ASYMPTOTIC ESTIMATES FOR ROOTS OF THE CUBOID CHARACTERISTIC EQUATION IN THE NONLINEAR REGION.

Ruslan Sharipov

Abstract. A perfect cuboid is a rectangular parallelepiped. Its edges, its face diagonals, and its space diagonal are of integer lengths. None of such cuboids is known thus far, though the system of Diophantine equations describing them is easily written. The cuboid characteristic equation is a twelfth degree Diophantine equation derived from the initial cuboid equations and equivalent to them. In the case of the second cuboid conjecture it reduces to a tenth degree equation. This equation comprises two parameters. Previously various asymptotics for roots of this equation were studied as its parameters tend to infinity either separately or simultaneously provided some linear combination of them is preserved finite. In the present paper this linear combination is replaced by a certain nonlinear expression.

1. Introduction.

Omitting details, which can be found in [1–3], let's proceed to the tenth degree cuboid characteristic equation arising in the case of the second cuboid conjecture:

$$Q_{pq}(t)=0. \tag{1.1}$$

The tenth degree polynomial $Q_{pq}(t)$ in (1.1) is given by the explicit formula

$$\begin{aligned}Q_{pq}(t)&=t^{10}+(2\,q^2+p^2)\,(3\,q^2-2\,p^2)\,t^8+(q^8+10\,p^2\,q^6+\\&+4\,p^4\,q^4-14\,p^6\,q^2+p^8)\,t^6-p^2\,q^2\,(q^8-14\,p^2\,q^6+4\,p^4\,q^4+\\&+10\,p^6\,q^2+p^8)\,t^4-p^6\,q^6\,(q^2+2\,p^2)\,(3\,p^2-2\,q^2)\,t^2-q^{10}\,p^{10}.\end{aligned}\tag{1.2}$$

More details concerning the polynomial (1.2) and its background can be found in [4–8]. For the history and various approaches to the problem of perfect cuboids the reader is referred to [9–55]. The papers [56–68] constitute a separate stream using a symmetry approach to the perfect cuboid equations. This approach is different from the approach of the present paper. Therefore we do not consider the results of the papers [56–68] below.

The tenth degree polynomial (1.2) is related to the perfect cuboid problem through the following theorem (see Theorem 8.1 in [1] or in [2]).

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Theorem 1.1. *A triple of positive integer numbers p, q, and t satisfying the equation (1.1) and such that $p \neq q$ are coprime produces a perfect cuboid if and only if the following inequalities are fulfilled:*

$$t > p^2, \qquad t > p\,q, \qquad t > q^2, \qquad (p^2 + t)\,(p\,q + t) > 2\,t^2.$$

The mechanism associating the numbers p, q, t with perfect cuboids was found in [4] and [5]. It is described in brief in [1], [2], and [3]. We shall not reproduce this description in the present paper.

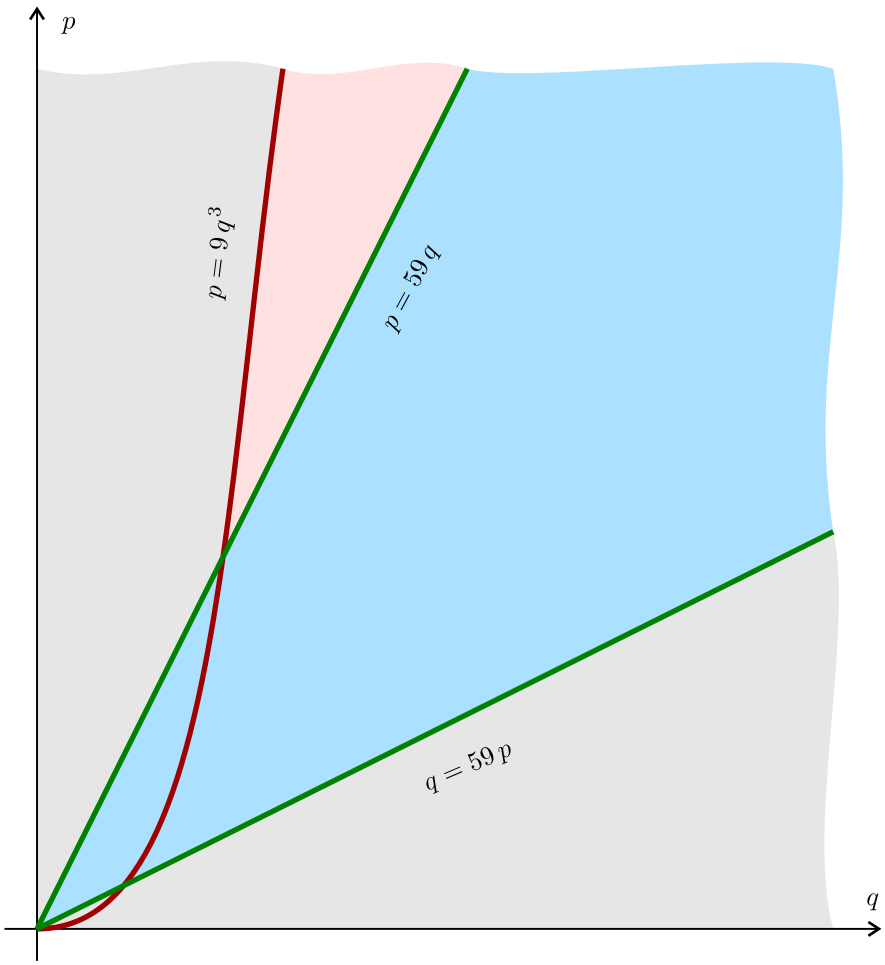

Fig. 1.1

Relying on Theorem 1.1, in [1], [2] two forms of asymptotics for the roots of the tenth degree polynomial equation (1.1) were studied:

$$\begin{aligned} &p = \text{const}, && q \to +\infty, \\ &q = \text{const}, && p \to +\infty. \end{aligned} \tag{1.3}$$

As a result of studying limits (1.3) in paper [2] the following three regions in the positive quadrant of the $p\,q$ - coordinate plane were defined:

1) **linear region** given by the linear inequalities

$$\frac{q}{59} < p, \qquad\qquad p < 59\,q; \tag{1.4}$$

2) **nonlinear region** given by the nonlinear inequalities

$$59\,q \leqslant p, \qquad\qquad p \leqslant 9\,q^3; \tag{1.5}$$

3) **no cuboid region** which is the rest of the positive $p\,q$ - quadrant.

These regions are schematically shown in Fig. 1.1 above. The linear region (1.4) is shown in sky blue. The nonlinear region (1.5) is shown in faded pink. And finally, the no cuboid region is shown in gray.

The linear region was studied in [3]. As a result a narrow strip surrounding the bisector line $p = q$ was cut off from the linear region. It is given by the inequalities

$$q - \frac{q}{97} \leqslant p, \qquad\qquad p \leqslant q + \min\Bigl(\frac{q}{97}, \sqrt[3]{\frac{q}{74}}\Bigr). \tag{1.6}$$

The strip (1.6) was annexed to the no cuboid region. Unfortunately the rest of the linear region (1.4) still remains uncertain. Perfect cuboids can potentially be found in it, but none of them is actually found. The main goal of the present paper is to study the nonlinear region (1.5).

2. Nonlinear transformation of parameters.

The upper boundary of the nonlinear region is given by the cubic parabola $p = 9\,q^3$ (see (1.5) and Fig. 1.1 above). For this reason we consider the following cubic transformation of the parameters p and q:

$$\tilde{p} = B\,q^3 - p, \qquad\qquad \tilde{q} = q. \tag{2.1}$$

Here B is some positive integer number. The transformation (2.1) is invertible:

$$p = B\,\tilde{q}^3 - \tilde{p}, \qquad\qquad q = \tilde{q}. \tag{2.2}$$

The transformations (2.1) and the transformation (2.2) map the integer $p\,q$ - grid onto the integer $\tilde{p}\,\tilde{q}$ - grid and vice versa.

Note that the curve given by the condition $\tilde{p} = \text{const}$ is a cubic parabola. In particular, if $B = 9$ and $\tilde{p} = 0$, it coincides with the upper boundary of the nonlinear region in Fig. 1.1. For this reason, instead of (1.3), we set

$$\tilde{p} = \text{const}, \qquad\qquad \tilde{q} \to +\infty \tag{2.3}$$

In order to study the limit (2.3) let's substitute (2.2) into the polynomial (1.2). As a result we get another polynomial $Q_{\tilde{p}\tilde{q}}(t)$. This polynomial is given by an explicit formula. However, the formula for the polynomial $Q_{\tilde{p}\tilde{q}}(t)$ is rather huge.

In the fully expanded form it has 108 summands. For this reason it is placed to the ancillary file strategy_formulas_04.txt in a machine-readable form.

Using the polynomial $Q_{\tilde{p}\tilde{q}}(t)$, one can write an equation similar to (1.1):

$$Q_{\tilde{p}\tilde{q}}(t) = 0. \tag{2.4}$$

It is clear that the equation (2.4) has the same roots as the equation (1.1), though they are expressed through different parameters. Like $Q_{pq}(t)$, the polynomial $Q_{\tilde{p}\tilde{q}}(t)$ in (2.4) is even with respect to t. Along with each root t it has the opposite root $-t$. Therefore we use the condition

$$\begin{cases} t > 0 \ \text{ if } \ t \ \text{ is a real root,} \\ \operatorname{Im}(t) > 0 \ \text{ if } \ t \ \text{ is a complex root,} \end{cases} \tag{2.5}$$

in order to divide the roots of the equation (2.4) into two groups. We denote through t_1, t_2, t_3, t_4, t_5 the roots that obey the conditions (2.5). Then t_6, t_7, t_8, t_9, t_{10} are opposite roots of the equation (2.4):

$$t_6 = -t_1, \qquad t_7 = -t_2, \qquad t_8 = -t_3, \qquad t_9 = -t_4, \qquad t_{10} = -t_5.$$

3. Parabolic expansions.

Typically, asymptotic expansions for roots of a polynomial equation look like power series (see [69]). By analogy to (2.3) in [1] and according to (2.3), we write

$$t_i(\tilde{p},\tilde{q}) = C_i\,\tilde{q}^{\,\alpha_i}\left(1 + \sum_{s=1}^{\infty}\beta_{is}\,\tilde{q}^{-s}\right) \ \text{ as } \ \tilde{q} \to +\infty. \tag{3.1}$$

The coefficients C_i in (3.1) should be nonzero: $C_i \neq 0$. The expansions (3.1) here are called parabolic expansions since they occur along cubic parabolas in the plane of the original parameters p and q.

The main tool for studying asymptotic expansions of the form (3.1) for roots of a polynomial equation is the Newton polygon of the corresponding polynomial.

Definition 3.1. For any polynomial of two variables $P(t,q)$ the convex hull of all integer nodes (m,r) on the coordinate plane associated with nonzero monomials $A_{m\,r}\,q^r\,t^m$ of this polynomial is called the Newton polygon of $P(t,q)$.

Remark. Note that in our case the polynomial $Q_{\tilde{p}\tilde{q}}(t)$ depends on three variables $\tilde{p}$, $\tilde{q}$, and $\tilde{t}$. However, we treat $\tilde{p}$ as a parameter and consider $Q_{\tilde{p}\tilde{q}}(t)$ as a polynomial of two variables when applying Definition 3.1 to it.

The Newton polygon of $Q_{\tilde{p}\tilde{q}}(t)$ is shown in Fig. 3.1 below. Its boundary consists of three parts — the upper part, the lower part, and the vertical part. The upper part is drawn in green, the lower part is drawn in red. Here are the coefficients of monomials associated with the nodes on the upper part of the boundary:

$$\begin{aligned} &A_{0\,40} = -B^{10}, && A_{2\,36} = -6\,B^{10}, && A_{4\,32} = -B^{10}, \\ &A_{6\,24} = B^{8}, && A_{8\,12} = -2\,B^{4}, && A_{10\,0} = 1. \end{aligned} \tag{3.2}$$

Theorem 3.1. *The values of exponents α_i in the expansion (3.1) for roots of the* *equation* (2.4) *are determined according to the formula* $\alpha_i = -k$*, where* k *stands for slopes of segments of the polygonal line being the upper boundary of the Newton polygon in Fig. 3.1.*

Theorem 3.1 is a standard fact in Newton polygons application to studying bivariate polynomials. Its proof was given in [1] for the sake of reader's convenience.

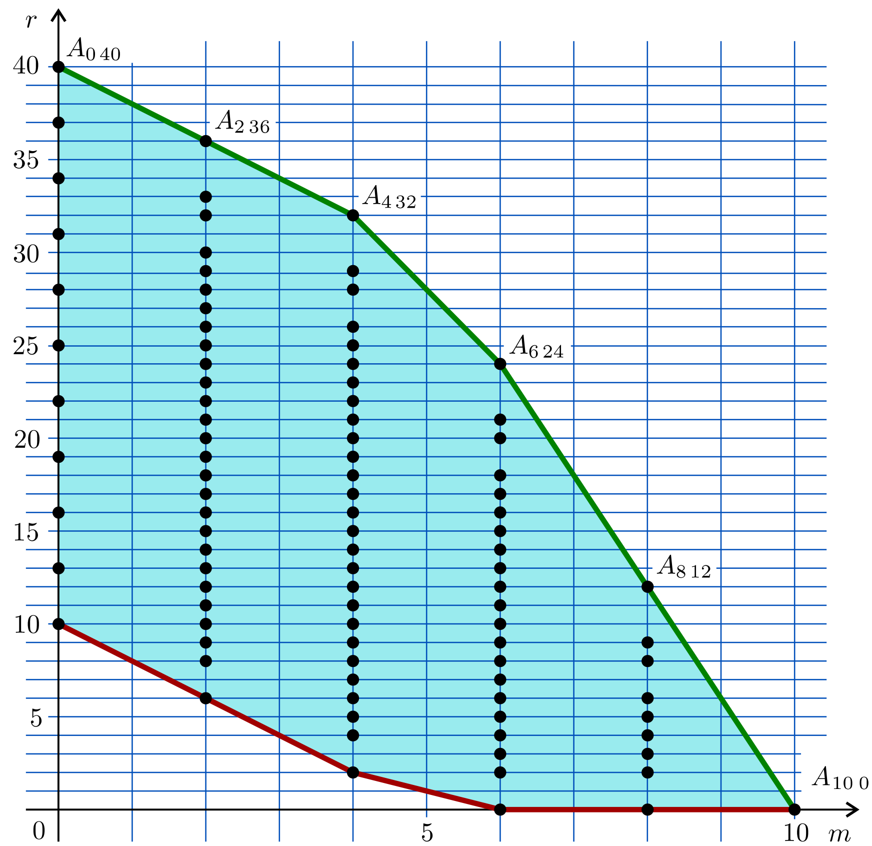

Fig. 3.1

In our particular case, applying Theorem 3.1, we get

$$\alpha_i = 2, \qquad \alpha_i = 4, \qquad \alpha_i = 6. \tag{3.3}$$

The options (3.3) determine the growth rates for the roots of the equation (2.4) as $\tilde{q} \to +\infty$. They grow as the second, the fourth, and the sixth powers of $\tilde{q}$.

The case $\alpha_i = 2$. This case corresponds to the topmost segment on the upper boundary of the Newton polygon in Fig. 3.1. This segment comprises three nodes $A_{0\,40}$, $A_{2\,36}$, and $A_{4\,32}$. Therefore, substituting the expansion (3.1) with $\alpha_i = 2$ into the equation (2.4), we get the following equation for C_i:

$$A_{4\,32}\, {C_i}^4 + A_{2\,36}\, {C_i}^2 + A_{0\,40} = 0. \tag{3.4}$$

Taking into account (3.2), the equation (3.4) is transformed to

$$B^{10}\,{C_i}^4+6\,B^{10}\,{C_i}^2+B^{10}=0. \tag{3.5}$$

Since $B\neq 0$ in (2.1) and (2.2), we can cancel B^{10} in (3.5) and write (3.5) as

$${C_i}^4+6\,{C_i}^2+1=0. \tag{3.6}$$

The equation (3.6) has four purely complex roots:

$$C_i=(\sqrt{2}+1)\,\mathrm{i}, \qquad\qquad C_i=(\sqrt{2}-1)\,\mathrm{i}, \tag{3.7}$$

$$C_i=-(\sqrt{2}+1)\,\mathrm{i}, \qquad\qquad C_i=-(\sqrt{2}-1)\,\mathrm{i}. \tag{3.8}$$

Here $\mathrm{i}=\sqrt{-1}$. The roots (3.8) are excluded by the condition (2.5). The remain is two root (3.7) of multiplicity 1. They yield the following asymptotic expansions:

$$\begin{aligned}
&t_i(\tilde{p},\tilde{q})=(\sqrt{2}+1)\,\mathrm{i}\,\tilde{q}^2\left(1+\sum_{s=1}^{\infty}\beta_{is}\,\tilde{q}^{\,-s}\right),\\
&t_i(\tilde{p},\tilde{q})=(\sqrt{2}-1)\,\mathrm{i}\,\tilde{q}^2\left(1+\sum_{s=1}^{\infty}\beta_{is}\,\tilde{q}^{\,-s}\right).
\end{aligned} \tag{3.9}$$

The case $\alpha_i=4$. This case corresponds to the middle segment in the upper boundary of the Newton polygon in Fig. 3.1. It comprises two nodes $A_{4\,32}$ and $A_{6\,24}$. Therefore, substituting the expansion (3.1) with $\alpha_i=4$ into the equation (2.4), we get the following equation for C_i:

$$A_{6\,24}\,{C_i}^6+A_{4\,32}\,{C_i}^4=0. \tag{3.10}$$

Applying (3.2), the equation (3.10) is transformed to

$$B^{8}\,{C_i}^6-B^{10}\,{C_i}^4=0. \tag{3.11}$$

Since $C_i\neq 0$ in (3.1) and since $B\neq 0$ in (2.1) and (2.2), we can cancel the common divisor $B^{8}\,{C_i}^4$ of two terms in (3.11). As a result (3.11) takes the form

$${C_i}^2-B^{2}=0. \tag{3.12}$$

The equation (3.12) has two real roots, which are simple:

$$C_i=B, \qquad\qquad C_i=-B. \tag{3.13}$$

The second root (3.13) is excluded by the condition (2.5). The remain is one simple positive root. It yields the following asymptotic expansion:

$$t_i(\tilde{p},\tilde{q})=B\,\tilde{q}^4\left(1+\sum_{s=1}^{\infty}\beta_{is}\,\tilde{q}^{\,-s}\right). \tag{3.14}$$

The case $\alpha_i = 6$. This case corresponds to the lowermost segment in the upper boundary of the Newton polygon in Fig. 3.1. It comprises three nodes $A_{6\,24}$, $A_{8\,12}$, and $A_{10\,0}$. Therefore, substituting the expansion (3.1) with $\alpha_i = 6$ into the equation (2.4), we get the following equation for C_i:

$$A_{10\,0}\,C_i^{\,10} + A_{8\,12}\,C_i^{\,8} + A_{6\,24}\,C_i^{\,6} = 0. \tag{3.15}$$

Applying the formulas (3.2), the equation (3.15) is transformed to

$$C_i^{\,10} - 2\,B^4\,C_i^{\,8} + B^8\,C_i^{\,6} = 0. \tag{3.16}$$

Since $C_i \neq 0$ in (3.1), we can cancel the common divisor $C_i^{\,6}$ of the three terms in (3.16). As a result the equation (3.16) takes the form

$$C_i^{\,4} - 2\,B^4\,C_i^{\,2} + B^8 = 0. \tag{3.17}$$

The equation (3.17) has two real roots of multiplicity 2:

$$C_i = B^2, \qquad\qquad C_i = -B^2. \tag{3.18}$$

The second root (3.18) is excluded by the condition (2.5). The remain is one simple positive root. It yields the following asymptotic expansion:

$$t_i(\tilde{p},\tilde{q}) = B^2\,\tilde{q}^{\,6}\biggl(1 + \sum_{s=1}^{\infty}\beta_{is}\,\tilde{q}^{\,-s}\biggr). \tag{3.19}$$

The results (3.9), (3.14), (3.19) are summed up in the following theorem.

Theorem 3.2. *For sufficiently large positive values of the parameter* $\tilde{q}$*, i. e. for* $\tilde{q} > \tilde{q}_{\text{min}}$*, the tenth-degree equation* (2.4) *has five roots of multiplicity one satisfying the condition* (2.5)*. Three of them* t_1*,* t_2*, and* t_3 *are real roots. Their asymptotics as* $\tilde{q} \to +\infty$ *are given by the formulas*

$$t_1 \sim B^2\,\tilde{q}^{\,6}, \qquad\qquad t_2 \sim B^2\,\tilde{q}^{\,6}, \qquad\qquad t_3 \sim B\,\tilde{q}^{\,4}. \tag{3.20}$$

The rest two roots t_4 *and* t_5 *of the equation* (2.4) *are complex. Their asymptotics as* $\tilde{q} \to +\infty$ *are given by the formulas*

$$t_4 \sim (\sqrt{2} + 1)\,\mathrm{i}\,\tilde{q}^2, \qquad\qquad t_5 \sim (\sqrt{2} - 1)\,\mathrm{i}\,\tilde{q}^2. \tag{3.21}$$

The complex roots (3.21) do not provide perfect cuboids. However, they are important for determining the exact number of real roots in asymptotic intervals.

4. Asymptotic estimates for real roots.

Acting by analogy to [1], we replace the series expansion of the form (3.1) by finite sum expansions with remainder terms. In the case of the fast growing root t_1 we replace the expansion (3.19) by the following sum:

$$\begin{aligned} t_1 = B^2\,\tilde{q}^{\,6} + 2\,B\,\tilde{q}^4 - 2\,B\,\tilde{p}\,\tilde{q}^3 - 2\,\tilde{q}^2 -\\ - 2\,\tilde{p}\,\tilde{q} + \tilde{p}^2 + \frac{5}{B} - \frac{20}{B^2\,\tilde{q}^2} + R_1(\tilde{p},\tilde{q},B). \end{aligned} \tag{4.1}$$

The formula (4.1) is in agreement with (3.19). It means that

$$\beta_{11}=0,\qquad \beta_{12}=\frac{2}{B},\qquad \beta_{13}=-\frac{2\,\tilde p}{B},\qquad \beta_{14}=-\frac{2}{B^2},$$
$$\beta_{15}=-\frac{2\,\tilde p}{B^2},\qquad \beta_{16}=\frac{\tilde p^2}{B^2}+\frac{5}{B^3},\qquad \beta_{17}=0\qquad \beta_{18}=-\frac{20}{B^4}$$

in the formula (3.19). Our goal is to derive an estimate of the form

$$|R_1(\tilde p,\tilde q,B)|<\frac{C}{\tilde q^3} \tag{4.2}$$

for the remainder term in (4.1). In order to get such an estimate we substitute

$$t=B^2\,\tilde q^6+2\,B\,\tilde q^4-2\,B\,\tilde p\,\tilde q^3-2\,\tilde q^2-2\,\tilde p\,\tilde q+\tilde p^2+\frac{5}{B}-\frac{20}{B^2\,\tilde q^2}+\frac{c}{\tilde q^3}. \tag{4.3}$$

into the equation (2.4). Immediately after that we perform another substitution into the equation obtained by substituting (4.3) into (2.4):

$$\tilde q=\frac{1}{z}. \tag{4.4}$$

Upon two substitutions (4.3) and (4.4) and upon removing denominators the equation (2.4) is written as a polynomial equation in the new variables c and z:

$$16\,c\,B^{37}=80\,\tilde p\,B^{35}+\varphi(c,z,\tilde p,B). \tag{4.5}$$

Here $\varphi(c,z,\tilde p,B)$ is a polynomial of c, z, $\tilde p$, and B with integer coefficients. The explicit expression for $\varphi(c,z,\tilde p,B)$ comprises 1612 monomials. Therefore it is placed to the ancillary file `strategy_formulas_04.txt` in a machine-readable form.

Let's recall that B is a positive integer constant. If $B\geqslant 10$, then the curve $\tilde p=\text{const}$ goes to infinity outside the nonlinear region (see (2.1), (2.3), and Fig. 1.1). Therefore B takes the following finite set of values:

$$B=1,\,2,\,3,\,\ldots\,,\,9. \tag{4.6}$$

For each B in (4.6) assume that c obeys the condition

$$\begin{aligned}&-\frac{10\,|\tilde p|}{B^2}<c<0\quad\text{if}\quad \tilde p<0,\\ &0<c<\frac{10\,|\tilde p|}{B^2}\quad\text{if}\quad \tilde p>0.\end{aligned} \tag{4.7}$$

The case $\tilde p=0$ is exceptional. It should be studied separately.

Since $\tilde p$ is integer and since $\tilde p\neq 0$ in our present case, we have the inequality

$$|\tilde p|\geqslant 1. \tag{4.8}$$

Since we study the asymptotics of the roots t_i as $\tilde q\to+\infty$, we assume that

$$\tilde q\geqslant 20\,\sqrt[3]{|\tilde p|}. \tag{4.9}$$

Let's apply (4.9) to (4.4). As a result we derive the inequality $|z| \leqslant 1/20\,|\tilde{p}|^{-1/3}$. Applying this inequality along with the inequalities (4.7) and (4.8) to the polynomial $\varphi(c, z, \tilde{p}, B)$, we derive the following estimate for it:

$$|\varphi(c, z, \tilde{p}, B)| \leqslant 72\,|\tilde{p}|\,B^{35}. \tag{4.10}$$

The inequality (4.10) holds for each value of B in (4.6).

Now we apply the inequality (4.10) to the equation (4.5). If $\tilde{p} < 0$, it means that the right hand side of the equation (4.5) is a continuous function of c that varies from $-152\,|\tilde{p}|\,B^{35}$ to $-8\,|\tilde{p}|\,B^{35}$ while c runs over the negative interval (4.7). As for the left hand side of this equation, it is also a continuous function of c that monotonically increases from $-160\,|\tilde{p}|\,B^{35}$ to 0 while c runs over this interval. Therefore the equation (4.5) has at least one root within the negative interval (4.7).

Similarly, if $\tilde{p} > 0$, the right hand side of the equation (4.5) varies from $8\,|\tilde{p}|\,B^{35}$ to $152\,|\tilde{p}|\,B^{35}$ while c runs over the positive interval (4.7) and hence the equation (4.5) has at least one root within this interval.

The variable c is related to the original variable t through the formula (4.3). Therefore from the above considerations we derive the following inequalities for t:

$$\begin{aligned} &B^2\,\tilde{q}^6 + 2\,B\,\tilde{q}^4 - 2\,B\,\tilde{p}\,\tilde{q}^3 - 2\,\tilde{q}^2 - 2\,\tilde{p}\,\tilde{q} + \tilde{p}^2 + \frac{5}{B} - \\ &-\frac{20}{B^2\,\tilde{q}^2} + \frac{10\,\tilde{p}}{B^2\,\tilde{q}^3} < t < B^2\,\tilde{q}^6 + 2\,B\,\tilde{q}^4 - 2\,B\,\tilde{p}\,\tilde{q}^3 - \\ &-2\,\tilde{q}^2 - 2\,\tilde{p}\,\tilde{q} + \tilde{p}^2 + \frac{5}{B} - \frac{20}{B^2\,\tilde{q}^2} \quad \text{in the case} \quad \tilde{p} < 0, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &B^2\,\tilde{q}^6 + 2\,B\,\tilde{q}^4 - 2\,B\,\tilde{p}\,\tilde{q}^3 - 2\,\tilde{q}^2 - 2\,\tilde{p}\,\tilde{q} + \tilde{p}^2 + \frac{5}{B} - \\ &-\frac{20}{B^2\,\tilde{q}^2} < t < B^2\,\tilde{q}^6 + 2\,B\,\tilde{q}^4 - 2\,B\,\tilde{p}\,\tilde{q}^3 - 2\,\tilde{q}^2 - \\ &-2\,\tilde{p}\,\tilde{q} + \tilde{p}^2 + \frac{5}{B} - \frac{20}{B^2\,\tilde{q}^2} + \frac{10\,\tilde{p}}{B^2\,\tilde{q}^3} \quad \text{in the case} \quad \tilde{p} > 0. \end{aligned} \tag{4.12}$$

As a result we have proved the following theorem.

Theorem 4.1. *If $\tilde{p} \neq 0$, then for each $\tilde{q} \geqslant 20\,\sqrt[3]{|\tilde{p}|}$ and for each value of B in (4.6) there is at least one real root of the equation (2.4) satisfying the inequalities (4.11) or (4.12) respectively.*

Theorem 4.1 means that we have derived the estimate (4.2) with $C = 10\,|\tilde{p}|/B^2$ for the remainder term $R_1(\tilde{p}, \tilde{q}, B)$ in the asymptotic expansion (4.1) for at least one root of the equation (2.4).

The root t_2 in (3.20) is handled in a similar way. The asymptotic expansion analogous to the expansion (4.1) for it is written as follows:

$$\begin{aligned} t_2 = B^2\,\tilde{q}^6 - 2\,B\,\tilde{q}^4 - 2\,B\,\tilde{p}\,\tilde{q}^3 - 2\,\tilde{q}^2 + &\\ + 2\,\tilde{p}\,\tilde{q} + \tilde{p}^2 - \frac{5}{B} - \frac{20}{B^2\,\tilde{q}^2} + R_2(\tilde{p}, \tilde{q}, B).& \end{aligned} \tag{4.13}$$

The formula (4.13) is in agreement with (3.19). It means that

$$\beta_{21} = 0, \qquad \beta_{22} = -\frac{2}{B}, \qquad \beta_{23} = -\frac{2\,\tilde{p}}{B}, \qquad \beta_{24} = -\frac{2}{B^2},$$

$$\beta_{25}=\frac{2\,\tilde p}{B^{\,2}},\qquad \beta_{26}=\frac{\tilde p^{\,2}}{B^{\,2}}-\frac{5}{B^{\,3}},\qquad \beta_{17}=0\qquad \beta_{18}=-\frac{20}{B^{\,4}}$$

The formula (4.3) in this case is replaced by the following one:

$$t=B^{\,2}\,\tilde q^{\,6}-2\,B\,\tilde q^{\,4}-2\,B\,\tilde p\,\tilde q^{\,3}-2\,\tilde q^{\,2}+2\,\tilde p\,\tilde q+\tilde p^{\,2}-\frac{5}{B}-\frac{20}{B^{\,2}\,\tilde q^{\,2}}+\frac{c}{\tilde q^{\,3}}.\tag{4.14}$$

Upon substituting (4.14) and (4.4) into the equation (2.4) and upon removing denominators the equation (2.4) is written as an equation very similar to (4.5):

$$16\,c\,B^{\,37}=-80\,\tilde p\,B^{\,35}+\psi(c,z,\tilde p,B).\tag{4.15}$$

Here $\psi(c,z,\tilde p,B)$ is a polynomial of c, z, $\tilde p$, and B with integer coefficients. The explicit expression for $\psi(c,z,\tilde p,B)$ comprises 1612 monomials. Therefore it is placed to the ancillary file strategy_formulas_04.txt in a machine-readable form.

Like in the previous case, for each B in (4.6) assume that c obeys the inequalities

$$\begin{aligned}&0<c<\frac{10\,|\tilde p|}{B^{\,2}}\ \text{ if }\ \tilde p<0,\\ &-\frac{10\,|\tilde p|}{B^{\,2}}<c<0\ \text{ if }\ \tilde p>0.\end{aligned}\tag{4.16}$$

Then assume that $\tilde q$ obeys the inequality (4.9). Under these assumptions, taking into account (4.8), one can derive an estimate similar to the estimate (4.10):

$$|\psi(c,z,\tilde p,B)|\leqslant 72\,|\tilde p|\,B^{\,35}.\tag{4.17}$$

Now, writing the inequalities

$$\begin{aligned}&B^{\,2}\,\tilde q^{\,6}-2\,B\,\tilde q^{\,4}-2\,B\,\tilde p\,\tilde q^{\,3}-2\,\tilde q^{\,2}+2\,\tilde p\,\tilde q+\tilde p^{\,2}-\frac{5}{B}-\\ &-\frac{20}{B^{\,2}\,\tilde q^{\,2}}<t<B^{\,2}\,\tilde q^{\,6}-2\,B\,\tilde q^{\,4}-2\,B\,\tilde p\,\tilde q^{\,3}-2\,\tilde q^{\,2}+\\ &+2\,\tilde p\,\tilde q+\tilde p^{\,2}-\frac{5}{B}-\frac{20}{B^{\,2}\,\tilde q^{\,2}}-\frac{10\,\tilde p}{B^{\,2}\,\tilde q^{\,3}}\quad\text{in the case}\quad \tilde p<0,\end{aligned}\tag{4.18}$$

$$\begin{aligned}&B^{\,2}\,\tilde q^{\,6}-2\,B\,\tilde q^{\,4}-2\,B\,\tilde p\,\tilde q^{\,3}-2\,\tilde q^{\,2}+2\,\tilde p\,\tilde q+\tilde p^{\,2}-\frac{5}{B}-\\ &-\frac{20}{B^{\,2}\,\tilde q^{\,2}}-\frac{10\,\tilde p}{B^{\,2}\,\tilde q^{\,3}}<t<B^{\,2}\,\tilde q^{\,6}-2\,B\,\tilde q^{\,4}-2\,B\,\tilde p\,\tilde q^{\,3}-\\ &-2\,\tilde q^{\,2}+2\,\tilde p\,\tilde q+\tilde p^{\,2}-\frac{5}{B}-\frac{20}{B^{\,2}\,\tilde q^{\,2}}\quad\text{in the case}\quad \tilde p>0,\end{aligned}\tag{4.19}$$

and then applying the estimate (4.17) to the equation (4.15) and taking into account (4.16), we can easily prove the following theorem.

Theorem 4.2. *If $\tilde p\neq 0$, then for each $\tilde q\geqslant 20\,\sqrt[3]{|\tilde p|}$ and for each value of B in (4.6) there is at least one real root of the equation (2.4) satisfying the inequalities (4.18) or (4.19) respectively.*

Theorem 4.2 is similar to Theorem 4.1. It means that we have got the estimate

$$|R_2(\tilde{p},\tilde{q},B)| < \frac{C}{\tilde{q}^3}, \text{ where } C = \frac{10\,|\tilde{p}|}{B^2}, \tag{4.20}$$

for the remainder term $R_2(\tilde{p},\tilde{q},B)$ of the asymptotic expansion (4.13).

The root t_3 in (3.20) is somewhat different from t_1 and t_2. The analog of the asymptotic expansions (4.1) and (4.13) for this root is written as

$$t_3 = B\,\tilde{q}^4 - \tilde{p}\,\tilde{q} + \frac{16}{B} + R_3(\tilde{p},\tilde{q},B). \tag{4.21}$$

Like in (4.2) and (4.20), in this case we shall derive the inverse cubic estimate

$$|R_3(\tilde{p},\tilde{q},B)| < \frac{C}{\tilde{q}^3} \tag{4.22}$$

for the remainder term in (4.21). For this purpose we substitute

$$t = B\,\tilde{q}^4 - \tilde{p}\,\tilde{q} + \frac{16}{B} + \frac{c}{\tilde{q}^3} \tag{4.23}$$

into the equation (2.4). Immediately after that we perform the substitution (4.4) into the equation obtained by substituting (4.23) into (2.4). Upon removing denominators, the resulting equation can be written in the following form:

$$2\,c\,B^{23} = 32\,\tilde{p}\,B^{21} + f(c,z,\tilde{p},B). \tag{4.24}$$

Here $f(c,z,\tilde{p},B)$ is a polynomial of c, z, $\tilde{p}$, and B with integer coefficients. The explicit expression for $f(c,z,\tilde{p},B)$ comprises 490 monomials. Therefore it is placed to the ancillary file `strategy_formulas_04.txt` in a machine-readable form.

For each B in (4.6) assume that c obeys the condition

$$\begin{aligned} -\frac{32\,|\tilde{p}|}{B^2} < c < 0 \quad &\text{if} \quad \tilde{p} < 0,\\ 0 < c < \frac{32\,|\tilde{p}|}{B^2} \quad &\text{if} \quad \tilde{p} > 0. \end{aligned} \tag{4.25}$$

Again, the case $\tilde{p} = 0$ is exceptional. It should be studied separately.

Like in (4.9), assume that $\tilde{q}$ obeys the inequality

$$\tilde{q} \geqslant 7\,\sqrt[3]{|\tilde{p}|}. \tag{4.26}$$

Let's apply (4.26) to (4.4). As a result we derive the inequality $|z| \leqslant 1/7\,|\tilde{p}|^{-1/3}$. Applying this inequality along with the inequalities (4.25) and (4.8) to the polynomial $f(c,z,\tilde{p},B)$ in (4.24), we derive the following estimate for it:

$$|f(c,z,\tilde{p},B)| \leqslant 26\,|\tilde{p}|\,B^{21}. \tag{4.27}$$

The inequality (4.27) holds for each value of B in (4.6).

Now we apply the inequality (4.27) to the equation (4.24). If $\tilde p < 0$, it means that the right hand side of the equation (4.24) is a continuous function of c that varies from $-58\,|\tilde p|\,B^{35}$ to $-6\,|\tilde p|\,B^{35}$ while c runs over the negative interval (4.25). As for the left hand side of this equation, it is also a continuous function of c that monotonically increases from $-64\,|\tilde p|\,B^{35}$ to 0 while c runs over this interval. Therefore the equation (4.24) has at least one root within the negative interval (4.25). Similarly, if $\tilde p > 0$, the right hand side of the equation (4.24) varies from $6\,|\tilde p|\,B^{35}$ to $58\,|\tilde p|\,B^{35}$ while c runs over the positive interval (4.25) and hence the equation (4.24) has at least one root within this interval.

The variable c is related to the original variable t through the formula (4.23). Therefore from the above considerations we derive the following inequalities for t:

$$B\,\tilde q^4 - \tilde p\,\tilde q + \frac{16}{B} + \frac{32\,\tilde p}{B^2\,\tilde q^3} < t < B\,\tilde q^4 - \tilde p\,\tilde q + \frac{16}{B} \quad \text{if} \quad \tilde p < 0, \tag{4.28}$$

$$B\,\tilde q^4 - \tilde p\,\tilde q + \frac{16}{B} < t < B\,\tilde q^4 - \tilde p\,\tilde q + \frac{16}{B} + \frac{32\,\tilde p}{B^2\,\tilde q^3} \quad \text{if} \quad \tilde p > 0. \tag{4.29}$$

As a result we have proved the following theorem.

Theorem 4.3. *If $\tilde p \neq 0$, then for each $\tilde q \geqslant 7\,\sqrt[3]{|\tilde p|}$ and for each value of B in (4.6) there is at least one real root of the equation (2.4) satisfying the inequalities (4.28) or (4.29) respectively.*

Theorem 4.3 means that we have derived the estimate (4.22) with $C = 32\,|\tilde p|/B^2$ for the remainder term $R_3(\tilde p, \tilde q, B)$ in the asymptotic expansion (4.21) for at least one root of the equation (2.4).

5. Asymptotic estimates for complex roots.

There are two complex roots t_4 and t_5 of the equation (2.4) satisfying the condition (2.5). The leading terms of their asymptotics as $\tilde q \to +\infty$ are given by the formulas (3.21). Specifying the first formula (3.21), we write

$$t_4 = (\sqrt{2} + 1)\,\mathrm{i}\,\tilde q^2 - \frac{(10 + 7\,\sqrt{2}\,)\,\mathrm{i}}{B^2\,\tilde q^2} + R_4(\tilde p, \tilde q, B). \tag{5.1}$$

Here $\mathrm{i} = \sqrt{-1}$. Our goal is to derive an estimate of the form

$$|R_4(\tilde p, \tilde q, B)| < \frac{C}{\tilde q^5} \tag{5.2}$$

for the remainder term in (5.1). In order to get such an estimate we substitute

$$t = (\sqrt{2} + 1)\,\mathrm{i}\,\tilde q^2 - \frac{(10 + 7\,\sqrt{2}\,)\,\mathrm{i}}{B^2\,\tilde q^2} + \frac{c\,\mathrm{i}}{\tilde q^5} \tag{5.3}$$

into the equation (2.4). Immediately after that we perform the substitution (4.4) into the equation obtained by substituting (5.3) into (2.4). Upon removing denominators, the resulting equation can be written in the following form:

$$16\,c\,B^{30} = -32\,(10 + 7\,\sqrt{2})\,\tilde p\,B^{27} + \eta(c, z, \tilde p, B). \tag{5.4}$$

Here $\eta(c,z,\tilde p,B)$ is a polynomial of c, z, $\tilde p$, and B. The fully expanded expression for $\eta(c,z,\tilde p,B)$ comprises 1031 monomials. Therefore it is placed to the ancillary file `strategy_formulas_04.txt` in a machine-readable form.

Note that the value of the irrational coefficient in (5.4) obeys the inequalities

$$636<32\,(10+7\sqrt{2})\approx 636.78<640=16\cdot 40. \tag{5.5}$$

Therefore for each B in (4.6) assume that c obeys the condition

$$\begin{aligned} &0<c<\frac{80\,|\tilde p|}{B^{3}}\ \text{ if }\ \tilde p<0,\\ -&\frac{80\,|\tilde p|}{B^{3}}<c<0\ \text{ if }\ \tilde p>0. \end{aligned} \tag{5.6}$$

Like in (4.7), (4.16), and (4.25) above, the case $\tilde p=0$ is exceptional. It should be studied separately.

Assume that $\tilde q$ obeys the following inequality similar to (4.9) and (4.26):

$$\tilde q\geqslant 15\,\sqrt[3]{|\tilde p|}. \tag{5.7}$$

Let's apply (5.7) to (4.4). As a result we derive the inequality $|z|\leqslant 1/15\,|\tilde p|^{-1/3}$. Applying this inequality along with the inequalities (5.5) and (4.8) to the polynomial $\eta(c,z,\tilde p,B)$ in (5.4), we derive the following estimate for it:

$$|\eta(c,z,\tilde p,B)|\leqslant 512\,|\tilde p|\,B^{27}. \tag{5.8}$$

Let's compare (5.8) with (5.5) and then apply the inequality (5.8) to the equation (5.4). If $\tilde p<0$, it means that the right hand side of the equation (5.4) is a continuous function of c that varies from $124\,|\tilde p|\,B^{27}$ to $1152\,|\tilde p|\,B^{27}$ while c runs over the positive interval (5.6). As for the left hand side of this equation, it is also a continuous function of c that monotonically increases from 0 to $1280\,|\tilde p|\,B^{27}$ while c runs over this interval. Therefore the equation (5.4) has at least one root within the positive interval (5.6). Similarly, if $\tilde p>0$, the right hand side of the equation (5.4) varies from $-1152\,|\tilde p|\,B^{27}$ to $-124\,|\tilde p|\,B^{27}$ while c runs over the negative interval (5.6) and hence the equation (5.4) has at least one root within this interval.

The variable c is related to the original variable t through the formula (5.3). Therefore from the above considerations we derive the following inequalities for t:

$$\begin{aligned} (\sqrt{2}+1)\,\tilde q^{2}-\frac{10+7\sqrt{2}}{B^{2}\,\tilde q^{2}}<\operatorname{Im}t<(\sqrt{2}+1)\,\tilde q^{2}-\\ -\frac{10+7\sqrt{2}}{B^{2}\,\tilde q^{2}}-\frac{80\,\tilde p}{B^{3}\,\tilde q^{5}}\quad\text{in the case}\quad \tilde p<0, \end{aligned} \tag{5.9}$$

$$\begin{aligned} (\sqrt{2}+1)\,\tilde q^{2}-\frac{10+7\sqrt{2}}{B^{2}\,\tilde q^{2}}-\frac{80\,\tilde p}{B^{3}\,\tilde q^{5}}<\operatorname{Im}t<(\sqrt{2}+1)\,\tilde q^{2}-\\ -\frac{10+7\sqrt{2}}{B^{2}\,\tilde q^{2}}\quad\text{in the case}\quad \tilde p>0. \end{aligned} \tag{5.10}$$

The inequalities (5.9) and (5.10) are analogs of the corresponding inequalities in the case of the real roots, see (4.11) and (4.12), (4.18) and (4.19), or (4.28) and

(4.29). They mean that we have proved the following theorem.

Theorem 5.1. *If $\tilde{p}\neq 0$, then for each $\tilde{q}\geqslant 15\,\sqrt[3]{|\tilde{p}|}$ and for each value of B in* (4.6) *there is at least one purely imaginary root of the equation* (2.4) *satisfying the inequalities* (5.9) *or* (5.10) *respectively.*

Theorem 5.1 means that we have derived the estimate (5.2) with $C=80\,|\tilde{p}|/B^{\,3}$ for the remainder term $R_4(\tilde{p},\tilde{q},B)$ in the asymptotic expansion (5.1) for at least one root of the equation (2.4).

The complex root t_5 is similar to the root t_4. For this root we write

$$t_5=(\sqrt{2}-1)\,\mathrm{i}\,\tilde{q}^2+\frac{(10-7\sqrt{2}\,)\,\mathrm{i}}{B^{\,2}\,\tilde{q}^2}+R_5(\tilde{p},\tilde{q},B). \tag{5.11}$$

Here $\mathrm{i}=\sqrt{-1}$. Like in (5.1), our goal is to derive an estimate of the form

$$|R_5(\tilde{p},\tilde{q},B)|<\frac{C}{\tilde{q}^5} \tag{5.12}$$

for the remainder term in (5.11). In order to get such an estimate we substitute

$$t=(\sqrt{2}-1)\,\mathrm{i}\,\tilde{q}^2+\frac{(10-7\sqrt{2}\,)\,\mathrm{i}}{B^{\,2}\,\tilde{q}^2}+\frac{c\,\mathrm{i}}{\tilde{q}^5} \tag{5.13}$$

into the equation (2.4). Immediately after that we perform the substitution (4.4) into the equation obtained by substituting (5.13) into (2.4). Upon removing denominators, the resulting equation can be written in the following form:

$$16\,c\,B^{\,30}=32\,(10-7\sqrt{2})\,\tilde{p}\,B^{\,27}+\zeta(c,z,\tilde{p},B). \tag{5.14}$$

Here $\zeta(c,z,\tilde{p},B)$ is a polynomial of c, z, $\tilde{p}$, and B. The fully expanded expression for $\zeta(c,z,\tilde{p},B)$ comprises 1031 monomials. Therefore it is placed to the ancillary file `strategy_formulas_04.txt` in a machine-readable form.

Note that the value of the irrational coefficient in (5.14) obeys the inequality

$$3<32\,(10-7\sqrt{2})\approx 3.22<4=16\cdot\frac{1}{4}. \tag{5.15}$$

The number in (5.15) is substantially smaller than the number in (5.5). Therefore, instead of (5.6) we write the following inequalities:

$$\begin{aligned}&-\frac{|\tilde{p}|}{2\,B^3}<c<0\quad\text{if}\quad\tilde{p}<0,\\&0<c<\frac{|\tilde{p}|}{2\,B^3}\quad\text{if}\quad\tilde{p}>0.\end{aligned} \tag{5.16}$$

The inequalities (5.16) should be fulfilled for each value of B in (4.6).

Assume that $\tilde{q}$ obeys the following inequality similar to (4.9), (4.26), and (5.7):

$$\tilde{q}\geqslant 3600\,\sqrt[3]{|\tilde{p}|}. \tag{5.17}$$

The coefficient 3600 in (5.17) is much greater than the coefficient 15 in (5.7). Applying (5.17) to (4.4), we derive $|z| \leqslant 1/3600\,|\tilde{p}|^{-1/3}$. Applying this inequality along with the inequalities (5.15) and (4.8) to the polynomial $\zeta(c, z, \tilde{p}, B)$ in the equation (5.14), we derive the following estimate for it:

$$|\zeta(c, z, \tilde{p}, B)| \leqslant 2\,|\tilde{p}|\,B^{27}. \tag{5.18}$$

Using the estimate (5.18), we can easily prove that under the assumption (5.17) the equation (5.14) has at least one root obeying the corresponding inequalities (5.16). The variable c is related to the original variable t through the formula (5.13). Therefore from (5.16) we derive the following inequalities for t:

$$\begin{gathered}(\sqrt{2}-1)\,\tilde{q}^{\,2}+\frac{10-7\sqrt{2}}{B^2\,\tilde{q}^2}+\frac{\tilde{p}}{2\,B^3\,\tilde{q}^5}<\operatorname{Im} t<(\sqrt{2}-1)\,\tilde{q}^{\,2}+\\ +\frac{10-7\sqrt{2}}{B^2\,\tilde{q}^2}\quad\text{in the case}\quad \tilde{p}<0,\end{gathered} \tag{5.19}$$

$$\begin{gathered}(\sqrt{2}-1)\,\tilde{q}^{\,2}+\frac{10-7\sqrt{2}}{B^2\,\tilde{q}^2}<\operatorname{Im} t<(\sqrt{2}-1)\,\tilde{q}^{\,2}+\\ +\frac{10-7\sqrt{2}}{B^2\,\tilde{q}^2}+\frac{\tilde{p}}{2\,B^3\,\tilde{q}^5}\quad\text{in the case}\quad \tilde{p}>0.\end{gathered} \tag{5.20}$$

As a result we have proved the following theorem.

Theorem 5.2. *If $\tilde{p} \neq 0$, then for each $\tilde{q} \geqslant 3600\,\sqrt[3]{|\tilde{p}|}$ and for each value of B in (4.6) there is at least one purely imaginary root of the equation (2.4) satisfying the inequalities (5.19) or (5.20) respectively.*

Theorem 5.2 means that we have derived the estimate (5.12) with the coefficient $C = |\tilde{p}|/(2\,B^3)$ for the remainder term $R_5(\tilde{p}, \tilde{q}, B)$ in the asymptotic expansion (5.11) for at least one root of the equation (2.4).

6. Non-intersection of asymptotic sites.

In the previous two sections we have used four inequalities (4.9), (4.26), (5.7), and (5.17). The inequality (5.17) is the strongest of them. It looks like

$$\tilde{q} \geqslant 3600\,\sqrt[3]{|\tilde{p}|}. \tag{6.1}$$

If the inequality (6.1) is fulfilled, then all of the inequalities (4.9), (4.26), (5.7), (5.17) are also fulfilled. Due to these inequalities, applying Theorems 4.1, 4.2, 4.3, 5.1, and 5.2, we find that there are five asymptotic sites, each of which comprises at least one root of the equation (2.4).

The site comprising the root t_3 is given by the inequalities (4.28) and (4.29). From (6.1) and (4.8) we derive the following inequalities:

$$\begin{aligned}&\tilde{q}^{\,3} \geqslant 3600^3\,|\tilde{p}| &&\tilde{q} \geqslant 3600,\\ &\frac{\tilde{q}^{\,3}}{|\tilde{p}|} \geqslant 3600^3, &&\frac{|\tilde{p}|}{\tilde{q}^{\,3}} \leqslant \frac{1}{3600^3}.\end{aligned} \tag{6.2}$$

From (4.6) we easily derive the following inequalities for B:

$$1\leqslant B\leqslant 9,\qquad\qquad \frac{1}{9}\leqslant\frac{1}{B}\leqslant 1. \tag{6.3}$$

Applying (6.2) and (6.3) to the left hand side of (4.28), we derive

$$\begin{gathered} B\,\tilde q^{4}-\tilde p\,\tilde q+\frac{16}{B}+\frac{32\,\tilde p}{B^{2}\,\tilde q^{3}}\geqslant B\,\tilde q^{4}\Bigl(1-\frac{1}{3600^{3}}\Bigr)+\frac{16}{B}-\frac{32}{B^{2}\,3600^{3}}\geqslant\\ \geqslant 3600^{4}\Bigl(1-\frac{1}{3600^{3}}\Bigr)+\frac{16}{9}-\frac{32}{3600^{3}}\approx 1.68\cdot 10^{14}>0. \end{gathered} \tag{6.4}$$

The left hand side of (4.29) is treated similarly:

$$\begin{gathered} B\,\tilde q^{4}-\tilde p\,\tilde q+\frac{16}{B}\geqslant B\,\tilde q^{4}\Bigl(1-\frac{1}{3600^{3}}\Bigr)+\frac{16}{B}\geqslant\\ \geqslant 3600^{4}\Bigl(1-\frac{1}{3600^{3}}\Bigr)+\frac{16}{9}\approx 1.67\cdot 10^{14}>0. \end{gathered} \tag{6.5}$$

The root t_2 is delimited by the inequalities (4.18) and (4.19). Let's compare the left hand side of (4.18) with the right hand side of (4.28). For their difference, using (6.2) and (6.3), we derive the following estimate:

$$\begin{gathered} \Bigl(B^{2}\,\tilde q^{6}-2\,B\,\tilde q^{4}-2\,B\,\tilde p\,\tilde q^{3}-2\,\tilde q^{2}+2\,\tilde p\,\tilde q+\tilde p^{2}-\frac{5}{B}-\frac{20}{B^{2}\,\tilde q^{2}}\Bigr)-\\ -\Bigl(B\,\tilde q^{4}-\tilde p\,\tilde q+\frac{16}{B}\Bigr)=B^{2}\,\tilde q^{6}\Bigl(1-\frac{3}{B\,\tilde q^{2}}-\frac{2\,\tilde p}{B\,\tilde q^{3}}-\frac{2}{B^{2}\,\tilde q^{4}}+\frac{3\,\tilde p}{B^{2}\,\tilde q^{5}}-\\ -\frac{21}{B^{3}\,\tilde q^{6}}-\frac{20}{B^{4}\,\tilde q^{8}}\Bigr)\geqslant B^{2}\,\tilde q^{6}\Bigl(1-\frac{3}{3600^{2}}-\frac{2}{3600^{3}}-\frac{2}{3600^{4}}-\frac{3}{3600^{5}}-\\ -\frac{21}{3600^{6}}-\frac{20}{3600^{8}}\Bigr)\approx 0.99\cdot B^{2}\,\tilde q^{6}\geqslant 0.99\cdot 3600^{6}\approx 2.18\cdot 10^{21}>0. \end{gathered} \tag{6.6}$$

The difference of the left hand side of (4.19) and the right hand side of (4.29) is treated similarly. For this difference we have

$$\begin{gathered} \Bigl(B^{2}\,\tilde q^{6}-2\,B\,\tilde q^{4}-2\,B\,\tilde p\,\tilde q^{3}-2\,\tilde q^{2}+2\,\tilde p\,\tilde q+\tilde p^{2}-\frac{5}{B}-\frac{20}{B^{2}\,\tilde q^{2}}-\\ -\frac{10\,\tilde p}{B^{2}\,\tilde q^{3}}\Bigr)-\Bigl(B\,\tilde q^{4}-\tilde p\,\tilde q+\frac{16}{B}+\frac{32\,\tilde p}{B^{2}\,\tilde q^{3}}\Bigr)\geqslant B^{2}\,\tilde q^{6}\Bigl(1-\frac{3}{3600^{2}}-\\ -\frac{2}{3600^{3}}-\frac{2}{3600^{4}}-\frac{3}{3600^{5}}-\frac{21}{3600^{6}}-\frac{20}{3600^{8}}-\frac{42}{3600^{9}}\Bigr)\approx\\ \approx 0.99\cdot B^{2}\,\tilde q^{6}\geqslant 0.99\cdot 3600^{6}\approx 2.17\cdot 10^{21}>0. \end{gathered} \tag{6.7}$$

The root t_1 is delimited by the inequalities (4.11) and (4.12). Let's consider the difference of the left hand side of (4.11) and the right hand side of (4.18):

$$\begin{gathered} \Bigl(B^{2}\,\tilde q^{6}+2\,B\,\tilde q^{4}-2\,B\,\tilde p\,\tilde q^{3}-2\,\tilde q^{2}-2\,\tilde p\,\tilde q+\tilde p^{2}+\frac{5}{B}-\frac{20}{B^{2}\,\tilde q^{2}}+\\ +\frac{10\,\tilde p}{B^{2}\,\tilde q^{3}}\Bigr)-\Bigl(B^{2}\,\tilde q^{6}-2\,B\,\tilde q^{4}-2\,B\,\tilde p\,\tilde q^{3}-2\,\tilde q^{2}+2\,\tilde p\,\tilde q+\tilde p^{2}- \end{gathered}$$

$$-\frac{5}{B}-\frac{20}{B^2\,\tilde q^2}-\frac{10\,\tilde p}{B^2\,\tilde q^3}\Bigr)=4\,B\,\tilde q^4-4\,\tilde p\,\tilde q+\frac{10}{B}+\frac{20\,\tilde p}{B^2\,\tilde q^3}.$$

Applying (6.2) and (6.3) to the above expression, we get the following estimate:

$$\begin{gathered}4\,B\,\tilde q^4-4\,\tilde p\,\tilde q+\frac{10}{B}+\frac{20\,\tilde p}{B^2\,\tilde q^3}=4\,B\,\tilde q^4\Bigl(1-\frac{\tilde p}{B\,\tilde q^3}+\frac{5}{2\,B^2\,\tilde q^4}+\\+\frac{5\,\tilde p}{B^3\,\tilde q^7}\Bigr)\geqslant 4\,B\,\tilde q^4\Bigl(1-\frac{1}{3600^3}-\frac{5}{2\cdot 3600^4}-\frac{5}{3600^7}\Bigr)\approx\\\approx 4.00\cdot B\,\tilde q^4\geqslant 4.00\cdot 3600^4\approx 6.71\cdot 10^{14}>0.\end{gathered}\tag{6.8}$$

The difference of the left hand side of (4.12) and the right hand side of (4.19) is treated similarly. For this difference we have

$$\begin{gathered}\Bigl(B^2\,\tilde q^6+2\,B\,\tilde q^4-2\,B\,\tilde p\,\tilde q^3-2\,\tilde q^2-2\,\tilde p\,\tilde q+\tilde p^2+\frac{5}{B}-\frac{20}{B^2\,\tilde q^2}\Bigr)-\\-\Bigl(B^2\,\tilde q^6-2\,B\,\tilde q^4-2\,B\,\tilde p\,\tilde q^3-2\,\tilde q^2+2\,\tilde p\,\tilde q+\tilde p^2-\frac{5}{B}-\frac{20}{B^2\,\tilde q^2}\Bigr)=\\=4\,B\,\tilde q^4-4\,\tilde p\,\tilde q+\frac{10}{B}\geqslant 4\,B\,\tilde q^4\Bigl(1-\frac{1}{3600^3}-\frac{5}{2\cdot 3600^4}\Bigr)\approx\\\approx 4.00\cdot B\,\tilde q^4\geqslant 4.00\cdot 3600^4\approx 6.71\cdot 10^{14}>0.\end{gathered}\tag{6.9}$$

The inequalities (6.4), (6.5), (6.6), (6.7), (6.8), and (6.9) mean that if the inequality (6.1) is fulfilled, then the asymptotic sites for the roots t_1, t_2, t_3 do not intersect with each other and are located within the positive half-line of the real axis.

Now let's proceed to the purely imaginary roots t_4 and t_5. The root t_5 is delimited by the inequalities (5.19) and (5.20). Applying (6.2) and (6.3) to the left hand side of (5.19), we derive the following estimate for it:

$$\begin{gathered}(\sqrt{2}-1)\,\tilde q^2+\frac{10-7\sqrt{2}}{B^2\,\tilde q^2}+\frac{\tilde p}{2\,B^3\,\tilde q^5}=(\sqrt{2}-1)\,\tilde q^2\times\\\times\Bigl(1+\frac{3\sqrt{2}-4}{B^2\,\tilde q^4}+\frac{\tilde p\,(\sqrt{2}+1)}{2\,B^3\,\tilde q^7}\Bigr)\geqslant(\sqrt{2}-1)\,\tilde q^2\Bigl(1-\frac{3\sqrt{2}-4}{3600^4}-\\-\frac{\tilde p\,(\sqrt{2}+1)}{2\cdot 3600^7}\Bigr)\approx 0.41\cdot\tilde q^2\geqslant 0.41\cdot 3600^2\approx 5.31\cdot 10^6>0.\end{gathered}\tag{6.10}$$

The left hand side of (5.20) is treated similarly:

$$\begin{gathered}(\sqrt{2}-1)\,\tilde q^2+\frac{10-7\sqrt{2}}{B^2\,\tilde q^2}=(\sqrt{2}-1)\,\tilde q^2\Bigl(1+\frac{3\sqrt{2}-4}{B^2\,\tilde q^4}\Bigr)\geqslant\\\geqslant(\sqrt{2}-1)\,\tilde q^2\Bigl(1-\frac{3\sqrt{2}-4}{3600^4}\Bigr)\geqslant 0.41\cdot 3600^2\approx 5.31\cdot 10^6>0.\end{gathered}\tag{6.11}$$

The root t_4 is delimited by the inequalities (5.9) and (5.10). Let's compare the left hand side of (5.9) with the right hand side of (5.19). Here is their difference:

$$\Bigl((\sqrt{2}+1)\,\tilde q^2-\frac{10+7\sqrt{2}}{B^2\,\tilde q^2}\Bigr)-\Bigl((\sqrt{2}-1)\,\tilde q^2+\frac{10-7\sqrt{2}}{B^2\,\tilde q^2}\Bigr)=2\,\tilde q^2-\frac{20}{B^2\,\tilde q^2}.$$

Applying (6.2) and (6.3) to the above expression, we get the following estimate:

$$\begin{gathered}
2\,\tilde q^2-\frac{20}{B^2\,\tilde q^2}=2\,\tilde q^2\left(1-\frac{10}{B^2\,\tilde q^4}\right)\geqslant 2\,\tilde q^2\left(1-\frac{10}{3600^4}\right)\approx\\
\approx 1.99\cdot\tilde q^2\geqslant 1.99\cdot 3600^2\approx 2.59\cdot 10^7>0.
\end{gathered}\tag{6.12}$$

The difference of the left hand side of (5.10) and the right hand side of (5.20) is treated similarly. For this difference we have

$$\begin{gathered}
\left((\sqrt{2}+1)\,\tilde q^2-\frac{10+7\sqrt{2}}{B^2\,\tilde q^2}-\frac{80\,\tilde p}{B^3\,\tilde q^5}\right)-\left((\sqrt{2}-1)\,\tilde q^2+\right.\\
\left.+\frac{10-7\sqrt{2}}{B^2\,\tilde q^2}+\frac{\tilde p}{2\,B^3\,\tilde q^5}\right)=2\,\tilde q^2-\frac{20}{B^2\,\tilde q^2}-\frac{81\,\tilde p}{B^3\,\tilde q^5}=\\
=2\,\tilde q^2\left(1-\frac{10}{B^2\,\tilde q^4}-\frac{81\,\tilde p}{2\,B^3\,\tilde q^7}\right)\geqslant 2\,\tilde q^2\left(1-\frac{10}{3600^4}-\right.\\
\left.-\frac{81}{2\cdot 3600^7}\right)\approx 1.99\cdot\tilde q^2\geqslant 1.99\cdot 3600^2\approx 2.59\cdot 10^7>0.
\end{gathered}\tag{6.13}$$

The inequalities (6.10), (6.11), (6.12), and (6.13) mean that if the inequality (6.1) is fulfilled, then the asymptotic sites for the roots t_4 and t_5 do not intersect with each other and are located within the positive half-line of the imaginary axis. Summarizing this result with the above result for the real roots t_1, t_2, t_3 we can formulate the following two theorems.

Theorem 6.1. *If $\tilde p<0$ and $\tilde q\geqslant 3600\,\sqrt[3]{|\tilde p|}$, then the equation (2.4) has five simple roots obeying the condition (2.5). Three of them are real and positive. These three positive real roots are located within three disjoint asymptotic sites given by the inequalities (4.11), (4.18), and (4.28) respectively.*

Theorem 6.2. *If $\tilde p>0$ and $\tilde q\geqslant 3600\,\sqrt[3]{|\tilde p|}$, then the equation (2.4) has five simple roots obeying the condition (2.5). Three of them are real and positive. These three positive real roots are located within three disjoint asymptotic sites given by the inequalities (4.12), (4.19), and (4.29) respectively.*

7. Integer points of asymptotic sites.

Let's consider the inequalities (4.11) and (4.12) delimiting the root t_1 of the equation (2.4). Most of the terms in them are integer, except for following ones:

$$\frac{5}{B},\qquad\qquad \frac{20}{B^2\,\tilde q^2},\qquad\qquad \frac{10\,\tilde p}{B^2\,\tilde q^3}.\tag{7.1}$$

The first term (7.1) is optionally non-integer. If the inequalities (6.2) and (6.3) are fulfilled, then the other two terms (7.1) are certainly non-integer.

Let's begin with the case where $5/B$ is not integer. In this case due to (6.3) it is separated from the nearest integer number by a distance not less than $1/9$:

$$\left|\frac{5}{B}-n\right|\geqslant\frac{1}{9}.\tag{7.2}$$

The other two terms in (7.1) are substantially smaller. From (6.2) and (6.3) we get

$$\Bigl|\frac{20}{B^2\,\tilde q^2}\Bigr| < \frac{20}{3600^2}, \qquad\qquad \frac{10\,\tilde p}{B^2\,\tilde q^3} < \frac{10}{3600^3}. \tag{7.3}$$

Combining (7.2) and (7.3), we obtain the following inequality:

$$\Bigl|\frac{5}{B} \pm \frac{20}{B^2\,\tilde q^2} \pm \frac{10\,\tilde p}{B^2\,\tilde q^3} - n\Bigr| \geqslant \frac{1}{9} - \frac{20}{3600^2} - \frac{10}{3600^3} > \frac{1}{10}. \tag{7.4}$$

In the case where $5/B$ is integer, i.e. where $B = 5$, we treat the inequalities (4.11) and (4.12) separately. If $\tilde p < 0$, from (6.2) and (6.3) we derive

$$-1 < -\frac{20}{3600^2} - \frac{10}{3600^3} \leqslant -\frac{20}{B^2\,\tilde q^2} + \frac{10\,\tilde p}{B^2\,\tilde q^3} < -\frac{20}{B^2\,\tilde q^2} < 0. \tag{7.5}$$

If $\tilde p > 0$ we need an additional condition for $\tilde p$ and $\tilde q$:

$$2\,\tilde q > |\tilde p| \tag{7.6}$$

Provided the condition (7.6) is fulfilled, from (6.2) and (6.3) we derive

$$-1 < -\frac{20}{3600^2} \leqslant -\frac{20}{B^2\,\tilde q^2} < -\frac{20}{B^2\,\tilde q^2} + \frac{10\,\tilde p}{B^2\,\tilde q^3} < 0. \tag{7.7}$$

If the condition (7.6) is not fulfilled, then from (6.2) and (6.3) we derive

$$-1 < -\frac{20}{3600^2} \leqslant -\frac{20}{B^2\,\tilde q^2} < -\frac{20}{B^2\,\tilde q^2} + \frac{10\,\tilde p}{B^2\,\tilde q^3} < \frac{20}{3600^2} + \frac{10}{3600^3} < 1. \tag{7.8}$$

Theorem 7.1. *If $\tilde p \neq 0$, then for each $\tilde q \geqslant 3600\,\sqrt[3]{|\tilde p|}$ and for each $B \neq 5$ in (4.6) the asymptotic site given by the inequalities (4.11) and (4.12) has no integer points.*

Theorem 7.1 is proved by applying (7.4) to (4.11) and (4.12).

Theorem 7.2. *If $\tilde p < 0$ and $B = 5$, then for each $\tilde q \geqslant 3600\,\sqrt[3]{|\tilde p|}$ the asymptotic site given by the inequalities (4.11) has no integer points.*

Theorem 7.2 is proved by applying (7.5) to (4.11).

Theorem 7.3. *If $\tilde p > 0$ and $B = 5$, then for each $\tilde q$ such that $\tilde q \geqslant 3600\,\sqrt[3]{|\tilde p|}$ and $2\,\tilde q > \tilde p$ the asymptotic site given by the inequalities (4.12) has no integer points.*

Theorem 7.3 is proved by applying (7.7) to (4.12).

Theorem 7.4. *If $\tilde p \neq 0$ and $B = 5$, then for $\tilde q \geqslant 3600\,\sqrt[3]{|\tilde p|}$ the asymptotic site given by the inequalities (4.11) and (4.12) has at most one integer point given by the formula $t = 25\,\tilde q^6 + 10\,\tilde q^4 - 10\,\tilde p\,\tilde q^3 - 2\,\tilde q^2 - 2\,\tilde p\,\tilde q + \tilde p^2 + 1$, where $\tilde p > 0$.*

Theorem 7.4 is proved by applying (7.8) to (4.12).

Note that the inequalities (4.18) and (4.19) are quite similar to (4.11) and (4.12). Applying the estimates (7.4), (7.5), (7.7), and (7.8) to them we can prove the following four theorems for the corresponding asymptotic site.

Theorem 7.5. *If $\tilde{p}\neq 0$, then for each $\tilde{q}\geqslant 3600\,\sqrt[3]{|\tilde{p}|}$ and for each $B\neq 5$ in (4.6) the asymptotic site given by the inequalities (4.18) and (4.19) has no integer points.*

Theorem 7.6. *If $\tilde{p}>0$ and $B=5$, then for each $\tilde{q}\geqslant 3600\,\sqrt[3]{|\tilde{p}|}$ the asymptotic site given by the inequalities (4.19) has no integer points.*

Theorem 7.7. *If $\tilde{p}<0$ and $B=5$, then for each $\tilde{q}$ such that $\tilde{q}\geqslant 3600\,\sqrt[3]{|\tilde{p}|}$ and $2\,\tilde{q}>|\tilde{p}|$ the asymptotic site given by the inequalities (4.18) has no integer points.*

Theorem 7.8. *If $\tilde{p}\neq 0$ and $B=5$, then for $\tilde{q}\geqslant 3600\,\sqrt[3]{|\tilde{p}|}$ the asymptotic site given by the inequalities (4.18) and (4.19) has at most one integer point given by the formula $t=25\,\tilde{q}^6-10\,\tilde{q}^4-10\,\tilde{p}\,\tilde{q}^3-2\,\tilde{q}^2+2\,\tilde{p}\,\tilde{q}+\tilde{p}^2-1$, where $\tilde{p}<0$.*

Let's proceed to the inequalities (4.28) and (4.29) and let's begin with the case where $16/B$ is not integer. In this case due to (6.3) the number $16/B$ is separated from the nearest integer number by a distance not less than $1/9$:

$$\left|\frac{16}{B}-n\right|\geqslant\frac{1}{9}. \tag{7.9}$$

From (6.2) and (6.3) for the other fractional term in (4.28) and (4.29) we derive

$$\left|\frac{32\,\tilde{p}}{B^2\,\tilde{q}^3}\right|\leqslant\frac{32}{3600^3} \tag{7.10}$$

Combining (7.9) and (7.10), we obtain the following inequality:

$$\left|\frac{16}{B}\pm\frac{32\,\tilde{p}}{B^2\,\tilde{q}^3}-n\right|\geqslant\frac{1}{9}-\frac{32}{3600^3}\geqslant\frac{1}{10}. \tag{7.11}$$

The case where $16/B$ is integer is more simple. In this case we write the inequality (7.10) in the following slightly modified form:

$$0<\left|\frac{32\,\tilde{p}}{B^2\,\tilde{q}^3}\right|\leqslant\frac{32}{3600^3}<1. \tag{7.12}$$

In both cases, applying either (7.11) or (7.12) to the inequalities (4.28) and (4.29), we can prove the following theorem.

Theorem 7.9. *If $\tilde{p}\neq 0$, then for each $\tilde{q}\geqslant 3600\,\sqrt[3]{|\tilde{p}|}$ and for each B in (4.6) the asymptotic site given by the inequalities (4.28) and (4.29) has no integer points.*

The exceptional case.

As we noted above the case $\tilde{p}=0$ is exceptional, see Theorems 7.1 through 7.9. This case should be studied separately. Let's substitute $\tilde{p}=0$ into the equation (2.4). The resulting equation can be written explicitly:

$$\begin{aligned}&t^{10}+(6\,\tilde{q}^4-2\,B^4\,\tilde{q}^{12}-\tilde{q}^8\,B^2)\,t^8+(B^8\,\tilde{q}^{24}+10\,\tilde{q}^{12}\,B^2+4\,\tilde{q}^{16}\,B^4-\\&\quad-14\,\tilde{q}^{20}\,B^6+\tilde{q}^8)\,t^6+(14\,\tilde{q}^{20}\,B^4-4\,\tilde{q}^{24}\,B^6-\tilde{q}^{16}\,B^2-\tilde{q}^{32}\,B^{10}-\\&\quad-10\,\tilde{q}^{28}\,B^8)\,t^4+(2\,\tilde{q}^{28}\,B^6-6\,\tilde{q}^{36}\,B^{10}+\tilde{q}^{32}\,B^8)\,t^2-\tilde{q}^{40}\,B^{10}=0.\end{aligned} \tag{8.1}$$

Theorem 8.1. *For each B in (4.6) the polynomial in the left hand side of the equation (8.1) is irreducible in the ring $\mathbb{Z}[t]$.*

Theorem 8.1 is proved by means of direct computations. It means that for each B in (4.6) the equation (8.1) has no integer roots.

9. Application to the cuboid problem.

Theorems 7.1 through 7.9 are based on the inequality (6.1), where $\tilde{p} \neq 0$. Theorem 8.1 means that we can omit the condition $\tilde{p} \neq 0$. Transforming (6.1) back to the initial variables p and q with the use of (2.1), we get the inequalities

$$B\,q^3 - \frac{q^3}{3600^3} \leqslant p \leqslant B\,q^3 + \frac{q^3}{3600^3}. \tag{9.1}$$

From Theorems 6.2, 7.1, 7.5, and 7.9 we derive the following result.

Theorem 9.1. *For each $B \neq 5$ in (4.6) if the inequalities (9.1) are fulfilled, then the tenth degree cuboid characteristic equation (1.1) produces no perfect cuboids.*

The case $B = 5$ is special. In this case we have the additional condition (7.6). Upon transforming (7.6) back to the initial variables p and q it looks like

$$B\,q^3 - 2\,q < p < B\,q^3 + 2\,q. \tag{9.2}$$

From Theorems 6.2, 7.2, 7.3, 7.6, 7.7, and 7.9 we derive the following result.

Theorem 9.2. *For $B = 5$ if the inequalities (9.1) and (9.2) are fulfilled, then the tenth degree cuboid characteristic equation (1.1) produces no perfect cuboids.*

The inequalities (9.2) do not follow from (9.1). They become very restrictive for large q as $q \to +\infty$. Theorems 7.4 and 7.8 can be applied in order to remove this restriction. However, they do not change the state of affairs in general. Therefore this step is left for one of the future papers.

10. Conclusions.

The main result of this paper is presented by Theorems 9.1 and 9.2. Theorems 9.1 and 9.2 shrink the nonlinear region on the $p\,q$ - coordinate plane by cutting off nine narrow strips surrounding nine cubic parabolas

$$p = B\,q^3, \quad \text{where} \quad B = 1, 2, \ldots, 9.$$

These strips are outlined by the inequalities (9.1) and (9.2). They are annexed to the no cuboid region (see Fig. 1.1) thus reducing the area where perfect cuboids are still potentially possible[1].

11. Acknowledgments

On May 19, 2015, I have reported the papers [1–3] in the Ufa all-city seminar on differential equations of mathematical physics named after A. M. Ilyin. This

[1] The paper [70] has been recently published in ArXiv. It says that there are no perfect cuboids. However this paper is not yet verified by the mathematical community. Therefore alternative approaches to the perfect cuboid problem can be developed for some while.

seminar brings together many experts in the field of asymptotics residing in our city. I am grateful to L. A. Kalyakin, the chairman of the seminar, for the opportunity to give my talk. I am also grateful to V. Yu. Novokshenov, the other chairman, and to all participants of the seminar for their attention and comments.

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Bashkir State University, 32 Zaki Validi street, 450074 Ufa, Russia
E-mail address: r-sharipov@mail.ru