

ON POSITIVE BIVARIATE QUARTIC FORMS.

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ABSTRACT. A bivariate quartic form is a homogeneous bivariate polynomial of degree four. A criterion of positivity for such a form is known. In the present paper this criterion is reformulated in terms of pseudotensorial invariants of the form.

1. INTRODUCTION.

Let $a(x^1, x^2)$ be a bivariate quartic form¹. It is given by the following formula:

$$a(x^1, x^2) = A_{1111} (x^1)^4 + 4 A_{1112} (x^1)^3 x^2 + 6 A_{1122} (x^1)^2 (x^2)^2 + 4 A_{1222} x^1 (x^2)^3 + A_{2222} (x^2)^4. \quad (1.1)$$

Interpreting x^1 and x^2 as coordinates of a vector \mathbf{x} in some two-dimensional vector space V , one can find that A_{1111} , A_{1112} , A_{1122} , A_{1222} , A_{2222} are components of a symmetric tensor A of the type $(0, 4)$. Then the formula (1.1) is written as

$$a(x^1, x^2) = \sum_{i_1=1}^2 \sum_{i_2=1}^2 \sum_{i_3=1}^2 \sum_{i_4=1}^2 A_{i_1 i_2 i_3 i_4} x^{i_1} x^{i_2} x^{i_3} x^{i_4}. \quad (1.2)$$

The form $a(x^1, x^2)$ in (1.1) and in (1.2) is called positive if it takes positive values for all values of its arguments x^1 and x^2 not both zero. The inequalities

$$A_{1111} > 0, \quad A_{2222} > 0. \quad (1.3)$$

are necessary conditions for the positivity of the form (1.1). However they are not sufficient. Necessary and sufficient conditions for the positivity of the form (1.1) do exist. They are derived from L. E. Dickson's and E. L. Rees's results concerning a univariate quartic polynomial (see [2] and [3]). The main goal of the present paper is to express these necessary and sufficient conditions in terms of some pseudotensorial invariants associated with the tensor A whose components are used in (1.1).

2. DICKSON'S AND REES'S RESULTS.

In [2] L. E. Dickson considers a quartic equation in its reduced form with $r \neq 0$:

$$z^4 + q z^2 + r z + s = 0. \quad (2.1)$$

Among other things, in his book [2] one can find the following result.

2000 *Mathematics Subject Classification.* 15A48, 14L24, 11E76.

¹ Upper indices for numerating the variables x^1 and x^2 in (1.1) are used according to Einstein's tensorial notation, see [1].

Theorem 2.1. *The reduced quartic equation (2.1) with real coefficients such that $r \neq 0$ has no real roots if and only if its discriminant $D_4 > 0$ and one of the following two conditions is fulfilled:*

- 1) $4s \geq q^2$;
- 2) $4s < q^2$ and $q \geq 0$.

The discriminant D_4 of the reduced quartic equation (2.1) is given by the formula

$$D_4 = 256s^3 - 4q^3r^2 - 27r^4 + 16q^4s - 128q^2s^2 + 144qsr^2. \quad (2.2)$$

The case $r = 0$ was considered by E. L. Rees in [3]. In this case we have.

Theorem 2.2. *The reduced quartic equation (2.1) with real coefficients such that $r = 0$ has no real roots if and only if one of the following two conditions is fulfilled:*

- 1) $4s > q^2$;
- 2) $0 < 4s \leq q^2$ and $q > 0$.

The case $r = 0$ is rather simple. In this case, denoting $z^2 = y$, one can reduce the equation (2.1) to the following quadratic equation:

$$y^2 + qy + s = 0. \quad (2.3)$$

The discriminant of the quadratic equation (2.3) is given by the formula

$$D_2 = q^2 - 4s. \quad (2.4)$$

In the case $r = 0$ the discriminant (2.2) of the quartic equation (2.1) reduces to

$$D_4 = 16s(q^2 - 4s)^2. \quad (2.5)$$

Comparing (2.5) with two conditions in Theorem 2.2, we see that the first of them implies $D_4 > 0$, while the second one implies $D_4 \geq 0$.

3. BRINGING TO REDUCED QUARTIC POLYNOMIALS.

Note that the reduced quartic polynomial in the left hand side of (2.1) is positive if and only if the equation (2.1) has no real roots. Therefore bringing the form (1.1) to a reduced quartic polynomial is a method for testing its positivity.

Since x^1 and x^2 are not both zero in positivity tests, we consider two cases — the case $x^1 \neq 0$ and the case $x^2 \neq 0$. If $x^2 \neq 0$, dividing $a(x^1, x^2)$ by $(x^2)^4 > 0$ and denoting $t = x^1/x^2$, we derive the following polynomial from (1.1):

$$P_1(t) = A_{1111}t^4 + 4A_{1112}t^3 + 6A_{1122}t^2 + 4A_{1222}t + A_{2222}. \quad (3.1)$$

If $x^1 \neq 0$, dividing $a(x^1, x^2)$ by $(x^1)^4 > 0$ and denoting $t = x^2/x^1$, we get

$$P_2(t) = A_{1111} + 4A_{1112}t + 6A_{1122}t^2 + 4A_{1222}t^3 + A_{2222}t^4. \quad (3.2)$$

The discriminants D_{P_1} and D_{P_2} of the polynomials (3.1) and (3.2) do coincide.

They determine the quantity I_0 by means of the following formula

$$I_0 = \frac{DP_1}{256} = \frac{DP_2}{256}. \quad (3.3)$$

Writing the formula (3.3) explicitly, we get

$$\begin{aligned} I_0 = & 81 A_{1111} (A_{1122})^4 A_{2222} - 18 (A_{1111})^2 (A_{2222})^2 (A_{1122})^2 - \\ & - 27 (A_{1112})^4 (A_{2222})^2 - 12 (A_{1111})^2 A_{1112} A_{1222} (A_{2222})^2 - \\ & - 54 A_{1111} (A_{1122})^3 (A_{1222})^2 + 108 A_{1111} A_{1112} (A_{1222})^3 A_{1122} - \\ & - 64 (A_{1112})^3 (A_{1222})^3 + 54 A_{1111} (A_{2222})^2 A_{1122} (A_{1112})^2 + \\ & + (A_{1111})^3 (A_{2222})^3 + 54 (A_{1111})^2 A_{2222} A_{1122} (A_{1222})^2 + \\ & + 36 A_{1122}^2 (A_{1112})^2 (A_{1222})^2 - 54 (A_{1122})^3 (A_{1112})^2 A_{2222} - \\ & - 27 (A_{1111})^2 (A_{1222})^4 - 180 A_{1111} A_{1112} A_{1222} (A_{1122})^2 A_{2222} + \\ & + 108 (A_{1112})^3 A_{1222} A_{1122} A_{2222} - 6 A_{1111} (A_{1112})^2 (A_{1222})^2 A_{2222}. \end{aligned} \quad (3.4)$$

We can bring the polynomial (3.1) to the reduced form (2.1) by substituting

$$t = z - \frac{A_{1112}}{A_{1111}}.$$

As a result we get the following expressions for the coefficients in (2.1):

$$q_1 = \frac{6 A_{1122}}{A_{1111}} - \frac{6 (A_{1112})^2}{(A_{1111})^2}, \quad (3.5)$$

$$r_1 = \frac{4 A_{1222}}{A_{1111}} - \frac{12 A_{1122} A_{1112}}{(A_{1111})^2} + \frac{8 (A_{1112})^3}{(A_{1111})^3}, \quad (3.6)$$

$$s_1 = \frac{A_{2222}}{A_{1111}} - \frac{4 A_{1222} A_{1112}}{(A_{1111})^2} + \frac{6 A_{1122} (A_{1112})^2}{(A_{1111})^3} - \frac{3 (A_{1112})^4}{(A_{1111})^4}. \quad (3.7)$$

We can bring the polynomial (3.2) to the reduced form (2.1) by substituting

$$t = z - \frac{A_{1222}}{A_{2222}}.$$

As a result we get the following expressions for the coefficients in (2.1):

$$q_2 = \frac{6 A_{1122}}{A_{2222}} - \frac{6 (A_{1222})^2}{(A_{2222})^2}, \quad (3.8)$$

$$r_2 = \frac{4 A_{1112}}{A_{2222}} - \frac{12 A_{1122} A_{1222}}{(A_{2222})^2} + \frac{8 (A_{1222})^3}{(A_{2222})^3}, \quad (3.9)$$

$$s_2 = \frac{A_{1111}}{A_{2222}} - \frac{4 A_{1222} A_{1112}}{(A_{2222})^2} + \frac{6 A_{1122} (A_{1222})^2}{(A_{2222})^3} - \frac{3 (A_{1222})^4}{(A_{2222})^4}. \quad (3.10)$$

4. POSITIVITY CRITERIA FOR A BIVARIATE QUARTIC FORM.

The coefficient r of the reduced polynomial (2.1) is used as a testing parameter in Theorems 2.1 and 2.2. Therefore, relying upon (1.3) and using (3.6) and (3.9), we define the following two testing parameters:

$$I_1 = \frac{(A_{1111})^3 r_1}{4}, \quad I_2 = \frac{(A_{2222})^3 r_2}{4}. \quad (4.1)$$

Here are the explicit expressions for the parameters (4.1):

$$I_1 = (A_{1111})^2 A_{1222} - 3 A_{1111} A_{1112} A_{1122} + 2 (A_{1112})^3, \quad (4.2)$$

$$I_2 = (A_{2222})^2 A_{1112} - 3 A_{2222} A_{1222} A_{1122} + 2 (A_{1222})^3. \quad (4.3)$$

Another parameter used in Theorems 2.1 and 2.2 is q . Therefore, relying upon (1.3) and using (3.5) and (3.8), we introduce

$$I_3 = \frac{(A_{1111})^2 q_1}{6}, \quad I_4 = \frac{(A_{2222})^2 q_2}{6}. \quad (4.4)$$

Here are the explicit expressions for the parameters (4.4):

$$I_3 = A_{1111} A_{1122} - (A_{1112})^2, \quad (4.5)$$

$$I_4 = A_{2222} A_{1122} - (A_{1222})^2. \quad (4.6)$$

The third parameter used in Theorems 2.1 and 2.2 is s . Therefore, relying upon (1.3) and using (3.7) and (3.9), we introduce the following two parameters:

$$I_5 = (A_{1111})^4 s_1, \quad I_6 = (A_{2222})^4 s_2. \quad (4.7)$$

Here are the explicit expressions for the parameters (4.7):

$$I_5 = 6 A_{1111} A_{1122} (A_{1112})^2 - 3 (A_{1112})^4 - 4 A_{1111}^2 A_{1222} A_{1112} + (A_{1111})^3 A_{2222}, \quad (4.8)$$

$$I_6 = 6 A_{2222} A_{1122} (A_{1222})^2 - 3 (A_{1222})^4 - 4 A_{2222}^2 A_{1112} A_{1222} + (A_{2222})^3 A_{1111}. \quad (4.9)$$

Apart from separate entries of q and s , Theorems 2.1 and 2.2 comprise their combination (2.4). Applying (3.5), (3.7) and (3.8), (3.10) to (2.4), we derive

$$D_{2(1)} = \frac{36 (A_{1122})^2}{(A_{1111})^2} - \frac{96 A_{1122} (A_{1112})^2}{(A_{1111})^3} + \frac{48 (A_{1112})^4}{(A_{1111})^4} + \frac{16 A_{1222} A_{1112}}{(A_{1111})^2} - \frac{4 A_{2222}}{A_{1111}}, \quad (4.10)$$

$$\begin{aligned}
D_{2(2)} &= \frac{36 (A_{1122})^2}{(A_{2222})^2} - \frac{96 A_{1122} (A_{1222})^2}{(A_{2222})^3} + \\
&+ \frac{48 (A_{1222})^4}{(A_{2222})^4} + \frac{16 A_{1112} A_{1222}}{(A_{2222})^2} - \frac{4 A_{1111}}{A_{2222}},
\end{aligned} \tag{4.11}$$

Now, relying upon (1.3) and using (4.10) and (4.11), we introduce

$$I_7 = \frac{(A_{1111})^4 D_{2(1)}}{4}, \quad I_8 = \frac{(A_{2222})^4 D_{2(2)}}{4}. \tag{4.12}$$

Here are the explicit expressions for the parameters (4.12):

$$\begin{aligned}
I_7 &= 9 (A_{1122})^2 (A_{1111})^2 - 24 A_{1111} A_{1122} (A_{1112})^2 + \\
&+ 12 (A_{1112})^4 + 4 (A_{1111})^2 A_{1222} A_{1112} - (A_{1111})^3 A_{2222},
\end{aligned} \tag{4.13}$$

$$\begin{aligned}
I_8 &= 9 (A_{2222})^2 (A_{1122})^2 - 24 A_{2222} A_{1122} (A_{1222})^2 + \\
&+ 12 (A_{1222})^4 + 4 (A_{2222})^2 A_{1112} A_{1222} - (A_{2222})^3 A_{1111}.
\end{aligned} \tag{4.14}$$

Using the parameters (3.4), (4.2), (4.3), (4.5), (4.6), (4.6), (4.8), (4.9), (4.13), and (4.14), we can formulate the following theorems.

Theorem 4.1. *A bivariate quartic form (1.1) is positive if and only if $A_{1111} > 0$ and one of the following four conditions for its parameters is fulfilled:*

- 1) $I_1 \neq 0, I_0 > 0, I_7 \leq 0$;
- 2) $I_1 \neq 0, I_0 > 0, I_7 > 0, I_3 \geq 0$;
- 3) $I_1 = 0, I_7 < 0$;
- 4) $I_1 = 0, I_7 \geq 0, I_3 > 0, I_5 > 0$.

Theorem 4.2. *A bivariate quartic form (1.1) is positive if and only if $A_{2222} > 0$ and one of the following four conditions for its parameters is fulfilled:*

- 1) $I_2 \neq 0, I_0 > 0, I_8 \leq 0$;
- 2) $I_2 \neq 0, I_0 > 0, I_8 > 0, I_4 \geq 0$;
- 3) $I_2 = 0, I_8 < 0$;
- 4) $I_2 = 0, I_8 \geq 0, I_4 > 0, I_6 > 0$.

Theorems 4.1 and 4.2 are immediate from Theorems 2.1 and 2.2. Each of them is a criterion of positivity for the bivariate quartic form (1.1). Hence they are equivalent to each other. However, deriving one of these theorems from the other seems to be rather difficult.

5. PSEUDOTENSORS AND PSEUDOSCALARS.

Let V be some n -dimensional linear vector space. Assume that $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_n$ are arbitrary two bases in V . In this context they are usually called the old basis and the new basis respectively (see [4]). The bases are related to each other by means of two mutually inverse matrices square S and T :

$$\tilde{\mathbf{e}}_i = \sum_{j=1}^n S_i^j \mathbf{e}_j, \quad \mathbf{e}_i = \sum_{j=1}^n T_i^j \tilde{\mathbf{e}}_j. \tag{5.1}$$

The matrices S and T are called direct and inverse transition matrices respectively, while the formulas (5.1) are called direct and inverse transition formulas.

Definition 5.1. A pseudotensor of the type (r, s) and of the weight m is a geometrical and/or physical object in a linear vector space V presented by an array of quantities $F_{j_1 \dots j_s}^{i_1 \dots i_r}$ in each basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of V and transformed as follows under any change of basis given by the formulas (5.1):

$$F_{j_1 \dots j_s}^{i_1 \dots i_r} = (\det T)^m \sum_{\substack{p_1 \dots p_r \\ q_1 \dots q_s}}^n S_{p_1}^{i_1} \dots S_{p_r}^{i_r} T_{j_1}^{q_1} \dots T_{j_s}^{q_s} \tilde{F}_{q_1 \dots q_s}^{p_1 \dots p_r}.$$

Pseudotensors of the weight $m = 0$ are known as tensors (see [5]).

Definition 5.2. Pseudotensors of the type $(0, 0)$ are called pseudoscalars. Pseudoscalars of the weight $m = 0$ are called scalars.

Definition 5.1 can be found in [6], though of course it was known much prior to [6] (see [7] for instance). This definition is slightly different from that of [8].

In our case $\dim V = 2$. There is a fundamental pseudotensor \mathbf{d} of the type $(0, 2)$ and of the weight -1 in each two-dimensional linear vector space V . Its components are given by the same skew-symmetric matrix

$$d_{ij} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}$$

in any basis $\mathbf{e}_1, \mathbf{e}_2$ of V . The dual object for \mathbf{d} is given by the same matrix

$$d^{ij} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \quad (5.2)$$

in any basis $\mathbf{e}_1, \mathbf{e}_2$ of V . This dual object is denoted by the same symbol \mathbf{d} as the initial one. It is a pseudotensor of the type $(2, 0)$, its weight is equal to 1.

6. PSEUDOTENSORIAL INVARIANTS.

The tensor A , whose components are used in (1.1) and (1.2), is a true tensor, i. e. its weight is zero. Combining A with the pseudotensor \mathbf{d} defined by (5.2), we can compose various pseudotensorial objects. Let's begin with the following one:

$$B_{i_1 i_2 i_3 i_4} = \sum_{\substack{k_3, k_4 \\ j_3, j_4}}^2 A_{i_1 i_2 j_2 k_1} d^{k_1 j_1} A_{i_3 i_4 j_1 k_2} d^{k_2 j_2} \quad (6.1)$$

The formula (6.1) defines a pseudotensorial object of the type $(0, 4)$ and of the weight 2. Its components can be calculated explicitly. Two of them are associated with the parameters I_2 and I_4 in (4.5) and (4.6):

$$I_3 = -\frac{B_{1111}}{2}, \quad I_4 = -\frac{B_{2222}}{2}. \quad (6.2)$$

Definition 6.1. Any pseudotensorial object constructed with the use of A and \mathbf{d} is called a pseudotensorial invariant of the quartic form (1.1).

So, according to Definition 6.1, the pseudotensor B is a pseudotensorial invariant of the quartic form (1.1). As for I_3 and I_4 in (6.2), they are not pseudotensorial invariants. They are just certain components of a pseudotensorial invariant.

The next pseudotensorial invariant \hat{C} is constructed with the use of A , B , and \mathbf{d} by means of the following formula:

$$\hat{C}_{i_1 i_2 i_3 i_4 i_5 i_6} = \sum_{j_4, k_4}^2 B_{i_1 i_2 i_3 j_4} d^{j_4 k_4} A_{i_4 i_5 i_6 k_4}. \quad (6.3)$$

The formula (6.3) defines a pseudotensorial object of the type (0,6) and of the weight 3. Its components can be calculated explicitly. We need only two of them. They are associated with the parameters I_1 and I_2 in (4.2) and (4.3):

$$I_1 = \hat{C}_{111111}, \quad I_2 = -\hat{C}_{222222}. \quad (6.4)$$

In what follows we need the pseudoscalar object β given by the formula

$$\beta = \sum_{\substack{i_1, i_2 \\ j_1, j_2}}^2 B_{i_1 i_2 j_1 j_2} d^{i_1 j_1} d^{i_2 j_2}. \quad (6.5)$$

The pseudoscalar β in (6.5) can be calculated explicitly:

$$\beta = 8 A_{1112} A_{1222} - 6 (A_{1122})^2 - 2 A_{1111} A_{2222}. \quad (6.6)$$

Its weight is equal to 4. Along with β in (6.6), we need the following pseudotensorial object of the type (0,8) and of the weight 4:

$$\begin{aligned} D_{i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8} &= \sum_{\substack{k_1, k_2, k_3, k_4 \\ j_1, j_2, j_3, j_4}}^2 A_{i_1 i_2 j_4 k_1} d^{k_1 j_1} A_{i_3 i_4 j_1 k_2} d^{k_2 j_2} \times \\ &\times A_{i_5 i_6 j_2 k_3} d^{k_3 j_3} A_{i_7 i_8 j_3 k_4} d^{k_4 j_4}. \end{aligned} \quad (6.7)$$

All of the components of the pseudotensorial object (6.7) can be calculated explicitly. We need only two of them:

$$\begin{aligned} D_{11111111} &= 2 (A_{1112})^4 + 2 (A_{1111})^2 (A_{1122})^2 - 4 A_{1111} A_{1122} (A_{1112})^2, \\ D_{22222222} &= 2 (A_{1222})^4 + 2 (A_{2222})^2 (A_{1122})^2 - 4 A_{2222} A_{1122} (A_{1222})^2. \end{aligned} \quad (6.8)$$

Comparing (6.8) with (4.8) and (4.9) and using (6.6), we can write

$$\begin{aligned} I_5 &= -\frac{3 D_{11111111}}{2} - \frac{\beta (A_{1111})^2}{2}, \\ I_6 &= -\frac{3 D_{22222222}}{2} - \frac{\beta (A_{2222})^2}{2}. \end{aligned} \quad (6.9)$$

The formulas (6.9) mean that the parameters I_5 and I_6 are not pseudoscalars. They are just components of the pseudotensorial invariant given by the formula

$$-\frac{3}{2}D_{i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8} - \frac{\beta}{2}A_{i_1 i_2 i_3 i_4}A_{i_5 i_6 i_7 i_8}.$$

The parameters I_7 and I_8 are similar to I_5 and I_6 . Comparing (6.8) with (4.13) and (4.14) and taking into account (6.6), we can write

$$\begin{aligned} I_7 &= 6D_{11111111} + \frac{\beta(A_{1111})^2}{2}, \\ I_8 &= 6D_{22222222} + \frac{\beta(A_{2222})^2}{2}. \end{aligned} \tag{6.10}$$

The formulas (6.10) mean that the parameters I_7 and I_8 are not pseudoscalars. They are just components of the pseudotensorial invariant given by the formula

$$6D_{i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8} + \frac{\beta}{2}A_{i_1 i_2 i_3 i_4}A_{i_5 i_6 i_7 i_8}.$$

7. TENSORIAL PRESENTATION OF THE DISCRIMINANT.

Let's proceed to the parameter I_0 in (3.4) which was produced from the discriminants D_{P_1} and D_{P_2} in (3.3). This parameter is much more complicated than the previous ones. In dealing with this parameter we need more sums like (6.1) and (6.7). It is convenient to associate some graphical images with such sums (see Fig. 7.1) where each entry of A corresponds to a node, while each entry of \mathbf{d} corresponds to a bond. Each index not used in summation is represented as a free bond. Since the tensor A has four indices, each node in Fig. 7.1 has exactly four bonds either bound or unbound. For example, the sum (6.1) is presented as dipole with two inner bonds and two free bonds at each end.

The pseudoscalar β is produced from B according to (6.5). Graphically the sum (6.5) corresponds to binding free bonds of B . Therefore β is presented as dipole with four inner bonds in Fig. 7.1. There is an intermediate object

$$\hat{B}_{i_1 i_2} = \sum_{k_1, j_1}^2 B_{i_1 j_1 i_2 k_1} d^{k_1 j_1}. \tag{7.1}$$

It is presented as a dipole with three inner bonds and with one free bond at each end. One can calculate the components of \hat{B} in (7.1) explicitly and find that

$$\hat{B}_{i_1 i_2} = \frac{\beta}{2}d_{i_1 i_2}. \tag{7.2}$$

Due to (7.2) the triple dipole \hat{B} can be replaced with a bond whenever it enters to a more complicated diagram.

The sum (6.7) corresponds to the square D with two free bonds at each node. There is one more square shape in Fig. 7.1. It is denoted through δ . The shapes with three nodes are presented by triangles C and γ and by the right angle \hat{C} . The right angle \hat{C} corresponds to the sum (6.3).

Below we shall derive formulas associated with each shape in Fig. 7.1. Let's begin with the triangular shape C . Like the square D , the triangle C has two free

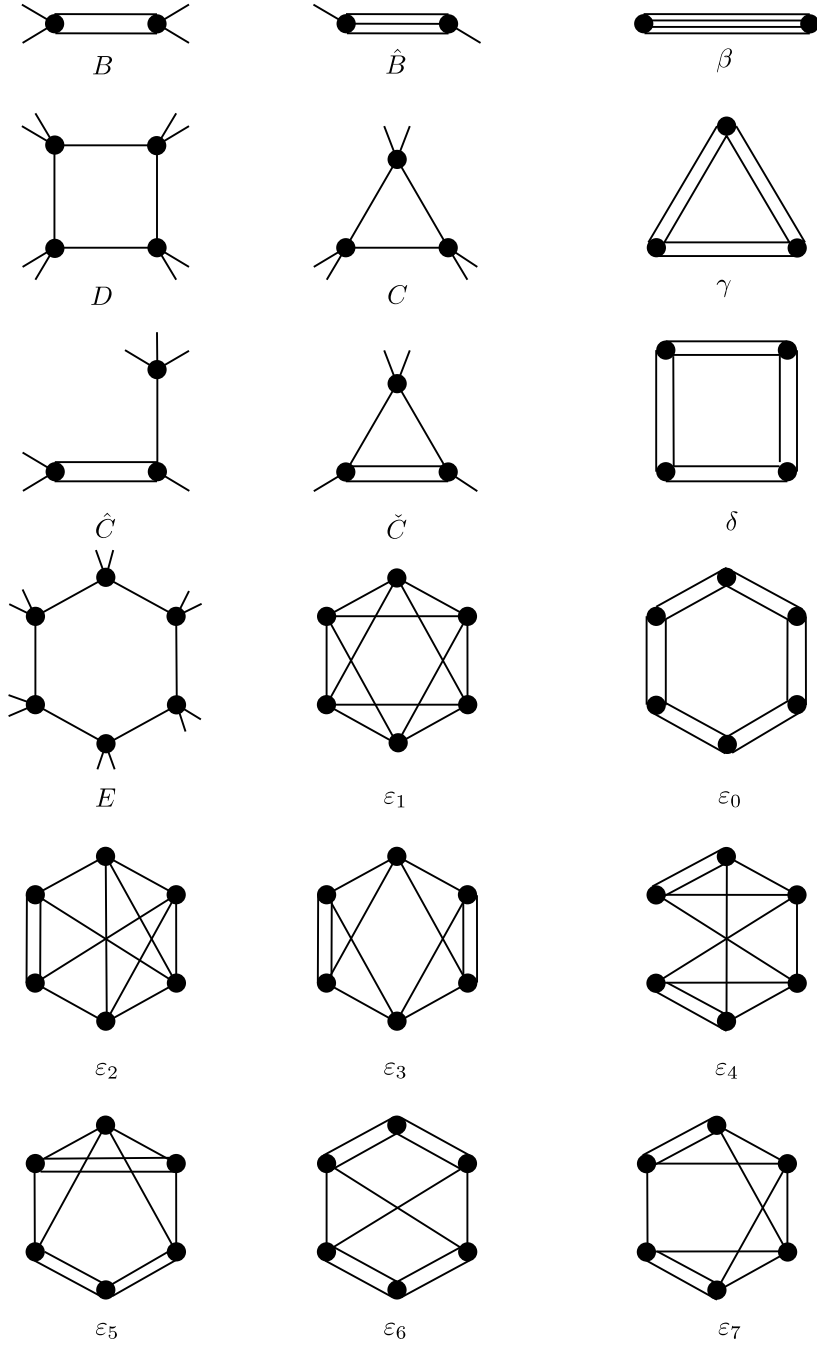


Fig. 7.1

bonds at each note. Here is the pseudotensorial object associated with C :

$$C_{i_1 i_2 i_3 i_4 i_5 i_6} = \sum_{\substack{k_1, k_2, k_3 \\ j_1, j_2, j_3}}^2 A_{i_1 i_2 j_3 k_1} d^{k_1 j_1} A_{i_3 i_4 j_1 k_2} d^{k_2, j_2} A_{i_5 i_6 j_2 k_3} d^{k_3 j_3}. \quad (7.3)$$

Its type is $(0, 6)$, its weight is 3. The components of the pseudotensor (7.3) can be calculated explicitly. Using them, we calculate the pseudoscalar γ :

$$\gamma = \sum_{\substack{k_1, k_2, k_3 \\ j_1, j_2, j_3}}^2 C_{j_3 k_1 j_1 k_2 j_2 k_3} d^{k_1 j_1} d^{k_2, j_2} d^{k_3 j_3}. \quad (7.4)$$

The weight of the pseudoscalar (7.4) is 6. Here is the explicit formula for γ :

$$\begin{aligned} \gamma &= 12 A_{1112} A_{1122} A_{1222} + 6 A_{1111} A_{1122} A_{2222} - \\ &- 6 (A_{1122})^3 - 6 (A_{1112})^2 A_{2222} - 6 A_{1111} (A_{1222})^2. \end{aligned}$$

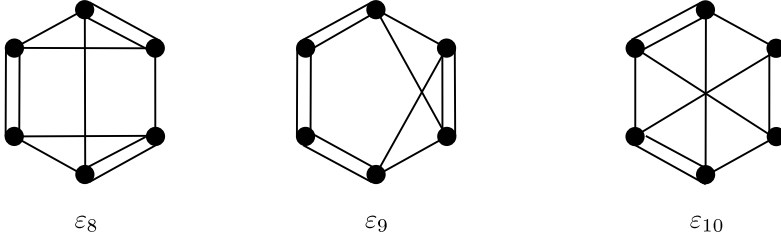


Fig. 7.2

There is one more triangular shape in Fig. 7.1. It is denoted \check{C} . Here is the formula for the associated pseudotensorial object:

$$\check{C}_{i_1 i_2 i_3 i_4} = \sum_{j_1, k_1}^2 C_{i_1 j_1 i_2 k_1 i_3 i_4} d^{k_1 j_1}. \quad (7.5)$$

The formula (7.5) defines a pseudotensorial object of the type $(0, 4)$ and of the weight 4. Its components can be calculated explicitly if needed.

The square shape δ in Fig. 7.1 is associated with a pseudoscalar object of the weight 8. It is calculated by means of the following formula:

$$\delta = \sum_{\substack{k_1, k_2, k_3, k_4 \\ j_1, j_2, j_3, j_4}}^2 D_{j_4 k_1 j_1 k_2 j_2 k_3 j_3 k_4} d^{k_1 j_1} d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4}. \quad (7.6)$$

It turns out that δ from (7.6) is expressed through β from (6.6):

$$\delta = \frac{\beta^2}{2}.$$

Pentagonal shapes cannot be associated with the parameter I_0 . Therefore we proceed to hexagonal ones. The shape E in Fig. 7.1 is presented by the formula

$$\begin{aligned}
E_{i_1 i_2 i_3 i_4 i_5 i_6 i_7 i_8 i_9 i_{10} i_{11} i_{12}} &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 A_{i_1 i_2 j_6 k_1} d^{k_1 j_1} \times \\
&\times A_{i_3 i_4 j_1 k_2} d^{k_2 j_2} A_{i_5 i_6 j_2 k_3} d^{k_3 j_3} A_{i_7 i_8 j_3 k_4} d^{k_4 j_4} \times \\
&\times A_{i_9 i_{10} j_4 k_5} d^{k_5 j_5} A_{i_{11} i_{12} j_5 k_6} d^{k_6 j_6}.
\end{aligned} \tag{7.7}$$

It defines a pseudotensorial object of the type $(0, 12)$ and the weight 6. Using (7.7), one can calculate the pseudoscalar ε_0 associated with the double hexagon:

$$\begin{aligned}
\varepsilon_0 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 E_{j_6 k_1 j_1 k_2 j_2 k_3 j_3 k_4 j_4 k_5 j_5 k_6} d^{k_1 j_1} \times \\
&\times d^{k_2 j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}.
\end{aligned} \tag{7.8}$$

Actually, using (7.7) is rather time consuming. For this reason, instead of (7.8), we use another formula for calculating ε_0 :

$$\begin{aligned}
\varepsilon_0 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 B_{j_5 j_6 k_1 k_2} B_{j_1 j_2 k_3 k_4} B_{j_3, j_4, k_5, k_6} d^{k_1 j_1} \times \\
&\times d^{k_2 j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}.
\end{aligned} \tag{7.9}$$

The quantity (7.9) is a pseudoscalar object of the weight 12. Here is an explicit expression for this pseudoscalar object:

$$\begin{aligned}
\varepsilon_0 &= 128 (A_{11112})^3 (A_{1222})^3 - 30 (A_{11111})^2 (A_{2222})^2 (A_{11122})^2 - \\
&- 30 A_{11111} (A_{11122})^4 A_{2222} - 24 A_{11111} (A_{11122})^3 (A_{1222})^2 - \\
&- 12 (A_{11112})^4 (A_{2222})^2 + 96 A_{11111} A_{11112} A_{1222} (A_{11122})^2 A_{2222} + \\
&+ 48 (A_{11112})^3 A_{1222} A_{11122} A_{2222} + 24 (A_{11111})^2 A_{11112} A_{1222} (A_{2222})^2 + \\
&+ 48 A_{11111} A_{11112} (A_{1222})^3 A_{11122} + 24 A_{11111} A_{2222}^2 A_{11122} (A_{11112})^2 + \\
&+ 24 A_{11111}^2 A_{2222} A_{11122} (A_{1222})^2 - 336 (A_{11122})^2 (A_{11112})^2 (A_{1222})^2 - \\
&- 24 (A_{11122})^3 (A_{11112})^2 A_{2222} - 120 A_{11111} (A_{11112})^2 (A_{1222})^2 A_{2222} - \\
&- 12 (A_{11111})^2 (A_{1222})^4 - 2 (A_{11111})^3 (A_{2222})^3 - 66 (A_{11122})^6 + \\
&+ 264 A_{11112} A_{1222} (A_{11122})^4.
\end{aligned} \tag{7.10}$$

Formulas similar to (7.9) are available for the quantities from ε_1 through ε_{10} :

$$\begin{aligned}
\varepsilon_1 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 C_{j_1 j_6 j_2 j_3 j_4 j_5} C_{k_1 k_2 k_3 k_4 k_5 k_6} d^{k_1 j_1} \times \\
&\times d^{k_2 j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6},
\end{aligned} \tag{7.11}$$

$$\begin{aligned} \varepsilon_2 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_1 j_6 j_3 j_2 j_4 j_5} C_{k_1 k_2 k_3 k_4 k_5 k_6} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}, \end{aligned} \quad (7.12)$$

$$\begin{aligned} \varepsilon_3 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_1 j_2 j_3 j_4 j_5 k_1} \hat{C}_{k_6 k_5 k_4 j_6 k_3 k_2} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}, \end{aligned} \quad (7.13)$$

$$\begin{aligned} \varepsilon_4 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_1, j_2, j_3, j_4, j_5, k_1} \hat{C}_{k_4, k_6, k_3, j_6, k_2, k_5} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}, \end{aligned} \quad (7.14)$$

$$\begin{aligned} \varepsilon_5 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_1 j_3 j_5 j_6 j_4 k_1} \hat{C}_{k_3 k_4 j_2 k_5 k_6 k_2} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}, \end{aligned} \quad (7.15)$$

$$\begin{aligned} \varepsilon_6 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_3, j_4, j_1, j_5, j_6, k_1} \hat{C}_{k_3, k_5, j_2, k_4, k_6, k_2} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}, \end{aligned} \quad (7.16)$$

$$\begin{aligned} \varepsilon_7 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_1 j_6 j_3 j_4 j_5 k_1} \hat{C}_{j_2 k_6 k_4 k_3 k_5 k_2} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}, \end{aligned} \quad (7.17)$$

$$\begin{aligned} \varepsilon_8 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_1 j_3 j_4 j_5 j_6 k_1} \hat{C}_{j_2 k_3 k_4 k_5 k_6 k_2} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}, \end{aligned} \quad (7.18)$$

$$\begin{aligned} \varepsilon_9 &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_1 j_3 j_4 j_5 j_6 k_1} \hat{C}_{k_3 k_4 j_2 k_5 k_6 k_2} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}, \end{aligned} \quad (7.19)$$

$$\begin{aligned} \varepsilon_{10} &= \sum_{\substack{k_1, k_2, k_3, k_4, k_5, k_6 \\ j_1, j_2, j_3, j_4, j_5, j_6}}^2 \hat{C}_{j_1 j_2 j_3 j_4 j_5 j_6} \hat{C}_{k_1 k_6 k_3 k_2 k_4 k_5} d^{k_1 j_1} \times \\ &\quad \times d^{k_2, j_2} d^{k_3 j_3} d^{k_4 j_4} d^{k_5 j_5} d^{k_6 j_6}. \end{aligned} \quad (7.20)$$

It turns out that ε_2 in (7.12) vanishes, i. e we have the equality

$$\varepsilon_2 = 0.$$

Using (7.11), (7.13), (7.14), (7.15), (7.16),(7.17), (7.18), (7.19), (7.20), one can derive explicit formulas for ε_1 and for the quantities from ε_3 through ε_{10} . For instance, in the case of the quantity ε_5 we have

$$\begin{aligned} \varepsilon_5 = & 12 A_{1111} (A_{1122})^4 A_{2222} - 6 (A_{1111})^2 (A_{2222})^2 (A_{1122})^2 - \\ & - 12 A_{1111} (A_{1122})^3 (A_{1222})^2 - 6 (A_{1112})^4 (A_{2222})^2 - \\ & - 24 A_{1111} A_{1112} A_{1222} (A_{1122})^2 A_{2222} + 24 (A_{1112})^3 A_{1222} A_{1122} A_{2222} + \\ & + 24 A_{1111} A_{1112} (A_{1222})^3 A_{1122} + 12 A_{1111} (A_{2222})^2 A_{1122} (A_{1112})^2 + \quad (7.21) \\ & + 12 A_{1111})^2 A_{2222} A_{1122} (A_{1222})^2 - 24 (A_{1122})^2 (A_{1112})^2 (A_{1222})^2 - \\ & - 12 (A_{1122})^3 (A_{1112})^2 A_{2222} - 12 A_{1111} (A_{1112})^2 (A_{1222})^2 A_{2222} - \\ & - 6 (A_{1111})^2 (A_{1222})^4 - 6 (A_{1122})^6 + 24 A_{1112} A_{1222} (A_{1122})^4. \end{aligned}$$

Fortunately there is no need to calculate the rest of the quantities $\varepsilon_1, \dots, \varepsilon_{10}$. They are expressed as linear combinations of (7.10) and (7.21):

$$\begin{aligned} \varepsilon_1 &= \varepsilon_5, & \varepsilon_3 &= 2\varepsilon_5, \\ \varepsilon_4 &= -\varepsilon_5, & \varepsilon_6 &= -\varepsilon_0 + 3\varepsilon_5, \\ \varepsilon_7 &= \varepsilon_5, & \varepsilon_8 &= -2\varepsilon_5, \\ \varepsilon_9 &= 2\varepsilon_5, & \varepsilon_{10} &= -\varepsilon_0 + 4\varepsilon_5. \end{aligned} \quad (7.22)$$

There is a formula similar to (7.22) for the parameter I_0 :

$$I_0 = -\frac{1}{2}\varepsilon_0 + \frac{11}{2}\varepsilon_5. \quad (7.23)$$

It turns out that ε_0 and ε_5 are expressed through β^3 and γ^2 :

$$\varepsilon_0 = \frac{1}{4}\beta^3 - \frac{1}{3}\gamma^2, \quad \varepsilon_5 = -\frac{1}{6}\gamma^2. \quad (7.24)$$

Substituting (7.24) into (7.23), we derive

$$I_0 = -\frac{1}{8}\beta^3 - \frac{3}{4}\gamma^2, \quad (7.25)$$

where β and γ are given by (6.5) and (7.4). The equality (7.25) shows that I_0 is a pseudoscalar object of the weight 12 unlike $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8$, which are just certain components of pseudotensorial objects. The equality (7.25) is not surprising since β and γ are two basic invariants of a quartic form (see [9]).

8. CONCLUSIONS.

The main result of the present paper consists in revealing the tensorial nature of the parameters $I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8$ in Dickson's and Rees's positivity test for quartic forms. This result is expressed by the formulas (6.2), (6.4), (6.9), and (6.10). It can be further generalized to trivariate quartic forms and to quartic forms with greater number of variables. As for the tensorial nature of the parameter I_0

expressed through the formulas (7.25), (6.5) and (7.4), it is a classical result known probably since Issai Schur, F. Franklin, J. J. Sylvester, and David Hilbert.

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