provides as many expo-polynomials as the exponential functions we have.

Theorem 4.2. *For a set of mutually distinct complex quantities λ_{ij}, where $i=1,\,\ldots\,,m$ and where $j=1,\,\ldots\,,k_i$, tending to m mutually distinct complex numbers $\Lambda_1,\,\ldots\,,\,\Lambda_m$ so that $\lambda_{ij}\to\Lambda_i$ there is a set of functions $f_{ij}(x)$, where $i=1,\,\ldots\,,m$ and $j=1,\,\ldots\,,k_i$, belonging to the subspace*

$$L=\operatorname{Span}(\{e^{\lambda_{ij}x}\}_{j=1,\,\ldots\,,\,k_i}^{i=1,\,\ldots\,,\,m})$$

of the Hilbert space of square integrable functions $\mathcal{H}=L^2([-\pi,+\pi])$ and such that

$$\|f_{ij}(x)-x^{j-1}\,e^{\Lambda_i x}\|\to 0 \quad as \quad \lambda_{ij}\to\Lambda_i. \tag{4.2}$$

It is important to note that the norm convergences in (4.1) and (4.2) are irrespective of any mutual relations between λ_{ij} and are irrespective of the individual convergence rates of $\lambda_{ij}\to\Lambda_i$.

The quantities λ_{ij} tending to Λ_i within their clusters can go through some discrete sets of values or their convergence can be a continuous process. In both cases, relying on the definitions 2.1 and 2.2, we can derive the following theorem.

Theorem 4.3. *For a set of mutually distinct complex quantities λ_{ij}, where $i=1,\,\ldots\,,m$ and $j=1,\,\ldots\,,k_i$, tending to m mutually distinct complex numbers $\Lambda_1,\,\ldots\,,\,\Lambda_m$ so that $\lambda_{ij}\to\Lambda_i$ the span of exponential functions*

$$L=\operatorname{Span}(\{e^{\lambda_{ij}x}\}_{j=1,\,\ldots\,,\,k_i}^{i=1,\,\ldots\,,\,m})$$

converges to the span of expo-polynomials

$$M=\operatorname{Span}(\{x^{j-1}\,e^{\Lambda_i x}\}_{j=1,\,\ldots\,,\,k_i}^{i=1,\,\ldots\,,\,m})$$

in the Hilbert space of square integrable functions $\mathcal{H}=L^2([-\pi,+\pi])$.

Theorem 4.3 is the main result of this paper. It is immediate from Theorem 4.2.

5. Acknowledgment.

I am grateful to my friend A. S. Vishnevskiy who stimulated my interest to a mathematical problem originated in his business. This paper is the first step in my attempts toward solving this problem.

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