ON SOME HIGHER DEGREE SIGN-DEFINITE MULTIVARIATE POLYNOMIALS ASSOCIATED WITH DEFINITE QUADRATIC FORMS.

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Abstract. Positive and negative quadratic forms are well known and widely used. They are multivariate homogeneous polynomials of degree two taking positive or negative values respectively for any values of their arguments not all zero. In the present paper a certain higher degree polynomial is associated with each quadratic form such that the form is definite if and only if this polynomial is sign-definite.

1. Introduction.

Quartic and higher order positive polynomials are of growing interest (see [1–5]). Trivial examples of them are constructed as sums of squares. But in general the polynomial non-negativity is an NP-hard problem (see [6], [7]). For this reason any examples of higher degree sign-definite polynomials are worthwhile.

Let $a(x^1,\ldots,x^n)$ be a quadratic form of n variables[1]. It is given by the formula

$$a(x^1,\ldots,x^n)=\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\,x^i\,x^j. \tag{1.1}$$

The form (1.1) is associated with its matrix

$$A=\begin{Vmatrix} a_{11} & \ldots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \ldots & a_{nn}\end{Vmatrix} \tag{1.2}$$

which is symmetric, i.e. $a_{ij}=a_{ji}$. If $x^1,\ldots,x^n$ are interpreted as components of a vector in some linear vector space V, then a_{ij} are components of a twice covariant tensor. Under a linear change of variables

$$x^i=\sum_{j=1}^{n}S^i_j\,\tilde x^j, \tag{1.3}$$

which is interpreted as a change of basis in V, the matrix (1.2) is transformed as

$$\tilde A=S^{\top}A\,S. \tag{1.4}$$

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[1] Upper indices for numerating variables in (1.1) are used according to Einstein's tensorial notation, see [8].

Here S is the transition matrix (see [8] or [9]). Its components are used in (1.3).

Let Λ be a skew-symmetric matrix of the same size as the matrix A in (1.2):

$$\Lambda=\begin{Vmatrix} 0 & \lambda_{12} & \dots & \lambda_{1n}\\ -\lambda_{12} & 0 & \dots & \lambda_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ -\lambda_{1n} & -\lambda_{2n} & \dots & 0\end{Vmatrix}. \tag{1.5}$$

Using the matrices (1.2) and (1.5), we define the polynomial

$$P(\lambda_{12},\dots,\lambda_{n-1\,n})=P(\Lambda)=\det(A-\Lambda). \tag{1.6}$$

For the definition (1.6) to be coordinate covariant we set

$$\tilde\Lambda=S^{\top}\Lambda\,S, \tag{1.7}$$

which is the same transformation rule as (1.4). From (1.4) and (1.7) we derive

$$P(\tilde\Lambda)=(\det S)^2\,P(\Lambda). \tag{1.8}$$

The components $\lambda_{12},\dots,\lambda_{n-1\,n}$ of the matrix Λ in (1.6) are interpreted as independent variables, i.e. as arguments of the polynomial $P(\Lambda)$. The polynomial $P(\Lambda)$ in (1.6) is of degree two in each particular variable λ_{ij}. However, its total degree is typically higher than two.

Note that the formula (1.6) is somewhat similar to the formula of the characteristic polynomial of a matrix. Therefore below we shall call $P(\Lambda)$ the skew-characteristic polynomial of the form (1.1). Studying some properties of this polynomial is the main goal of the present paper.

2. Proving the positivity.

Theorem 2.1. *If a quadratic form with the matrix A is positive, then its associated skew-characteristic polynomial $P(\Lambda)=\det(A-\Lambda)$ is positive, i. e. $P(\Lambda)>0$ for any skew-symmetric matrix Λ.*

It is known that the matrix of a positive quadratic form can be brought to the unit matrix at the expense of linear transformations of the form (1.3), i.e. in a proper basis (see [9]). Therefore, relying on (1.8), without loss of generality we can choose $A=\mathbf{1}$ and consider some examples.

The case n=2. In this case we easily calculate

$$P(\Lambda)=\begin{vmatrix} 1 & -\lambda_{12}\\ \lambda_{12} & 1\end{vmatrix}=1+\lambda_{12}^2>0. \tag{2.1}$$

The case n=3. This case is similar to the previous one:

$$P(\Lambda)=\begin{vmatrix} 1 & -\lambda_{12} & -\lambda_{13}\\ \lambda_{12} & 1 & -\lambda_{23}\\ \lambda_{13} & \lambda_{23} & 1\end{vmatrix}=1+\lambda_{12}^2+\lambda_{13}^2+\lambda_{23}^2>0. \tag{2.2}$$

The case n=4. This case is a little bit more complicated than (2.1) and (2.2):

$$
\begin{aligned}
P(\Lambda) = 1 + \lambda_{12}^{\,2} + \lambda_{13}^{\,2} + \lambda_{14}^{\,2} + \lambda_{23}^{\,2} + \lambda_{24}^{\,2} + \lambda_{34}^{\,2} + \\
+ (\lambda_{12}\,\lambda_{34} + \lambda_{23}\,\lambda_{14} - \lambda_{13}\,\lambda_{24})^{\,2} > 0.
\end{aligned}
\tag{2.3}
$$

Taking (2.1), (2.2), and (2.3) as a background, we proceed to proving Theorem 2.1.

Proof of Theorem 2.1. Interpreting $x^1, \ldots, x^n$ in (1.1) as the coordinates of a vector $\mathbf{x} \in V$, we can associate a symmetric bilinear form with the matrix A:

$$
a(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x^i\, y^j. \tag{2.4}
$$

The matrix Λ in (1.5) is not associated with a quadratic form. However, it is associated with a skew-symmetric bilinear form:

$$
\lambda(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_{ij}\, x^i\, y^j. \tag{2.5}
$$

Due to (2.4) and (2.5) the matrix $A - \Lambda$ in (1.6) is associated with the bilinear form

$$
b(\mathbf{x}, \mathbf{y}) = a(\mathbf{x}, \mathbf{y}) - \lambda(\mathbf{x}, \mathbf{y}), \tag{2.6}
$$

which is neither symmetric nor skew-symmetric.

Assume that the matrix A of the positive quadratic form $a(x^1, \ldots, x^n) = a(\mathbf{x}, \mathbf{x})$ is brought to the unit matrix by choosing some proper basis in V. Then for $\Lambda = 0$ in (1.6) we have the following inequality for $P(\Lambda)$:

$$
P(\Lambda) = P(0) = \det(\mathbf{1}) = 1 > 0. \tag{2.7}
$$

Further we shall prove that the polynomial $P(\Lambda)$ cannot vanish. The proof is by contradiction. Indeed, if $P(\Lambda) = 0$, then $\det(A - \Lambda) = 0$ and $A - \Lambda$ is a degenerate matrix. This means that the form (2.6) has a nonzero kernel[1]:

$$
\operatorname{Ker} b = \{\mathbf{x} \in V : \ b(\mathbf{x}, \mathbf{y}) = 0 \ \forall\, y \in V\} \neq \{\mathbf{0}\}. \tag{2.8}
$$

Let $\mathbf{x} \neq 0$ be a vector belonging to the kernel (2.8). Then

$$
b(\mathbf{x}, \mathbf{x}) = 0. \tag{2.9}
$$

[1] Actually the form b has two kernels — the left kernel and the right kernel, both being nonzero. We choose the left kernel in (2.8) for the sake of certainty.

Applying (2.6) to (2.9) and taking into account that $\lambda(\mathbf{x},\mathbf{x})=0$ since the bilinear form λ in (2.5) is skew-symmetric, we derive

$$a(\mathbf{x},\mathbf{x})=b(\mathbf{x},\mathbf{x})+\lambda(\mathbf{x},\mathbf{x})=0+0=0.$$

But the equality $a(\mathbf{x},\mathbf{x})=0$ for $\mathbf{x}\neq 0$ contradicts the positivity of the form a. The contradiction obtained proves that $P(\Lambda)$ cannot vanish.

Thus we know that the polynomial $P(\Lambda)$ is a continuous function of its arguments which is positive for $\Lambda=0$ due to (2.7) and which never vanishes. Therefore $P(\Lambda)$ is always positive. Theorem 2.1 is proved. □

3. A criterion of definiteness.

Theorem 3.1. *A quadratic form with the matrix A is definite if and only if its associated skew-characteristic polynomial $P(\Lambda)=\det(A-\Lambda)$ is sign-definite.*

Proof. Assume that the form a is definite. Then it is either positive or negative. If a is a positive form with the matrix A, then $P(\Lambda)>0$, which follows from Theorem 2.1. If a is negative, then the form $-a$ is positive. For this form we derive

$$P_{-a}(-\Lambda)=\det(-A+\Lambda)=(-1)^n\,\det(A-\Lambda)=(-1)^n\,P(\Lambda). \tag{3.1}$$

Applying Theorem 2.1 to (3.1), we find that $P(\Lambda)>0$ if the dimension $n=\dim V$ is even and $P(\Lambda)<0$ if n is odd. In both cases $P(\Lambda)$ is sign-definite. This means that the necessity is proved.

The proof of the sufficiency is by contradiction. Assume that $P(\Lambda)$ is sign-definite but the form a is not definite. Then a is either degenerate or non-degenerate. If a is degenerate, then $\det A=0$. Choosing $\Lambda=0$ in (1.6), we get $P(0)=\det(A)=0$. The equality $P(0)=0$ contradicts both $P(\Lambda)>0$ and $P(\Lambda)<0$.

If a is non-degenerate and indefinite, then its signature is $(m,n-m)$, where $m\neq 0$ and $n-m\neq 0$. In this case by mean of some proper choice of basis in V we can bring the matrix A to the following diagonal form:

$$A=\begin{Vmatrix} 1 & \dots & 0 & 0 & \dots & 0\\ \vdots & \ddots & \vdots & \vdots & & \vdots\\ 0 & \dots & 1 & 0 & \dots & 0\\ 0 & \dots & 0 & -1 & \dots & 0\\ \vdots & & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & 0 & \dots & -1 \end{Vmatrix}. \tag{3.2}$$

Relying on (3.2), we choose the following skew-symmetric matrix Λ:

$$\Lambda=\begin{Vmatrix} 0 & \dots & 0 & 0 & \dots & 0\\ \vdots & \ddots & \vdots & \vdots & & \vdots\\ 0 & \dots & 0 & \lambda_{m\,m+1} & \dots & 0\\ 0 & \dots & -\lambda_{m\,m+1} & 0 & \dots & 0\\ \vdots & & \vdots & \vdots & \ddots & \vdots\\ 0 & \dots & 0 & 0 & \dots & 0 \end{Vmatrix}. \tag{3.3}$$

Substituting (3.2) and (3.3) into (1.6) we derive

$$P(\Lambda) = (-1)^{n-m-1}\,(-1+\lambda^{\,2}_{m\,m+1}). \tag{3.4}$$

It is easy to see that the polynomial (3.4) is not sign-definite, which is again a contradiction. Thus, Theorem 3.1 is proved. □

Theorem 3.1 is the main result of the present paper. It can be further used as a background in deriving definiteness criteria for quartic and higher order forms.

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