A NOTE ON INVERTIBLE QUADRATIC TRANSFORMATIONS OF THE REAL PLANE.

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Abstract. A polynomial transformation of the real plane $\mathbb{R}^2$ is a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ given by two polynomials of two variables. Such a transformation is called quadratic if the degrees of its polynomials are not greater than two. In the present paper an exhaustive description of invertible quadratic transformations of the real plane is given. Their application to the perfect cuboid problem is discussed.

1. Introduction.

A perfect cuboid is a rectangular parallelepiped whose edges, whose face diagonals and whose space diagonal all are of integer lengths. None of such cuboids is found thus far. The problem of finding them or proving their non-existence is still open, see its history in [1–47].

Slanted (non-rectangular) perfect cuboids are known. Some of them were found in [48]. Infinite families of slanted perfect cuboids were found in [49] and [50].

In some approaches the search for rectangular perfect cuboids is reduced to a single Diophantine equation. One of such Diophantine equations was derived in [51]. On the basis of this equation in [52] three cuboid conjectures were formulated. The first cuboid conjecture is rather simple. Though it was not yet proved, in [53] it was shown that there no perfect cuboids in the case of the first cuboid conjecture.

The second and the third cuboid conjectures are more complicated. They were considered in [54] and [55], but were not proved. At the present time the case of the second cuboid conjecture is being intensively studied using the asymptotic approach (see [56–59]). The approach used in [60–72] is quite different. We do not consider this approach below in the present paper.

In the case of the second cuboid conjecture the search for rectangular perfect cuboids is reduced to the following Diophantine equation of tenth degree with respect to the positive integer variable $t > 0$:

$$\begin{gathered} t^{10} + (2\,q^2 + p^2)\,(3\,q^2 - 2\,p^2)\,t^8 + (q^8 + 10\,p^2\,q^6 + \\ + 4\,p^4\,q^4 - 14\,p^6\,q^2 + p^8)\,t^6 - p^2\,q^2\,(q^8 - 14\,p^2\,q^6 + 4\,p^4\,q^4 + \\ + 10\,p^6\,q^2 + p^8)\,t^4 - p^6\,q^6\,(q^2 + 2\,p^2)\,(3\,p^2 - 2\,q^2)\,t^2 - q^{10}\,p^{10} = 0. \end{gathered} \tag{1.1}$$

Two coprime positive integer numbers $p \neq q$ are parameters of the equation (1.1). They define a point (p, q) on the $p\,q$-coordinate plane $\mathbb{R}^2$.

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Within the asymptotic approach to the equation (1.1) its parameters p and q tend to infinity either separately like in [56] and [57] or simultaneously like in [58] and [59]. In order to arrange a proper passage to the limit $(p, q) \to \infty$ in [58] and [59] linear and nonlinear transformations of the form

$$\tilde{p} = \tilde{p}(p, q), \qquad\qquad \tilde{q} = \tilde{q}(p, q). \tag{1.2}$$

were used. There are two natural requirements for the transformations (1.2) applied to the parameters of the Diophantine equation (1.1):

1) they should be invertible;
2) they should map bijectively the integer $p\,q$ - greed onto itself.

Polynomial transformations are the best candidates for this role. For example in [59] the following cubic transformations of the form (1.2) were used:

$$\tilde{p} = B\,q^3 - p, \qquad\qquad \tilde{q} = q. \tag{1.3}$$

Unfortunately the coefficient B in (1.3) is restricted to integer numbers. Therefore the transformations (1.3) appeared to be insufficient for to cover all needs they were designed to serve in [59].

The main goal of the present paper is to extend the number of invertible polynomial transformations applicable to Diophantine equation (1.1). We consider quadratic transformations which are most simple after linear ones.

It is worth to note that complex polynomial transformations in $\mathbb{C}^n$ are associated with the Jacobian conjecture, which is another open problem in mathematics (see [73]). The corresponding real Jacobian conjecture in $\mathbb{R}^n$ is invalid. A counter example to it was given by S. I. Pinchuk in [74].

2. Quadratic transformations.

A general quadratic transformation of $\mathbb{R}^2$ is given by the following formulas:

$$\begin{aligned} &\tilde{p} = a_{20}\,p^2 + 2\,a_{11}\,p\,q + a_{02}\,q^2 + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ &\tilde{q} = b_{20}\,p^2 + 2\,b_{11}\,p\,q + b_{02}\,q^2 + 2\,b_{10}\,p + 2\,b_{01}\,q + b_{00}. \end{aligned} \tag{2.1}$$

Without loss of generality one can assume that

$$b_{11} = 0. \tag{2.2}$$

Indeed, if the condition (2.2) is not fulfilled and if $a_{11} \neq 0$, we can compose (2.1) with the following invertible linear transformation:

$$\hat{p} = \tilde{p}, \qquad\qquad \hat{q} = \tilde{q} - \frac{b_{11}}{a_{11}}\,\tilde{p}. \tag{2.3}$$

If $a_{11} = 0$, we can just exchange variables by applying the transformation

$$\hat{p} = \tilde{q}, \qquad\qquad \hat{q} = \tilde{p} \tag{2.4}$$

instead of (2.3). Like (2.3), the transformation (2.4) is invertible. In both cases the composite transformation will obey the condition (2.2).

Assume that the coefficients b_{20} and b_{11} are proportional to the coefficients a_{20} and a_{11} in (2.1), i. e. assume that they obey the relationship

$$\frac{b_{20}}{b_{11}}=\frac{a_{20}}{a_{11}}. \tag{2.5}$$

In this case, composing (2.3) with (2.1), we get a quadratic transformation of the form (2.1) with two coefficients b_{20} and b_{11} being equal to zero:

$$b_{20}=0, \qquad\qquad b_{11}=0. \tag{2.6}$$

Similarly, if the relationship

$$\frac{b_{02}}{b_{11}}=\frac{a_{02}}{a_{11}}. \tag{2.7}$$

is fulfilled, then upon composing (2.3) with (2.1) we shall have

$$b_{11}=0, \qquad\qquad b_{02}=0. \tag{2.8}$$

Apart from (2.5) and (2.7), there is a third option:

$$\frac{b_{02}}{b_{20}}=\frac{a_{02}}{a_{20}}. \tag{2.9}$$

In this case instead of (2.3) we choose the following linear transformation:

$$\hat{p}=\tilde{p}, \qquad\qquad \hat{q}=\tilde{q}-\frac{b_{20}}{a_{20}}\,\tilde{p}. \tag{2.10}$$

Provided (2.9) is fulfilled, upon composing (2.10) with (2.1) we shall have

$$b_{20}=0, \qquad\qquad b_{02}=0. \tag{2.11}$$

The relationships (2.5), (2.7), and (2.9) can be written in a denominator-free form:

$$\det\begin{Vmatrix} a_{20} & a_{11}\\ b_{20} & b_{11}\end{Vmatrix}=0, \qquad \det\begin{Vmatrix} a_{11} & a_{02}\\ b_{11} & b_{02}\end{Vmatrix}=0, \qquad \det\begin{Vmatrix} a_{20} & a_{02}\\ b_{20} & b_{02}\end{Vmatrix}=0. \tag{2.12}$$

Typically the relationships (2.12) are not fulfilled. However, we can use a linear transformation in order to change the coefficients in (2.1). Let's set

$$p=c_{11}\,\check{p}+c_{12}\,\check{q}, \qquad\qquad q=c_{21}\,\check{p}+c_{22}\,\check{q}. \tag{2.13}$$

Substituting (2.9) into (2.1), we get a transformation of the same form like (2.1):

$$\begin{aligned}\tilde{p}&=\check{a}_{20}\,\check{p}^2+2\,\check{a}_{11}\,\check{p}\,\check{q}+\check{a}_{02}\,\check{q}^2+2\,\check{a}_{10}\,\check{p}+2\,\check{a}_{01}\,\check{q}+\check{a}_{00},\\ \tilde{q}&=\check{b}_{20}\,\check{p}^2+2\,\check{b}_{11}\,\check{p}\,\check{q}+\check{b}_{02}\,\check{q}^2+2\,\check{b}_{10}\,\check{p}+2\,\check{b}_{01}\,\check{q}+\check{b}_{00}.\end{aligned}$$

Its coefficients depend on c_{11}, c_{12}, c_{21}, c_{22} and on the coefficients of (2.1). Upon substituting $\breve{a}_{20}$, $\breve{a}_{11}$, $\breve{a}_{02}$ and $\breve{b}_{20}$, $\breve{b}_{11}$, $\breve{b}_{02}$ for a_{20}, a_{11}, a_{02} and b_{20}, b_{11}, b_{02} into (2.8) we derive the following three relationships:

$$\det C\left(c_{11}^2\begin{vmatrix}a_{20} & a_{11}\\ b_{20} & b_{11}\end{vmatrix}+c_{11}\begin{vmatrix}a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix}c_{21}+\begin{vmatrix}a_{11} & a_{02}\\ b_{11} & b_{02}\end{vmatrix}c_{21}^2\right)=0,\tag{2.14}$$

$$\det C\left(c_{22}^2\begin{vmatrix}a_{20} & a_{11}\\ b_{20} & b_{11}\end{vmatrix}+c_{22}\begin{vmatrix}a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix}c_{12}+\begin{vmatrix}a_{11} & a_{02}\\ b_{11} & b_{02}\end{vmatrix}c_{21}^2\right)=0,\tag{2.15}$$

$$\begin{aligned}\det C\Biggl(c_{11}\begin{vmatrix}a_{20} & a_{11}\\ b_{20} & b_{11}\end{vmatrix}c_{12}+c_{11}\begin{vmatrix}a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix}c_{22}\,+\\ +\,c_{21}\begin{vmatrix}a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix}c_{12}+c_{21}\begin{vmatrix}a_{11} & a_{02}\\ b_{11} & b_{02}\end{vmatrix}c_{22}\Biggr)=0.\end{aligned}\tag{2.16}$$

Here $\det C$ is the determinant of the following matrix:

$$C=\begin{Vmatrix}c_{11} & c_{12}\\ c_{21} & c_{22}\end{Vmatrix}.\tag{2.17}$$

The matrix (2.17) is non-degenerate since the transformation (2.13) should be invertible. Hence $\det C\neq 0$ and we can cancel it in (2.14), (2.15), and (2.16).

Looking at (2.14) and (2.15), we define the quadratic form[1]

$$\omega_1(\mathbf{c})=c_1^2\begin{vmatrix}a_{20} & a_{11}\\ b_{20} & b_{11}\end{vmatrix}+c_1\begin{vmatrix}a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix}c_2+\begin{vmatrix}a_{11} & a_{02}\\ b_{11} & b_{02}\end{vmatrix}c_2^2,\tag{2.18}$$

where $\mathbf{c}\in\mathbb{R}^2$ is a vector with two components c_1 and c_2. Any quadratic form in a real vector space can be definite, semi-definite, or indefinite (see [75]).

3. The case where $\omega_1(\mathbf{c})$ is zero.

In this case all of the three determinants in (2.18) are zero. This means that the coefficients b_{20}, b_{11}, b_{02} are proportional to the coefficients a_{20}, a_{11}, a_{02}. Hence, applying some tricks like (2.3) and (2.4), we can bring (2.1) to the form

$$\begin{aligned}&\tilde{p}=a_{20}\,p^2+2\,a_{11}\,p\,q+a_{02}\,q^2+2\,a_{10}\,p+2\,a_{01}\,q+a_{00},\\ &\tilde{q}=2\,b_{10}\,p+2\,b_{01}\,q+b_{00}.\end{aligned}\tag{3.1}$$

Since the transformation (3.1) is assumed to be invertible, at least one of the two coefficients b_{10} and b_{01} is nonzero. Therefore, applying some linear transformation of the form (2.13) and a shift of origin, we can further simplify the formulas (3.1):

$$\begin{aligned}&\tilde{p}=a_{20}\,p^2+2\,a_{11}\,p\,q+a_{02}\,q^2+2\,a_{10}\,p+2\,a_{01}\,q,\\ &\tilde{q}=q.\end{aligned}\tag{3.2}$$

[1] Regular quadratic forms are tensors of the type $(0,2)$. The quadratic form $\omega_1(\mathbf{c})$ is a pseudotensor of the type $(0,2)$ in the sense of Definition 2.1 in [76].

Assume that $a_{20} \neq 0$ and assume that q is fixed. In this case the quadratic polynomial in (3.2) takes some values twice and does not take some other values at all. This fact contradicts the invertibility of the transformation (3.2). Hence $a_{20} = 0$ and the formulas (3.2) simplify to the following ones:

$$\tilde{p} = 2\,a_{11}\,p\,q + a_{02}\,q^2 + 2\,a_{10}\,p + 2\,a_{01}\,q, \qquad \tilde{q} = q. \tag{3.3}$$

If $a_{11} \neq 0$, then for some fixed q the first polynomial in (3.3) does not depend on p. This fact contradicts the invertibility of the transformation (3.3). Hence $a_{11} = 0$. The coefficient $2\,a_{10}$ in (3.3) is nonzero. Applying a linear transformation it can be reduced to the unity. Therefore the formula (3.3) simplifies to

$$\tilde{p} = p + a_{02}\,q^2 + 2\,a_{01}\,q, \qquad \tilde{q} = q. \tag{3.4}$$

The coefficient a_{02} in (3.4) is nonzero. Applying scaling transformations (which are linear) in p, q, $\tilde{p}$, $\tilde{q}$ and some origin shifts, we can bring a_{02} to the unity and can annul the coefficient $2\,a_{01}$. As a result (3.4) turns to

$$\tilde{p} = p + q^2, \qquad \tilde{q} = q. \tag{3.5}$$

This result is formulated as the following theorem.

Theorem 3.1. *In the case where the associated quadratic form* (2.18) *is zero any invertible quadratic transformation* (2.1) *reduces to the form* (3.5) *at the expense of composing it with linear transformations and origin shifts.*

4. The case where $\omega_1(\mathbf{c})$ is indefinite.

Each indefinite quadratic form in $\mathbb{R}^2$ has a basis of two linearly independent homogeneous vectors $\mathbf{c}_1$ and $\mathbf{c}_2$, i.e. two vectors in $\mathbb{R}^2$ such that

$$\omega_1(\mathbf{c}_1) = 0, \qquad \omega_1(\mathbf{c}_2) = 0. \tag{4.1}$$

In physics the homogeneous vectors $\mathbf{c}_1$ and $\mathbf{c}_2$ in (4.1) are called light vectors or light cone vectors (see [77]). Choosing

$$\mathbf{c}_1 = \begin{Vmatrix} c_{11} \\ c_{21} \end{Vmatrix}, \qquad \mathbf{c}_2 = \begin{Vmatrix} c_{12} \\ c_{22} \end{Vmatrix},$$

we construct a non-degenerate matrix C in (2.17) and apply it in (2.13). For such a matrix both equalities (2.14) and (2.15) are fulfilled. Hence two of the three equalities (2.12) are fulfilled. The third equality (2.12) is not fulfilled since otherwise we would return to the previous case where $\omega_1(\mathbf{c})$ is zero:

$$\det\begin{Vmatrix} a_{20} & a_{11} \\ b_{20} & b_{11} \end{Vmatrix} = 0, \qquad \det\begin{Vmatrix} a_{11} & a_{02} \\ b_{11} & b_{02} \end{Vmatrix} = 0, \qquad \det\begin{Vmatrix} a_{20} & a_{02} \\ b_{20} & b_{02} \end{Vmatrix} \neq 0. \tag{4.2}$$

The relationships mean that the first two of the three vectors

$$\begin{Vmatrix} a_{20} \\ b_{20} \end{Vmatrix}, \qquad \begin{Vmatrix} a_{02} \\ b_{02} \end{Vmatrix}, \qquad \begin{Vmatrix} a_{11} \\ b_{11} \end{Vmatrix} \tag{4.3}$$

in (4.3) are linearly independent, while the third vector (4.3) belongs to the span of each of the first two. This fact implies $a_{11} = 0$ and $b_{11} = 0$. It means that by applying some properly chosen linear transformation (2.13), we can bring our quadratic transformation (2.1) to the following form:

$$\begin{aligned}\tilde{p} &= a_{20}\,p^2 + a_{02}\,q^2 + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ \tilde{q} &= b_{20}\,p^2 + b_{02}\,q^2 + 2\,b_{10}\,p + 2\,b_{01}\,q + b_{00}.\end{aligned} \tag{4.4}$$

Now lets consider another linear transformation similar to (2.13):

$$\hat{p} = d_{11}\,\tilde{p} + d_{12}\,\tilde{q}, \qquad\qquad \hat{q} = d_{21}\,\tilde{p} + d_{22}\,\tilde{q}. \tag{4.5}$$

The transformations (2.3), (2.4), and (2.10) are special instances of the transformation (4.5). Since the third determinant in (4.2) is nonzero, choosing properly the coefficients of (4.5) and then applying (4.5) to (4.4), we can bring (4.4) to

$$\begin{aligned}\tilde{p} &= p^2 + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ \tilde{q} &= q^2 + 2\,b_{10}\,p + 2\,b_{01}\,q + b_{00}.\end{aligned} \tag{4.6}$$

Using origin shifts we can further simplify (4.6) bringing it to

$$\begin{aligned}\tilde{p} &= p^2 + 2\,a_{01}\,q,\\ \tilde{q} &= q^2 + 2\,b_{10}\,p.\end{aligned} \tag{4.7}$$

The coefficient $2\,b_{10}$ in (4.7) is nonzero. Indeed, otherwise we would have $\tilde{q} = q^2$. The equality $\tilde{q} = q^2$ cannot be resolved with respect to q if $\tilde{q} < 0$ thus contradicting the invertibility of the transformation (4.7). Similar arguments apply to $2\,a_{01}$ in (4.7). Therefore the following conditions are fulfilled:

$$a_{01} \neq 0, \qquad\qquad b_{10} \neq 0. \tag{4.8}$$

Relying on (4.8), we consider the scaling transformations

$$p = \alpha\,\check{p}, \qquad\qquad q = \beta\,\check{q}, \tag{4.9}$$

$$\hat{p} = \alpha^2\,\tilde{p}, \qquad\qquad \hat{q} = \beta^2\,\tilde{q}, \tag{4.10}$$

where α and β are given by the formulas

$$\alpha = 2\,\sqrt[3]{a_{01}^2\,b_{10}}, \qquad\qquad \beta = 2\,\sqrt[3]{a_{01}\,b_{10}^2}. \tag{4.11}$$

The transformations (4.9) and (4.10) are special instances of (2.13) and (4.5). Applying them to (4.7) and taking into account (4.11), we derive

$$\tilde{p} = p^2 + q, \qquad\qquad \tilde{q} = q^2 + p. \tag{4.12}$$

The second equality (4.12) can be resolved with respect to the variable p. It yields $p = \tilde{q} - q^2$. Substituting $p = \tilde{q} - q^2$ into the first equation (4.12), we derive the following quartic equation with respect to the variable q:

$$q^4 - 2\,\tilde{q}\,q^2 + q + \tilde{q}^2 - \tilde{p} = 0. \tag{4.13}$$

The invertibility of the transformation (4.12) means that for any real values of $\tilde{p}$ and $\tilde{q}$ the quartic equation (4.13) should have exactly one real root. Fortunately there is a criterion for a quartic equation with real coefficients to have exactly one real root. This criterion is given below by Theorem A.1 in Appendix A.

Theorem A.1 says that the discriminant of the equation (4.13) should be zero:

$$D_4 = 0. \tag{4.14}$$

The rest is to calculate the discriminant D_4 in (4.14) explicitly. Applying the formula (A.7) to the coefficients of the equation (4.13), we derive:

$$D_4 = -256\,\tilde{p}^3 + 256\,\tilde{q}^2\,\tilde{p}^2 + 288\,\tilde{q}\,\tilde{p} - 256\,\tilde{q}^3 - 27. \tag{4.15}$$

The discriminant (4.15) depends on $\tilde{p}$ and $\tilde{q}$. It is not identically zero. Therefore the equality (4.14) cannot be fulfilled for all real values of $\tilde{p}$ and $\tilde{q}$. This result is formulated as the following theorem.

Theorem 4.1. *There is no invertible quadratic transformation in $\mathbb{R}^2$ whose associated quadratic form $\omega_1(\mathbf{c})$ in (2.18) is indefinite.*

5. The case where $\omega_1(\mathbf{c})$ is semi-definite

Each semi-definite quadratic form in $\mathbb{R}^2$ up to a scalar factor has only one homogeneous vector $\mathbf{c}_1$, i. e. a vector in $\mathbb{R}^2$ such that $\omega_1(\mathbf{c}_1) = 0$. Choosing $\mathbf{c}_1$ for one of the two columns of the matrix (2.17), we can satisfy only one of the two relationships (2.14) or (2.15). Let's choose $\mathbf{c}_1$ for the first column of the matrix (2.17). Then the relationship (2.14) is fulfilled, while the relationship (2.15) is not fulfilled. Upon applying the transformation (2.13) to (2.1) we shall find that the first equality (2.12) is fulfilled, while the second equality (2.12) is broken:

$$\det\begin{Vmatrix} a_{20} & a_{11} \\ b_{20} & b_{11} \end{Vmatrix} = 0, \qquad\qquad \det\begin{Vmatrix} a_{11} & a_{02} \\ b_{11} & b_{02} \end{Vmatrix} \neq 0. \tag{5.1}$$

The second relationship (5.1) means that the vectors

$$\begin{Vmatrix} a_{02} \\ b_{02} \end{Vmatrix}, \qquad\qquad \begin{Vmatrix} a_{11} \\ b_{11} \end{Vmatrix}$$

are linearly independent. Hence they are nonzero. This means that $a_{11} \neq 0$ or $b_{11} \neq 0$. Applying the transformation (2.4) if needed, we can assume that $a_{11} \neq 0$. Hence we can apply (2.3) and derive $b_{11} = 0$ after that. The first relationship (5.1) means that we shall derive $b_{20} = 0$ along with $b_{11} = 0$, i. e. the relationships (2.6) will be fulfilled. Once they are fulfilled, we find that (2.1) is transformed to

$$\begin{aligned} \tilde{p} &= a_{20}\,p^2 + 2\,a_{11}\,p\,q + a_{02}\,q^2 + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00}, \\ \tilde{q} &= b_{02}\,q^2 + 2\,b_{10}\,p + 2\,b_{01}\,q + b_{00}. \end{aligned} \tag{5.2}$$

The coefficient b_{02} in (5.2) is nonzero, since otherwise we return to the previous case where the quadratic form $\omega_1(\mathbf{c})$ is zero:

$$b_{02} \neq 0. \tag{5.3}$$

Using (5.3) and composing (5.2) with the linear transformation

$$\hat{p} = p - \frac{a_{02}}{b_{02}}\,q, \qquad\qquad \hat{q} = q,$$

we can bring the transformation (5.2) to the form with $a_{02} = 0$:

$$\begin{aligned} &\tilde{p} = a_{20}\,p^2 + 2\,a_{11}\,p\,q + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ &\tilde{q} = b_{02}\,q^2 + 2\,b_{10}\,p + 2\,b_{01}\,q + b_{00}. \end{aligned} \tag{5.4}$$

Let's apply (5.4) to (2.18). The first determinant in (2.18) vanishes. The second and the third determinants for (5.4) are calculated explicitly:

$$\begin{vmatrix} a_{20} & a_{11}\\ b_{20} & b_{11}\end{vmatrix} = 0, \qquad \begin{vmatrix} a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix} = a_{20}\,b_{02}, \qquad \begin{vmatrix} a_{11} & a_{02}\\ b_{11} & b_{02}\end{vmatrix} = a_{11}\,b_{02}. \tag{5.5}$$

Due to (5.5) and (2.18) the quadratic form $\omega_1(\mathbf{c})$ is presented by the matrix

$$\Omega_1 = \begin{Vmatrix} 0 & \dfrac{a_{20}\,b_{02}}{2}\\[2ex] \dfrac{a_{20}\,b_{02}}{2} & a_{11}\,b_{02}\end{Vmatrix}, \qquad\qquad \det\Omega_1 = -\frac{a_{20}^2\,b_{02}^2}{4}. \tag{5.6}$$

It is known that a quadratic form in $\mathbb{R}^2$ is semi-definite if and only if its matrix is nonzero, but the determinant of its matrix is zero (see [75]). Applying this fact to (5.6) and taking into account (5.3), we derive

$$a_{20} = 0, \qquad\qquad a_{11} \neq 0. \tag{5.7}$$

Due to (5.7) the transformation (5.4) looks like

$$\begin{aligned} &\tilde{p} = 2\,a_{11}\,p\,q + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ &\tilde{q} = b_{02}\,q^2 + 2\,b_{10}\,p + 2\,b_{01}\,q + b_{00}. \end{aligned} \tag{5.8}$$

By means of scaling transformations and origin shifts we can bring (5.8) to

$$\begin{aligned} &\tilde{p} = p\,q + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ &\tilde{q} = q^2 + 2\,b_{10}\,p. \end{aligned} \tag{5.9}$$

Using the same arguments as in deriving (5.9), we find that $b_{10} \neq 0$ in (5.9). Hence, applying proper scaling transformation, we can further simplify (5.9):

$$\begin{aligned} &\tilde{p} = p\,q + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ &\tilde{q} = q^2 + p. \end{aligned} \tag{5.10}$$

Then by means of origin shifts in p, $\tilde{q}$, and $\tilde{p}$ we can annul a_{01} and a_{00} in (5.10):

$$\begin{aligned} \tilde{p} &= p\,q + 2\,a_{10}\,p, \\ \tilde{q} &= q^2 + p. \end{aligned} \tag{5.11}$$

The coefficient $2\,a_{10}$ in (5.11) can be either zero or nonzero. In both cases we can resolve the second equality (5.11) with respect to p:

$$p = \tilde{q} - q^2. \tag{5.12}$$

Substituting (5.12) back to the first equality (5.11), we derive the cubic equation

$$q^3 + 2\,a_{10}\,q^2 - \tilde{q}\,q - 2\,a_{10}\,\tilde{q} + \tilde{p} = 0 \tag{5.13}$$

for the variable q. The invertibility of the transformation (5.11) means that for any real values of $\tilde{p}$ and $\tilde{q}$ the cubic equation (4.14) should have exactly one real root. Fortunately there is a criterion for a general cubic equation with real coefficients

$$q^3 + \alpha_1\,q^2 + \alpha_2\,q + \alpha_3 = 0 \tag{5.14}$$

to have exactly one real root. It is formulated through its discriminant

$$D_3 = -27\,\alpha_3^2 + 18\,\alpha_3\,\alpha_1\,\alpha_2 + \alpha_1^2\,\alpha_2^2 - 4\,\alpha_1^3\,\alpha_3 - 4\,\alpha_2^3. \tag{5.15}$$

Theorem 5.1. *A cubic equation with real coefficients (5.14) has exactly one simple real root if and only if its discriminant (5.15) is negative: $D_3 < 0$.*

Theorem 5.1 is immediate from the following theorem.

Theorem 5.2. *A cubic equation with real coefficients (5.14) has three distinct real roots if and only if its discriminant (5.15) is positive: $D_3 > 0$.*

Indeed, the case $D_3 = 0$ is trivial. In this case the equation (5.14) has one real root (of multiplicity 3) or two real roots (one of which is simple and the other is double). As for Theorem 5.2, it is well-known. Its proof can be found in [78] (see Theorem 3.1 over there).

Now, applying Theorem 5.1 to the equation (5.13) and taking into account the case $D_3 = 0$ with one triple root, we write the inequality

$$D_3 \leqslant 0. \tag{5.16}$$

The rest is to calculate the discriminant D_3 of the equation (5.13) explicitly:

$$D_3 = 4\,\tilde{q}^3 - 32\,a_{10}^2\,\tilde{q}^2 + 8\,a_{10}\,(9\,\tilde{p} + 8\,a_{10}^3)\,\tilde{q} - \tilde{p}\,(32\,a_{10}^3 + 27\,\tilde{p}). \tag{5.17}$$

It is easy to see that the discriminant (5.17) is a cubic polynomial with respect to the variable $\tilde{q}$ with the non-vanishing leading coefficient 4. Such a polynomial takes values of both signs — positive and negative. Therefore the inequality (5.16) cannot be fulfilled for all $\tilde{p}$ and $\tilde{q}$.

Theorem 5.3. *There is no invertible quadratic transformation in $\mathbb{R}^2$ whose associated quadratic form $\omega_1(\mathbf{c})$ in (2.18) is semi-definite.*

5. The case where $\omega_1(\mathbf{c})$ is definite.

If $\omega_1(\mathbf{c})$ in (2.18) is definite, none of the equalities (2.14) and (2.15) can be fulfilled. Therefore we proceed to the equality (2.16). Looking at it, we define the second quadratic form $\omega_2(\mathbf{c})$ associated with a quadratic transformation of the form (2.1). Unlike $\omega_1(\mathbf{c})$, the form $\omega_2(\mathbf{c})$ is a quadratic form[1] in the four-dimensional vector space $\mathbb{R}^4$. It is given by the following formula:

$$\begin{gathered}\omega_2(\mathbf{c})=c_1\begin{vmatrix}a_{20} & a_{11}\\ b_{20} & b_{11}\end{vmatrix}c_3+c_1\begin{vmatrix}a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix}c_4+\\ +\,c_2\begin{vmatrix}a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix}c_3+c_2\begin{vmatrix}a_{11} & a_{02}\\ b_{11} & b_{02}\end{vmatrix}c_4.\end{gathered}\tag{6.1}$$

Here c_1, c_2, c_3, c_4 are the coordinates of a vector $\mathbf{c}\in\mathbb{R}^4$.

The determinant of the matrix C from (2.17) produces another quadratic form in R^4. It is given by the following formula:

$$\omega_3(\mathbf{c})=2\,c_1\,c_4-\,2\,c_2\,c_3=2\,\det C.\tag{6.2}$$

Let's denote through $2\,\alpha$, $2\,\beta$, and $2\,\gamma$ the determinants in (6.1) and (2.18):

$$2\,\alpha=\begin{vmatrix}a_{20} & a_{11}\\ b_{20} & b_{11}\end{vmatrix},\qquad 2\,\beta=\begin{vmatrix}a_{20} & a_{02}\\ b_{20} & b_{02}\end{vmatrix},\qquad 2\,\gamma=\begin{vmatrix}a_{11} & a_{02}\\ b_{11} & b_{02}\end{vmatrix}.\tag{6.3}$$

Then, like in (5.6), we can write the matrix presentation of (2.18):

$$\Omega_1=\begin{Vmatrix}2\,\alpha & \beta\\ \beta & 2\,\gamma\end{Vmatrix}.\tag{6.4}$$

The forms (6.1) and (6.2) have the following matrix presentations

$$\Omega_2=\begin{Vmatrix}0 & 0 & \alpha & \beta\\ 0 & 0 & \beta & \gamma\\ \alpha & \beta & 0 & 0\\ \beta & \gamma & 0 & 0\end{Vmatrix},\qquad\qquad \Omega_3=\begin{Vmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{Vmatrix}.$$

It is known that a quadratic form in $\mathbb{R}^2$ is definite if and only if the diagonal elements of its matrix are nonzero and of the same sign, and if the determinant of its matrix is positive (see Silvester's criterion in [75]). For (6.4) this yields

$$\alpha\,\gamma>0,\qquad\qquad 4\,\alpha\,\gamma-\beta^2>0.\tag{6.5}$$

The form $\omega_2(\mathbf{c})$ in (6.1) cannot be zero since otherwise the form $\omega_1(\mathbf{c})$ in (2.18) would be zero thus returning us to one of the previous cases. Note that there are

[1] Regular quadratic forms are tensors of the type $(0,2)$. Like $\omega_1(\mathbf{c})$, the quadratic form $\omega_2(\mathbf{c})$ is a pseudo-tensor of the type $(0,2)$ in the sense of Definition 2.1 in [76].

no squares of c_1, c_2, c_3, c_4 in (6.1). Hence the form $\omega_2(\mathbf{c})$ is either indefinite or semi-definite. In both cases there is at least one vector $\mathbf{c}_1 \in \mathbb{R}^4$ such that

$$\omega_2(\mathbf{c}_1) = 0. \tag{6.6}$$

Apart from (6.6), we need to fulfill the other condition

$$\omega_3(\mathbf{c}_1) \neq 0. \tag{6.7}$$

Therefore we shall calculate the components of $\mathbf{c}_1$ explicitly. Using the notations (6.3) and applying them to (6.1), we write (6.6) as

$$\alpha\, c_1\, c_3 + \beta\, c_1\, c_4 + \beta\, c_2\, c_3 + \gamma\, c_2\, c_4 = 0,$$

or equivalently

$$c_1\,(\alpha\, c_3 + \beta\, c_4) + c_2(\beta\, c_3 + \gamma\, c_4) = 0. \tag{6.8}$$

Let's choose the following values for c_1, c_2, c_3, c_4:

$$c_1 = \gamma, \qquad c_2 = -3\,\beta, \qquad c_3 = -2\,\beta, \qquad c_4 = \frac{3\,\beta^2 - \alpha\,\gamma}{\gamma}. \tag{6.9}$$

It is easy to see that (6.9) is a solution of the equation (6.8). Substituting (6.9) into (6.2) and taking into account (6.5), we derive

$$\omega_3(\mathbf{c}_1) = -2\,(\alpha\,\gamma + 3\,\beta^2) < -\frac{13}{2}\,\beta^2 < 0. \tag{6.10}$$

The inequality (6.10) means that the inequality (6.7) is fulfilled.

Now we use the quantities (6.9) as the components of the matrix C in (2.17):

$$\begin{aligned} &c_{11} = c_1 = \gamma, && c_{21} = c_2 = -3\,\beta,\\ &c_{12} = c_3 = -2\,\beta, && c_{22} = c_4 = \frac{3\,\beta^2 - \alpha\,\gamma}{\gamma}. \end{aligned} \tag{6.11}$$

The inequality (6.10) implying (6.7) means that $\det C \neq 0$. Therefore we can use the matrix C with the components (6.11) in (2.13). Upon applying the transformation (2.13) to (2.1) we shall find that the third equality is (2.12) fulfilled. As for the first two equalities (2.12), they are broken since otherwise the form $\omega_1(\mathbf{c})$ would not be definite thus returning us to one of the previous cases:

$$\det\begin{Vmatrix} a_{20} & a_{11}\\ b_{20} & b_{11}\end{Vmatrix} \neq 0, \qquad \det\begin{Vmatrix} a_{11} & a_{02}\\ b_{11} & b_{02}\end{Vmatrix} \neq 0, \qquad \det\begin{Vmatrix} a_{20} & a_{02}\\ b_{20} & b_{02}\end{Vmatrix} = 0. \tag{6.12}$$

The first relationship (6.12) means that the vectors

$$\begin{Vmatrix} a_{20}\\ b_{20}\end{Vmatrix}, \qquad\qquad \begin{Vmatrix} a_{11}\\ b_{11}\end{Vmatrix} \tag{6.13}$$

are linearly independent. Hence they are nonzero. This means that $a_{20} \neq 0$ or $b_{20} \neq 0$. Applying the transformation (2.4) if needed, we can assume that $a_{20} \neq 0$. Hence we can apply (2.10) and derive $b_{20} = 0$ after that. The last relationship (6.13) means that we shall derive $b_{02} = 0$ along with $b_{21} = 0$, i. e. the relationships (2.11) will be fulfilled. Once they are fulfilled, we find that (2.1) is transformed to

$$\begin{aligned} &\tilde{p} = a_{20}\,p^2 + 2\,a_{11}\,p\,q + a_{02}\,q^2 + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ &\tilde{q} = 2\,b_{11}\,p\,q + 2\,b_{10}\,p + 2\,b_{01}\,q + b_{00}. \end{aligned} \tag{6.14}$$

The coefficient b_{11} in (6.14) is nonzero, since otherwise we return to the previous case where the quadratic form $\omega_1(\mathbf{c})$ is zero:

$$b_{11} \neq 0. \tag{6.15}$$

Using (6.15) and composing (6.14) with a properly chosen linear transformation, we can bring the transformation (6.14) to the form with $a_{11} = 0$:

$$\begin{aligned} &\tilde{p} = a_{20}\,p^2 + a_{02}\,q^2 + 2\,a_{10}\,p + 2\,a_{01}\,q + a_{00},\\ &\tilde{q} = 2\,b_{11}\,p\,q + 2\,b_{10}\,p + 2\,b_{01}\,q + b_{00}. \end{aligned} \tag{6.16}$$

Let's apply (6.16) to (2.18). The second determinant in (2.18) vanishes. The first and the third determinants for (6.16) are calculated explicitly:

$$\begin{vmatrix} a_{20} & a_{11}\\ b_{20} & b_{11} \end{vmatrix} = a_{20}\,b_{11}, \qquad \begin{vmatrix} a_{20} & a_{02}\\ b_{20} & b_{02} \end{vmatrix} = 0, \qquad \begin{vmatrix} a_{11} & a_{02}\\ b_{11} & b_{02} \end{vmatrix} = -a_{02}\,b_{11}. \tag{6.17}$$

Due to (6.17) and (2.18) the quadratic form $\omega_1(\mathbf{c})$ is presented by the matrix

$$\Omega_1 = \left\| \begin{matrix} a_{20}\,b_{11} & 0\\ 0 & -a_{02}\,b_{11} \end{matrix} \right\|, \qquad \det\Omega_1 = -a_{20}\,a_{02}\,b_{11}^2. \tag{6.18}$$

It is known that a quadratic form in $\mathbb{R}^2$ is definite if and only if the diagonal elements of its matrix are nonzero and of the same sign, and if the determinant of its matrix is positive (see Silvester's criterion in [75]). Applying this fact to (6.18) and taking into account (6.15), we derive

$$a_{20} \neq 0, \qquad a_{02} \neq 0, \qquad a_{20}\,a_{02} < 0. \tag{6.19}$$

Using the inequalities (6.19), by means of scaling transformations and origin shifts we can bring (6.16) to the following form:

$$\tilde{p} = p^2 - q^2 + 2\,a_{10}\,p + 2\,a_{01}\,q, \qquad \tilde{q} = p\,q. \tag{6.20}$$

The second equality (6.20) is linear with respect to p. We can resolve it as

$$p = \frac{\tilde{q}}{q} \ \text{ if } \ q \neq 0. \tag{6.21}$$

Let's substitute (6.21) into the first equality (6.20). Removing denominators, we derive the following quartic equation with respect to q:

$$q^4 + 2\,a_{01}\,q^3 - \tilde{p}\,q^2 + 2\,a_{10}\,\tilde{q}\,q + \tilde{q}^2 = 0. \tag{6.22}$$

If $\tilde{q} \neq 0$, the exceptional value $q = 0$ is not a root of the quartic equation (6.22). For all $\tilde{p}$ and $\tilde{q} \neq 0$ the invertibility of the transformation (6.5) means that the quartic equation (6.22) has exactly one real root. Fortunately there is a criterion for a quartic equation with real coefficients to have exactly one real root. This criterion is given below by Theorem A.1 in Appendix A.

Theorem A.1 says that the discriminant of the equation (6.22) should be zero:

$$D_4 = 0. \tag{6.23}$$

The rest is to calculate the discriminant D_4 in (6.23) explicitly. Applying the formula (A.7) to the coefficients of the equation (6.22), we derive:

$$\begin{gathered} D_4 = 256\,\tilde{q}^6 - 768\,a_{01}\,a_{10}\,\tilde{q}^5 - (576\,\tilde{p}\,a_{01}^2 + 576\,a_{10}^2\,\tilde{p} + \\ + 432\,a_{01}^4 + 96\,a_{01}^2\,a_{10}^2 + 128\,\tilde{p}^2 + 432\,a_{10}^4)\,\tilde{q}^4 - (288\,a_{01}^3\,a_{10}\,\tilde{p} + \\ + 320\,a_{01}\,a_{10}\,\tilde{p}^2 + 256\,a_{01}^3\,a_{10}^3 + 288\,a_{01}\,a_{10}^3\,\tilde{p})\,\tilde{q}^3 + (16\,\tilde{p}^4 + \\ + 16\,\tilde{p}^3\,a_{10}^2 + 16\,\tilde{p}^2\,a_{01}^2\,a_{10}^2 + 16\,\tilde{p}^3\,a_{01}^2)\,\tilde{q}^2. \end{gathered} \tag{6.24}$$

As we see, the discriminant (6.24) is a polynomial of sixth degree. For $\tilde{q} \neq 0$ this polynomial is not identically zero. Therefore the equality (6.23) cannot be fulfilled for all $\tilde{p}$ and $\tilde{q} \neq 0$. This result leads to the following theorem.

Theorem 6.1. *There is no invertible quadratic transformation in $\mathbb{R}^2$ whose associated quadratic form $\omega_1(\mathbf{c})$ in (2.18) is definite.*

6. Conclusions.

Definition 6.1. Two quadratic mappings $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $f_2 : \mathbb{R}^2 \to \mathbb{R}^2$ are called equivalent if there are two invertible linear mappings $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $\varphi_2 \colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi_1 \circ f_1 = f_2 \circ \varphi_2$.

The main result of the present paper consist in subdividing all potentially invertible quadratic transformations of the real plane $\mathbb{R}^2$ into four groups and in finding up to the equivalence introduced in Definition 6.1 some pre-canonical presentations for quadratic transformations within these groups. These presentations are given by the formulas (3.5), (4.12), (5.11), and (6.20). Theorems 3.1, 4.1, 5.3, and 6.1 show that only transformations of the first of the four groups are actually invertible.

Some examples of quadratic transformations of $\mathbb{R}^2$ were studied in [79] as discrete dynamical systems. Some prospects of applying quadratic transformations to the perfect cuboid problem are discussed in the introductory section of this paper.

Appendix A. Quartic polynomials with exactly one real root.

Let $P_4(x)$ be a real quartic polynomial. Without loss of generality we can assume that $P_4(x)$ is monic, i. e. its leading coefficient is equal to one:

$$P_4(x) = x^4 + a_1\,x^3 + a_2\,x^2 + a_3\,x + a_4. \tag{A.1}$$

Our goal is to find a necessary and sufficient condition for the polynomial (A.1) to have a unique real root. The graph of such a polynomial is shown in Fig. A.1. Since $P_4(x) \to +\infty$ as $x \to \pm\infty$ the unique root x_0 of the polynomial (A.1) should be a root of multiplicity at least two. Hence we have

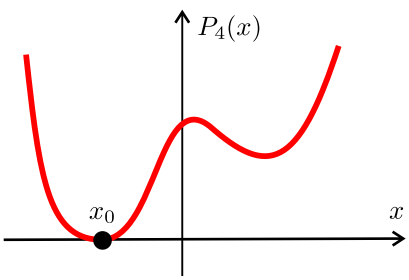

Fig. A.1

$$P_4(x) = (x - x_0)^2\,P_2(x). \tag{A.2}$$

Here $P_2(x)$ is a quadratic polynomial with real coefficients. We write it as

$$P_2(x) = x^2 + b_1\,x + b_2. \tag{A.3}$$

Substituting (A.3) into (A.2) and expanding the resulting expression we can express the coefficients of the polynomial $P_2(x)$ through a_1, a_2, and x_0:

$$b_1 = a_1 + 2\,x_0, \qquad\qquad b_2 = a_2 - x_0^2 + 2\,b_1\,x_0. \tag{A.4}$$

Moreover, we obtain two formulas expressing a_3 and a_4 through b_1, b_2, and x_0:

$$a_4 = b_2\,x_0^2, \qquad\qquad a_3 = b_1\,x_0^2 - 2\,b_2\,x_0. \tag{A.5}$$

The root x_0 in (A.2) is not simple. This means that the discriminant of the quartic polynomial $P_4(x)$ in (A.1) is equal to zero:

$$D_4 = 0. \tag{A.6}$$

The explicit formula for the discriminant D_4 in (A.6) looks like

$$\begin{gathered}
D_4 = 18\,a_1^3\,a_3\,a_2\,a_4 + 256\,a_4^3 - 6\,a_1^2\,a_3^2\,a_4 - 192\,a_1\,a_3\,a_4^2 + 18\,a_1\,a_3^3\,a_2 +\\
+ 144\,a_2\,a_1^2\,a_4^2 + a_2^2\,a_1^2\,a_3^2 - 4\,a_2^3\,a_1^2\,a_4 + 144\,a_4\,a_3^2\,a_2 - 4\,a_1^3\,a_3^3 - 27\,a_3^4 -\\
- 128\,a_2^2\,a_4^2 + 16\,a_2^4\,a_4 - 4\,a_2^3\,a_3^2 - 27\,a_1^4\,a_4^2 - 80\,a_1\,a_3\,a_2^2\,a_4.
\end{gathered} \tag{A.7}$$

The equality (A.6) is a necessary condition for the real quartic polynomial (A.1) to have a unique real root. But this condition is not sufficient.

The quadratic polynomial $P_2(x)$ in (A.2) should have at most one real root coinciding with x_0. Therefore its discriminant D_2 should be non-positive:

$$D_2 \leqslant 0. \tag{A.8}$$

The discriminant of the polynomial $P_2(x)$ is given by the formula

$$D_2 = b_1^2 - 4\,b_2. \tag{A.9}$$

Substituting (A.4) into (A.9), we derive the following formula:

$$D_2 = a_1^2 - 4\,a_2 - 8\,x_0^2 - 4\,x_0\,a_1. \tag{A.10}$$

The formula (A.10) comprises the root x_0. In order to write the inequality (A.8) in terms of the coefficients a_1, a_2, a_3, a_4 of the initial polynomial we need to express x_0 through them. Let's recall that the equality (A.6) means that x_0 is a common root of $P_4(x)$ and its first derivative $P_3(x) = P_4'(x)$:

$$P_3(x) = 4\,x^3 + 3\,a_1\,x^2 + 2\,a_2\,x + a_3. \tag{A.11}$$

Let's combine the polynomials (A.1) and (A.11) in the following way:

$$Q_3(x) = 4\,P_4(x) - x\,P_3(x). \tag{A.12}$$

It is easy to see that (A.12) is another cubic polynomial:

$$Q_3(x) = a_1\,x^3 + 2\,a_2\,x^2 + 3\,a_3\,x + 4\,a_4. \tag{A.13}$$

Using (A.13), we combine it with (A.11) as follows:

$$Q_2(x) = 4\,Q_3(x) - a_1\,P_3(x). \tag{A.14}$$

It is easy to see that (A.14) is a quadratic polynomial:

$$Q_2(x) = (8\,a_2 - 3\,a_1^2)\,x^2 + (12\,a_3 - 2\,a_1\,a_2)\,x + 16\,a_4 - a_1\,a_3. \tag{A.15}$$

Another quadratic polynomial is derived by means of the formula

$$R_2(x) = (8\,a_2 - 3\,a_1^2)\,P_3(x) - 4\,x\,Q_2(x). \tag{A.16}$$

The explicit formula for the polynomial (A.16) looks like

$$\begin{gathered} R_2(x) = (32\,a_1\,a_2 - 48\,a_3 - 9\,a_1^3)\,x^2 + (4\,a_1\,a_3 - 64\,a_4 - \\ - 6\,a_2\,a_1^2 + 16\,a_2^2)\,x + 8\,a_3\,a_2 - 3\,a_1^2\,a_3. \end{gathered} \tag{A.17}$$

Now we combine (A.15) and (A.17) by means of the formula

$$P_1(x) = \frac{8\,a_2 - 3\,a_1^2}{16}\,R_2(x) - \frac{32\,a_1\,a_2 - 48\,a_3 - 9\,a_1^3}{16}\,Q_2(x). \tag{A.18}$$

It is easy to see that (A.18) is a linear polynomial:

$$\begin{gathered} P_1(x) = (8\,a_2^3 + 36\,a_3^2 + 6\,a_1^3\,a_3 - 32\,a_2\,a_4 - 2\,a_1^2\,a_2^2 + \\ + 12\,a_1^2\,a_4 - 28\,a_1\,a_2\,a_3)\,x - 3\,a_1\,a_3^2 + 48\,a_4\,a_3 + 9\,a_1^3\,a_4 + \\ + 4\,a_3\,a_2^2 - a_3\,a_2\,a_1^2 - 32\,a_1\,a_4\,a_2. \end{gathered} \tag{A.19}$$

Due to $P_4(x_0)=0$ and $P_3(x_0)=0$, from (A.12), (A.14), (A.16), and (A.18) we derive that x_0 is a root of the linear polynomial (A.19):

$$P_1(x_0)=0. \tag{A.20}$$

Let's denote through A_0 the leading coefficient of the polynomial $P_1(x)$:

$$A_0=8\,a_2^3+36\,a_3^2+6\,a_1^3\,a_3-32\,a_2\,a_4-2\,a_1^2\,a_2^2+12\,a_1^2\,a_4-28\,a_1\,a_2\,a_3. \tag{A.21}$$

Similarly, lets denote through A_1 the constant term of the polynomial $P_1(x)$:

$$A_1=-3\,a_1\,a_3^2+48\,a_4\,a_3+9\,a_1^3\,a_4+4\,a_3\,a_2^2-a_3\,a_2\,a_1^2-32\,a_1\,a_4\,a_2. \tag{A.22}$$

Due to (A.21) and (A.22) the equation (A.20) is written as

$$A_0\,x_0+A_1=0. \tag{A.23}$$

Let's begin with the case $A_0\neq 0$. In this case we can resolve the linear equation (A.23) with respect to the variable x_0. Let's substitute

$$x_0=-\frac{A_1}{A_0} \tag{A.24}$$

into the formula (A.10) for the discriminant D_2. As a result we get the fraction

$$D_2=\frac{4\,B_2}{A_0^2}, \tag{A.25}$$

where the numerator term B_2 is given by the following explicit formula:

$$\begin{gathered}
B_2=552\,a_2^2\,a_1^4\,a_3^2-30\,a_1^6\,a_4\,a_2^2-64\,a_2^7+2208\,a_1\,a_3^3\,a_2^2-\\
-616\,a_2^3\,a_1^2\,a_3^2-704\,a_2^4\,a_1^2\,a_4+264\,a_2^3\,a_1^4\,a_4+1536\,a_4\,a_3^2\,a_2^2-\\
-336\,a_1^3\,a_2^4\,a_3+480\,a_1\,a_2^5\,a_3+78\,a_1^5\,a_2^3\,a_3-900\,a_2\,a_1^3\,a_3^3+\\
+144\,a_2\,a_1^4\,a_4^2-126\,a_2\,a_1^6\,a_3^2+900\,a_1^4\,a_4\,a_3^2-1152\,a_1^3\,a_4^2\,a_3+\\
+2304\,a_1\,a_3^3\,a_4-1296\,a_2\,a_3^4-1024\,a_2^3\,a_4^2+512\,a_2^5\,a_4-\\
-608\,a_2^4\,a_3^2-12\,a_1^4\,a_2^5+48\,a_1^2\,a_2^6-18\,a_1^6\,a_4^2+198\,a_1^2\,a_3^4-\\
-4608\,a_4^2\,a_3^2+90\,a_1^5\,a_3^3+a_1^6\,a_2^4+9\,a_1^8\,a_3^2+2112\,a_1^3\,a_3\,a_2^2\,a_4-\\
-1024\,a_1\,a_3\,a_2^3\,a_4-4032\,a_2\,a_1^2\,a_3^2\,a_4-828\,a_2\,a_1^5\,a_4\,a_3+\\
+4608\,a_4^2\,a_3\,a_1\,a_2+90\,a_1^7\,a_4\,a_3-6\,a_1^7\,a_2^2\,a_3.
\end{gathered} \tag{A.26}$$

Since the denominator in (A.25) is positive, the inequality (A.8) can be written as

$$B_2\leqslant 0. \tag{A.27}$$

Summarizing the above calculations, now we can formulate a lemma.

Lemma A.1. *If a quartic polynomial $P_4(x)$ with real coefficients in* (A.1) *has exactly one real root and if $A_0 \neq 0$ in* (A.21)*, then the discriminant $D_4 = 0$ in* (A.7) *and $B_2 \leqslant 0$ in* (A.26)*.*

The case $B_2 = 0$ in (A.27) or, equivalently, the case $D_2 = 0$ in (A.8) is exceptional. Omitting this case, we can formulate a lemma converse to Lemma A.1.

Lemma A.2. *If $D_4 = 0$ in* (A.7)*, if $A_0 \neq 0$ in* (A.21) *and if $B_2 < 0$ in* (A.26)*, then the quartic polynomial $P_4(x)$ with real coefficients in* (A.1) *has exactly one real root of multiplicity* 2*.*

Let's proceed to the case $A_0 = 0$ assuming the condition $D_4 = 0$ in (A.6) is fulfilled. The condition $D_4 = 0$ means that the polynomial $P_4(x)$ has a root x_0 of multiplicity at least two. Hence we can apply (A.2) and the formulas (A.4) and (A.5) following from (A.2) and (A.3). The formulas (A.4) can be written as

$$a_1 = b_1 - 2\,x_0, \qquad\qquad a_2 = b_2 + x_0^2 - 2\,b_1\,x_0. \tag{A.28}$$

Substituting (A.28) and (A.5) into (A.21), we derive

$$A_0 = -2\,(b_1^2 - 4\,b_2)\,(x_0^2 + b_1\,x_0 + b_2)^2. \tag{A.29}$$

Comparing the formula (A.29) with the formulas (A.3) and (A.9), we see that the equality $A_0 = 0$ implies at least one of the following two equalities:

$$D_2 = 0, \qquad\qquad P_2(x_0) = 0. \tag{A.30}$$

The equality $D_2 = 0$ means that the quadratic polynomial $P_2(x)$ has a double root:

$$P_2(x) = (x - x_1)^2. \tag{A.31}$$

The equality $P_2(x_0) = 0$ means that x_0 is one of two roots of the quadratic polynomial $P_2(x)$, i. e. the polynomial $P_2(x)$ is written as

$$P_2(x) = (x - x_0)\,(x - x_1). \tag{A.32}$$

In both cases (A.31) and (A.32) the quartic polynomial $P_4(x) = (x - x_0)^2\,P_2(x)$ has exactly one real root if and only if $x_1 = x_0$. Applying $x_1 = x_0$ back to (A.31) and (A.32), we derive $D_2 = 0$ and $P_2(x_0) = 0$, i. e. both equalities (A.30) are fulfilled. In other words we have the following lemma.

Lemma A.3. *If $A_0 = 0$ in* (A.21)*, then the equality $D_4 = 0$ in* (A.7) *and the couple of equalities* (A.30) *constitute a necessary and sufficient condition for the quartic polynomial $P_4(x)$ with real coefficients in* (A.1) *to have exactly one real root, which is of multiplicity* 4 *in this case.*

Note that the equalities (A.30) are not written in terms of the coefficients a_1, a_2, a_3, a_4 of the polynomial $P_4(x)$. In order to write them properly we apply $x_1 = x_0$ to the equalities (A.31) and (A.32) and derive from them

$$P_2(x) = (x - x_0)^2, \qquad\qquad P_4(x) = (x - x_0)^4. \tag{A.33}$$

The equalities (A.33) mean that the polynomials $P_2(x)$ and $x - x_0$ are expressed through the derivatives of the polynomial $P_4(x)$:

$$P_2(x) = \frac{P_4''(x)}{12}, \qquad\qquad x - x_0 = \frac{P_4'''(x)}{24}. \tag{A.34}$$

Substituting (A.1) into (A.34), we derive

$$P_2(x) = x^2 + \frac{a_1}{2}\,x + \frac{a_2}{6} \qquad\qquad x - x_0 = x + \frac{a_1}{4}. \tag{A.35}$$

Using (A.3), from (A.35) we derive the formulas for b_1, b_2, and x_0:

$$b_1 = \frac{a_1}{2}, \qquad\qquad b_2 = \frac{a_2}{6}, \qquad\qquad x_0 = -\frac{a_1}{4}. \tag{A.36}$$

The third formula (A.36) replaces the formula (A.24) in the present case $A_0 = 0$. Substituting it into (A.10), we derive the formula for D_2:

$$D_2 = \frac{3}{2}\,a_1^2 - 4\,a_2. \tag{A.37}$$

The formula (A.37) replaces the formula (A.25) in the present case $A_0 = 0$.

In order to express $P_2(x_0)$ through a_1, a_2, a_3, a_4 we first apply (A.4) to (A.3). As a result we get the following expression for $P_2(x)$:

$$P_2(x) = x^2 + (a_1 + 2\,x_0)\,x + a_2 + 3\,x_0^2 + 2\,x_0\,a_1. \tag{A.38}$$

Then we substitute $x = x_0$ into (A.38) and get

$$P_2(x_0) = 6\,x_0^2 + 3\,x_0\,a_1 + a_2. \tag{A.39}$$

And finally we apply the third equality (A.36) to (A.39). This yields

$$P_2(x_0) = -\frac{3}{8}\,a_1^2 + a_2. \tag{A.40}$$

Comparing (A.40) with (A.37), we see that two equalities (A.30) become equivalent to each other. Lemma A.3 now is reformulated as follows.

Lemma A.4. *If $A_0 = 0$ in (A.21), then the equality $D_4 = 0$ in (A.7) and the equality $D_2 = 0$ in (A.37) constitute a necessary and sufficient condition for the quartic polynomial $P_4(x)$ with real coefficients in (A.1) to have exactly one real root, which is of multiplicity 4 in this case.*

Theorem A.1. *A quartic polynomial $P_4(x)$ with real coefficients in (A.1) has exactly one real root if and only if its discriminant $D_4 = 0$ in (A.7) and if one of the following two conditions is fulfilled:*

1) *$A_0 \neq 0$ in (A.21) and $B_2 < 0$ in (A.26);*
2) *$A_0 = 0$ in (A.21) and $D_2 = 0$ in (A.37).*

Theorem A.1 is the ultimate result. It summarizes Lemmas A.1, A.2, and A.4. and provides a necessary and sufficient condition for a real quartic polynomial to have exactly one real root.

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