ON ROOT MEAN SQUARE APPROXIMATION BY EXPONENTIAL FUNCTIONS.

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Abstract. The problem of root mean square approximation of a square integrable function by finite linear combinations of exponential functions is considered. It is subdivided into linear and nonlinear parts. The linear approximation problem is solved. Then the nonlinear problem is studied in some particular example.

1. Introduction.

Exponential functions of the form $e^{\lambda\,x}$ arise as solutions of linear differential equations. In physics they describe oscillatory and damped oscillatory processes. Assume that $f(x)$ is a complex-valued square integrable function with the argument x belonging to the interval $[-\pi,\,+\pi]$ of the real line $\mathbb{R}$:

$$f(x)\in L^2([-\pi,\,+\pi]). \tag{1.1}$$

Let's consider a finite sequence of exponential functions

$$e^{\lambda_1 x},\,\ldots\,,\,e^{\lambda_n x}, \tag{1.2}$$

where $\lambda_1,\,\ldots\,,\,\lambda_n$ are distinct complex numbers, i. e. $\lambda_i\neq\lambda_j$. Using the functions (1.2), we compose a linear combination with complex coefficients:

$$\phi(x)=\sum_{i=1}^{n}a_i\,e^{\lambda_i x}. \tag{1.3}$$

We say that the function (1.3) approximates the square integrable function (1.1) if the L^2-norm of their difference is sufficiently small:

$$\|\,f-\phi\,\|=\sqrt{\frac{1}{2\,\pi}\int\limits_{-\pi}^{+\pi}|\,f(x)-\phi(x)\,|^2\,dx}. \tag{1.4}$$

The quantity (1.4) is also known as the root mean square deflection of ϕ from f. The problem of minimizing this deflection is called the root mean square approximation problem. It can be attributed to the class of variational problems.

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The problem of minimizing the root mean square deflection (1.4) is subdivided into linear and nonlinear parts. The linear problem consists in finding optimal coefficients $a_1, \dots, a_n$ in (1.3), provided $\lambda_1, \dots, \lambda_n$ are fixed. This problem is similar to those studied by A. F. Leontiev and his school (see [1]). In the case of a finite set of exponential functions (1.2) it is solved completely in an explicit form.

The nonlinear approximation problem consists in minimizing the solution of the linear problem by varying $\lambda_1, \dots, \lambda_n$ and choosing optimal values for them. It arises from the applied problem of numerical separation of a noised signal presumably being a mixture of oscillatory and damped oscillatory signals. In this form the problem was suggested by A. S. Vishnevskiy, president of PhysTech Co., the weighing technologies company. In the present paper the nonlinear problem is studied in the example of the very simple function $f(x) = \operatorname{sign}(x)$.

2. Solution of the linear approximation problem.

From (1.4) one can easily derive the following formula for the the root mean square deflection of the function (1.3) from f:

$$\begin{aligned} \|f-\phi\|^2 = \|f\|^2 - \sum_{j=1}^{n} a^j \left\langle f\middle|e^{\lambda_j x}\right\rangle - \\ - \sum_{i=1}^{n} \overline{a^i} \left\langle e^{\lambda_i x}\middle|f\right\rangle + \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\, \overline{a^i}\, a^j. \end{aligned} \tag{2.1}$$

By means of angular brackets in (2.1) we denote the L^2-scalar product

$$\left\langle a\middle|b\right\rangle = \frac{1}{2\,\pi} \int_{-\pi}^{+\pi} \overline{a(x)}\, b(x)\, dx \tag{2.2}$$

Overlined variables and functions in (2.1) and (2.2) mean complex conjugates. Through g_{ij} in (2.1) we denote the components of the Gram matrix G:

$$g_{ij} = \left\langle e^{\lambda_i x}\middle|\, e^{\lambda_j x}\right\rangle. \tag{2.3}$$

From (2.2) one can easily derive the following property of the L^2-scalar product

$$\overline{\left\langle a\middle|b\right\rangle} = \left\langle b\middle|a\right\rangle. \tag{2.4}$$

The property (2.4) implies the following relationships:

$$g_{ij} = \overline{g_{ji}}, \qquad\qquad \left\langle f\middle|\,e^{\lambda_i x}\right\rangle = \overline{\left\langle e^{\lambda_i x}\middle|f\right\rangle}. \tag{2.5}$$

If we denote $F = \|f-\phi\|^2$ and treat F as a function of the complex coefficients $a^1, \dots, a^m$ from (2.1), then the minimum of the function $F(a^1, \dots, a^m)$ is determined by the vanishing conditions for its partial derivatives:

$$\frac{\partial F}{\partial a^i} = 0, \quad \frac{\partial F}{\partial \overline{a^i}} = 0, \text{ where } i = 1, \dots, n. \tag{2.6}$$

Calculating the derivatives (2.6) for (2.1), we derive the equations

$$\sum_{i=1}^{n} g_{ij}\,\overline{a^i} = \bigl\langle f\,\big|\,e^{\lambda_j x}\bigr\rangle, \qquad\qquad \sum_{j=1}^{n} g_{ij}\,a^j = \bigl\langle e^{\lambda_i x}\,\big|\,f\bigr\rangle, \tag{2.7}$$

where $i = 1, \ldots, n$. Due to (2.5) two sets of equations (2.7) differ from each other only by complex conjugation. Hence it is sufficient to solve only one of them.

We choose for solving the second set of the equations (2.7). It is solved with the use of the inverse Gram matrix G^{-1}. Let's denote through g^{ij} the components of the transpose of the inverse Gram matrix $(G^{-1})^{\top}$. Then the quantities g_{ij} and g^{ij} are related to each other as follows:

$$\sum_{k=1}^{n} g_{ik}\,g^{jk} = \delta_i^j, \qquad\qquad \sum_{k=1}^{n} g^{kj}\,g_{ki} = \delta_i^j. \tag{2.8}$$

Here δ_i^j are the components of the unit matrix. They are called Kronecker's delta.

Applying the second relationship (2.8) to the second set of equations (2.7), we get their solution. This solution is given by the formula

$$a^j = \sum_{i=1}^{n} g^{ij}\,\bigl\langle e^{\lambda_i x}\,\big|\,f\bigr\rangle, \text{ where } j = 1, \ldots, n. \tag{2.9}$$

Substituting (2.9) into (2.1), we derive

$$F_{\min} = \|f\|^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}\,\bigl\langle e^{\lambda_i x}\,\big|\,f\bigr\rangle\,\bigl\langle f\,\big|\,e^{\lambda_j x}\bigr\rangle. \tag{2.10}$$

The formulas (2.9) and (2.10) yield a solution of the linear approximation problem.

3. One frequency approximation for the sign function.

The minimum value $F_{\min}$ in (2.10) is a function of $\lambda_1, \ldots, \lambda_n$. Let's denote

$$\Phi = \Phi(\lambda_1, \ldots, \lambda_n) = F_{\min}. \tag{3.1}$$

The nonlinear approximation problem consists in finding the absolute minimum of the function (3.1) as $\lambda_1, \ldots, \lambda_n$ run over $\mathbb{C}^n$. Though potentially this could be not a minimum, but infimum. In any case, since $0 \leqslant F_{\min} \leqslant \|f\|^2$, the minimal value of the function $\Phi(\lambda_1, \ldots, \lambda_n)$ does exist and is finite.

Let's consider the case $n = 1$. We call it the one frequency case since the quantities $\lambda_1, \ldots, \lambda_n$ are often associated with eigenfrequencies in applications. In order to study this case thoroughly we choose

$$f(x) = \operatorname{sign}(x) = \begin{cases} -1 & \text{for } x < 0, \\ 1 & \text{for } x \geqslant 0 \end{cases} \tag{3.2}$$

as an example. In the one frequency case the formula (2.10) simplifies. It turns to

$$F_{\min} = \|f\|^2 - \frac{\bigl\langle e^{\lambda_1 x}\,\big|\,f\bigr\rangle\,\bigl\langle f\,\big|\,e^{\lambda_1 x}\bigr\rangle}{g_{11}}. \tag{3.3}$$

The norm of the function (3.2) is easily calculated: $\|f\| = 1$. The denominator g_{11} in the formula (3.3) is also easily calculated:

$$g_{11} = \frac{1}{2\,\pi}\int\limits_{-\pi}^{+\pi} e^{(\lambda_1+\bar\lambda_1)\,x}\,dx = \frac{e^{(\lambda_1+\bar\lambda_1)\,\pi} - e^{-(\lambda_1+\bar\lambda_1)\,\pi}}{2\,\pi\,(\lambda_1+\bar\lambda_1)}. \tag{3.4}$$

Now let's calculate the quantities $\bigl\langle e^{\lambda_1 x}\bigr|f\bigr\rangle$ and $\bigl\langle f\bigr|e^{\lambda_1 x}\bigr\rangle$ in (3.3):

$$\bigl\langle e^{\lambda_1 x}\bigr|f\bigr\rangle = \frac{1}{2\,\pi}\int\limits_{-\pi}^{+\pi} e^{\bar\lambda_1 x}\,\mathrm{sign}(x)\,dx = \frac{e^{\bar\lambda_1\pi} + e^{-\bar\lambda_1\pi}}{2\,\pi\,\bar\lambda_1} - \frac{1}{\pi\,\bar\lambda_1}, \tag{3.5}$$

$$\bigl\langle f\bigr|e^{\lambda_1 x}\bigr\rangle = \frac{1}{2\,\pi}\int\limits_{-\pi}^{+\pi} \mathrm{sign}(x)\,e^{\lambda_1 x}\,dx = \frac{e^{\lambda_1\pi} + e^{-\lambda_1\pi}}{2\,\pi\,\lambda_1} - \frac{1}{\pi\,\lambda_1}. \tag{3.6}$$

The spectral parameter λ_1 is a complex variable. Therefore we write

$$\lambda_1 = u + \mathrm{i}\,v,\ \text{ where }\ \mathrm{i} = \sqrt{-1}. \tag{3.7}$$

Substituting (3.7) into the formula (3.4), we derive

$$g_{11} = \frac{\sinh(2\,\pi\,u)}{2\,\pi\,u}.$$

Similarly, using the formulas (3.5) and (3.6), we obtain

$$\bigl\langle e^{\lambda_1 x}\bigr|f\bigr\rangle\bigl\langle f\bigr|e^{\lambda_1 x}\bigr\rangle = \frac{\cosh(2\,\pi\,u) + \cos(2\,\pi\,v)}{2\,\pi^2(u^2+v^2)} + \\ + \frac{2 - 4\,\cosh(\pi\,u)\,\cos(\pi\,v)}{2\,\pi^2(u^2+v^2)} = \frac{(\cosh(\pi\,u) - \cos(\pi\,v))^2}{\pi^2(u^2+v^2)}.$$

As a result we obtain the following expression for the function (3.1):

$$\Phi(\lambda_1) = \Phi(u + \mathrm{i}\,v) = 1 - \frac{2\,u}{u^2+v^2}\,\frac{(\cosh(\pi\,u) - \cos(\pi\,v))^2}{\pi\,\sinh(2\,\pi\,u)}. \tag{3.8}$$

Passing to the limit $u \to 0$ in (3.8), we obtain the function

$$\Phi(0 + \mathrm{i}\,v) = 1 - \frac{(1 - \cos(\pi\,v))^2}{\pi^2\,v^2}. \tag{3.9}$$

The function (3.9) has two absolute minima at $v = \pm v_0$, where v_0 is a real irrational number being a solution of the equation

$$\sin(\pi\,v)\,\pi\,v + \cos(\pi\,v) = 1 \tag{3.10}$$

and such such that $0.1 < v_0 < 0.9$. Solving (3.10) numerically, we find that

$$v_0 = 0.742019\,\ldots. \tag{3.11}$$

Applying (3.10) and (3.11) to (3.9), we derive

$$\Phi_{\min} = \Phi(\pm\mathrm{i}\,v_0) = \cos^2(v_0) = 0.4749383\,.\,.\,.\,. \tag{3.12}$$

One can show that two complex numbers $\lambda = \pm\mathrm{i}\,v_0$ are absolute minima of the function (3.8) as well. The number (3.12) is its minimum value.

It is curious to note that the best one frequency root mean square approximation for the real-valued function $\operatorname{sign}(x)$ is given by a complex function, which is quite unlike to its Fourier expansion approximation.

4. Two frequencies approximation for the sign function.

The two frequencies case $n = 2$ subdivides into two subcases $\lambda_1 \neq \lambda_2$ and $\lambda_1 = \lambda_2$. The subcase $\lambda_1 = \lambda_2$ is a cluster case (see [2]). In this case two exponential functions $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are replaced by two expo-polynomials

$$\phi_1(x) = e^{\lambda_1 x}, \qquad\qquad \phi_2(x) = x\, e^{\lambda_1 x}.$$

The formula (2.3) is replaced by the formula

$$g_{ij} = \langle \phi_i \,|\, \phi_j \rangle.$$

Similarly, the formula (2.10) is replaced by the formula

$$F_{\min} = \|f\|^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}\,\langle \phi_i \,|\, f\rangle\,\langle f \,|\, \phi_j\rangle. \tag{4.1}$$

The formula (3.1) remains unchanged.

For the beginning we consider the cluster case $\lambda_1 = \lambda_2$ with $\lambda_1 = 0+\mathrm{i}\,v$. Applying the formulas (3.1) and (4.1) to this case, we derive

$$\begin{gathered}\Phi(0+\mathrm{i}\,v) = 1 - (1-\cos(\pi\,v))\,\times\\ \times\,\frac{(2\,\cos(\pi\,v)\,\pi^2\,v^2 - 3\,\cos(\pi\,v) + 3 + 4\,\pi^2\,v^2 - 6\,\sin(\pi\,v)\,\pi\,v)}{\pi^4\,v^4}.\end{gathered} \tag{4.2}$$

Unlike (3.9), the function (4.2) has exactly one absolute minimum at $v = 0$ (i. e. at the origin of the complex plane $\lambda_1 = 0 + \mathrm{i}\,0 = 0$) such that

$$\Phi_{\min} = \lim_{v\to 0} \Phi(0+\mathrm{i}\,v) = \frac{1}{4}. \tag{4.3}$$

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The general cluster case corresponds to $\lambda_1 = \lambda_2$ with $\lambda_1 = u + \mathrm{i}\,v$. In this case, applying (3.1) and (4.1), we obtain an explicit expression for the function $\Phi(\lambda_1) = \Phi(u+\mathrm{i}\,v)$. However, this expression is much more bulky than the expression (3.8). Analyzing this bulky expression numerically, we find that the function $\Phi(\lambda_1)$ has a unique minimum at the same point $\lambda_1 = 0$ as the function (4.2).

The next step is to proceed to the non-cluster case. In this case the formulas (2.10) and (3.1) yield a function of two complex variables $\Phi(\lambda_1, \lambda_2)$. The expression for $\Phi(\lambda_1, \lambda_2)$ is very bulky and complicated. Finding an absolute minimum numerically for such a function is also a complicated problem. Therefore at present moment we can only formulate a conjecture that

$$\lim_{\substack{\lambda_1\to 0\\ \lambda_2\to 0}} \Phi(\lambda_1,\lambda_2)=\inf_{\lambda_1\neq\lambda_2}\{\Phi(\lambda_1,\lambda_2)\}=\frac{1}{4}. \tag{4.4}$$

The formula (4.3) provides a support for our conjecture (4.4).

5. Conclusions.

Once a square integrable function $f(x)\in L^2([-\pi,\,+\pi])$ is given, the optimal values of $\lambda_1,\,\ldots\,,\,\lambda_n$ for f are called an n-frequencies spectrum of the function f. The above example of the sign function shows that the n-frequencies spectrum is not unique. Moreover, it is not stable. The spectral point $\lambda_1=\mathfrak{i}\,v_0$, which is present in one frequency spectrum, disappears in two frequencies spectrum.

References

1. Leontiev A. F., *Series of exponential functions*, Nauka publishers, Moscow, 1976.
2. Sharipov R. A., *Clusters of exponential functions in the space of square integrable functions*, e-print arXiv:1410.7202 in Electronic Archive http://arXiv.org.

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