# MULTIPLE DISCRIMINANTS AND CRITICAL VALUES OF A MULTIVARIATE POLYNOMIAL. 

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#### Abstract

A critical value of a function is the value of this function at one of its critical points. Each critical point of a differentiable multivariate function is described by the equations which consist in equating to zero all of its partial derivatives. However, in general case there is no equation for the corresponding critical value. The case of polynomials is different. In the present paper an equation for critical values of a polynomial is derived.


## 1. Introduction.

Let $f\left(x_{1}, \ldots, x_{n}\right)$ be a smooth real valued multivariate function in $\mathbb{R}^{n}$ or in some open domain $\Omega \subset \mathbb{R}^{n}$. Critical points of the function $f$ are determined by solving the following system of equations with respect to the variables $x_{1}, \ldots, x_{n}$ :

$$
\left\{\begin{align*}
& \frac{\partial f\left(x_{1}, \ldots, x_{n}\right)}{\partial x_{1}}=0  \tag{1.1}\\
& \ldots \cdots \cdots \\
& \frac{\partial f\left(x_{1}, \ldots, x_{n}\right)}{\partial x_{n}}=0
\end{align*}\right.
$$

They can be maxima, minima, saddle points, and other types of critical points.
Let $v$ be the value of $f\left(x_{1}, \ldots, x_{n}\right)$ at one of its critical points given by the equations (1.1). Then we can extend the system of equations (1.1) as follows:

$$
\left\{\begin{array}{l}
\frac{\partial f\left(x_{1}, \ldots, x_{n}\right)}{\partial x_{1}}=0  \tag{1.2}\\
\cdots \cdots \cdots \\
\frac{\partial f\left(x_{1}, \ldots, x_{n}\right)}{\partial x_{n}}=0 \\
f\left(x_{1}, \ldots, x_{n}\right)-v=0
\end{array}\right.
$$

The parameter $v$ in (1.2) is treated as a new variable independent of $x_{1} \ldots, x_{n}$, i.e. (1.2) is a system of $n+1$ equations for $n+1$ variables. Theoretically, one can eliminate the variables $x_{1} \ldots, x_{n}$ from the system (1.2) thus producing one equation for one variable $v$, which is the critical value of $f$ :

$$
\begin{equation*}
F(v)=0 \tag{1.3}
\end{equation*}
$$

[^0]In practice it is rather difficult to derive such an equation. In the present paper we derive the equation of the form (1.3) for the case where $f\left(x_{1}, \ldots, x_{n}\right)$ is a multivariate polynomial. For the sake of simplicity, from now on we treat $x_{1} \ldots, x_{n}$ as complex variables so that they constitute a point in $\mathbb{C}^{n}$.

## 2. The case of a univariate polynomial.

Let's begin with the case where $n=1$. Denoting $x_{1}=x$ for the sake of simplicity, assume that $f(x)$ is a univariate polynomial of $n$-th degree:

$$
f(x)=a_{0} x^{n}+a_{1} x^{n-1}+\ldots+a_{n-1} x+a_{n}
$$

Then $p(x)=f(x)-v$ is also a univariate polynomial of $n$-th degree:

$$
\begin{equation*}
p(x)=a_{0} x^{n}+a_{1} x^{n-1}+\ldots+a_{n-1} x+\left(a_{n}-v\right) \tag{2.1}
\end{equation*}
$$

Using $p(x)$, in this case we can write the system of equations (1.2) as follows:

$$
\left\{\begin{array}{l}
p^{\prime}(x)=0 \\
p(x)=0
\end{array}\right.
$$

It is known that a univariate polynomial $p(x)$ vanishes along with its first derivative $p^{\prime}(x)$ at some point $x \in \mathbb{C}$ if and only its discriminant is zero (see [1]):

$$
\begin{equation*}
D_{p}=0 \tag{2.2}
\end{equation*}
$$

The coefficients of the polynomial $p(x)$ in (2.1) depend on $v$. Therefore the discriminant $D_{p}$ in (2.2) depends on $v$. And we write (2.2) as

$$
\begin{equation*}
D_{p}(v)=0 \tag{2.3}
\end{equation*}
$$

The equation (2.3) is a required equation of the form (1.3).
Theorem 2.1. A complex number $v$ is a critical value of a univariate polynomial $f(x)$ if and only if it is a root of the equation (2.3), where $D_{p}(v)$ is the discriminant of the polynomial $p(x)=f(x)-v$.

Theorem 2.1 follows immediately from the above mentioned basic property of the discriminant of a univariate polynomial.

## 3. The case of a multivariate polynomial.

This case is somewhat similar to the previous one. Here we can replace the multivariate polynomial $f\left(x_{1}, \ldots, x_{n}\right)$ with the polynomial

$$
\begin{equation*}
p\left(x_{1}, \ldots, x_{n}\right)=f\left(x_{1}, \ldots, x_{n}\right)-v \tag{3.1}
\end{equation*}
$$

Due to (3.1) we can write the equations (1.2) as follows:

$$
\left\{\begin{array}{l}
\frac{\partial p\left(x_{1}, \ldots, x_{n}\right)}{\partial x_{1}}=0  \tag{3.2}\\
\cdots \cdots \cdots \cdots \\
\frac{\partial p\left(x_{1}, \ldots, x_{n}\right)}{\partial x_{n}}=0 \\
p\left(x_{1}, \ldots, x_{n}\right)=0
\end{array}\right.
$$

The concept of the discriminant of a multivariate polynomial is not commonly known. Its definition can be found in [2] (see also [3] and [4]).
Definition 3.1. The discriminant $D_{p}$ of a multivariate polynomial $p\left(x_{1}, \ldots, x_{n}\right)$ is a polynomial function of the coefficients of $p$ such that its own coefficients are integer, such that it is irreducible over $\mathbb{Z}$, and such that $D_{p}=0$ if and only if the system of equations (3.2) has an least one solution in complex numbers $x_{1}, \ldots, x_{n}$.

As it was said in [2] with the reference to [5], discriminants of multivariate polynomials were first considered by G. Boole. The reference [5] is taken from [2] "as is". It looks quite uncertain, since no information on publishers is provided (see more detailed historical research in [6] and [7]).

The coefficients of the polynomial $p\left(x_{1}, \ldots, x_{n}\right)$ in (3.1) depend on $v$. Therefore the discriminant $D_{p}$ depends on $v$. Hence we can write the equation

$$
\begin{equation*}
D_{p}(v)=0 \tag{3.3}
\end{equation*}
$$

The equation (3.3) is of the form (1.3). It solves our problem of reducing (1.2) to a single equation for $v$ through the following theorem which is immediate from the above Definition 3.1.

Theorem 3.1. A complex number $v$ is a critical value of a multivariate polynomial $f\left(x_{1}, \ldots, x_{n}\right)$ if and only if it is a root of the equation (3.3), where $D_{p}(v)$ is the discriminant of the polynomial (3.1).

However, the matter is that there is no simple formula for the discriminant $D_{p}$ in the case of a multivariate polynomial $p$. For this reason below we consider a different approach to deriving an equation of the form (1.3) from (1.2).

## 4. Critical points of discriminants.

Let $p(x, y)$ be a univariate polynomial of $n$-th degree in $x$, i. e.

$$
\begin{equation*}
p(x, y)=a_{n}(y) x^{n}+a_{n-1}(y) x^{n-1}+\ldots+a_{1}(y) x+a_{0}(y) \tag{4.1}
\end{equation*}
$$

whose coefficients are smooth function of a complex variable $y \in \mathbb{C}$. Assume that at some point $\left(x_{0}, y_{0}\right) \in \mathbb{C}^{2}$, where $a_{n}\left(y_{0}\right) \neq 0$, the following equations are fulfilled:

$$
\begin{equation*}
p_{x}^{\prime}\left(x_{0}, y_{0}\right)=0, \quad p_{y}^{\prime}\left(x_{0}, y_{0}\right)=0, \quad p\left(x_{0}, y_{0}\right)=0 \tag{4.2}
\end{equation*}
$$

Theorem 4.1. If $p$ is a univariate polynomial of the form (4.1) whose coefficients are smooth function of a complex variable $y$ and if the equations (4.2) are fulfilled at some point $\left(x_{0}, y_{0}\right) \in \mathbb{C}^{2}$, where $a_{n}\left(y_{0}\right) \neq 0$, then they imply the equations

$$
\begin{equation*}
D^{\prime}\left(y_{0}\right)=0, \quad D\left(y_{0}\right)=0 \tag{4.3}
\end{equation*}
$$

for the discriminant $D$ of $p$ which are fulfilled at the point $y_{0} \in \mathbb{C}$, where $y_{0}$ is the second coordinate of the corresponding point $\left(x_{0}, y_{0}\right) \in \mathbb{C}^{2}$.

Since $a_{n}\left(y_{0}\right) \neq 0$ in Theorem 4.1, we can divide the polynomial (4.1) by $a_{n}(y)$ and proceed to the following monic polynomial:

$$
\begin{equation*}
\tilde{p}(x, y)=x^{n}+\frac{a_{n-1}(y)}{a_{n}(y)} x^{n-1}+\ldots+\frac{a_{1}(y)}{a_{n}(y)} x+\frac{a_{0}(y)}{a_{n}(y)} . \tag{4.4}
\end{equation*}
$$

The discriminants of the polynomials (4.1) and (4.4) are related as follows:

$$
\begin{equation*}
D_{p}(y)=\left(a_{n}(y)\right)^{2 n-2} D_{\tilde{p}}(y) \tag{4.5}
\end{equation*}
$$

Relying on (4.5), we can write the monic polynomial

$$
\begin{equation*}
p(x, y)=x^{n}+a_{n-1}(y) x^{n-1}+\ldots+a_{1}(y) x+a_{0}(y) \tag{4.6}
\end{equation*}
$$

and then reformulate Theorem 4.1 in the following way.
Theorem 4.2. If $p$ is a monic univariate polynomial of the form (4.6) whose coefficients are smooth function of a complex variable $y$ and if the equations (4.2) are fulfilled at some point $\left(x_{0}, y_{0}\right) \in \mathbb{C}^{2}$, then they imply the equations (4.3) for the discriminant $D$ of $p$ which are fulfilled at the point $y_{0} \in \mathbb{C}$, where $y_{0}$ is the second coordinate of the corresponding point $\left(x_{0}, y_{0}\right) \in \mathbb{C}^{2}$.

Theorems 4.1 and 4.2 are equivalent to each other due to the relationship (4.5).

## 5. Proof of Theorem 4.2 in the case of a quadratic polynomial.

Assume that $n=2$ in (4.6). Then $p(x, y)$ is a quadratic polynomial:

$$
\begin{equation*}
p(x, y)=x^{2}+a_{1}(y) x+a_{0}(y) \tag{5.1}
\end{equation*}
$$

One can easily calculate the partial derivatives of $p(x, y)$ :

$$
\begin{equation*}
p_{x}^{\prime}(x, y)=2 x+a_{1}(y), \quad \quad p_{y}^{\prime}(x, y)=a_{1}^{\prime}(y) x+a_{0}^{\prime}(y) \tag{5.2}
\end{equation*}
$$

Substituting (5.1) and (5.2) into (4.2), we derive a system of three equations:

$$
\left\{\begin{array}{l}
2 x_{0}+a_{1}\left(y_{0}\right)=0  \tag{5.3}\\
a_{1}^{\prime}\left(y_{0}\right) x_{0}+a_{0}^{\prime}\left(y_{0}\right)=0 \\
\left(x_{0}\right)^{2}+a_{1}\left(y_{0}\right) x_{0}+a_{0}\left(y_{0}\right)=0
\end{array}\right.
$$

The discriminant of the polynomial (5.1) is calculated as follows:

$$
\begin{equation*}
D(y)=\left(a_{1}(y)\right)^{2}-4 a_{0}(y) \tag{5.4}
\end{equation*}
$$

Its derivative $D^{\prime}(y)$ is also easily calculated:

$$
\begin{equation*}
D^{\prime}(y)=2 a_{1}(y) a_{1}^{\prime}(y)-4 a_{0}^{\prime}(y) \tag{5.5}
\end{equation*}
$$

Note that the first equation in (5.3) can be resolved with respect to $x_{0}$ :

$$
\begin{equation*}
x_{0}=-\frac{a_{1}\left(y_{0}\right)}{2} . \tag{5.6}
\end{equation*}
$$

Substituting (5.6) into the third equation (5.3) we derive

$$
\begin{equation*}
-\frac{\left(a_{1}\left(y_{0}\right)\right)^{2}}{4}+a_{0}\left(y_{0}\right)=0 \tag{5.7}
\end{equation*}
$$

Comparing (5.7) with (5.4), we see that (5.3) implies

$$
\begin{equation*}
D\left(y_{0}\right)=0 \tag{5.8}
\end{equation*}
$$

Then we substitute (5.6) into the second equation (5.3). As a result we get

$$
\begin{equation*}
-\frac{a_{1}\left(y_{0}\right) a_{1}^{\prime}\left(y_{0}\right)}{2}+a_{0}^{\prime}\left(y_{0}\right)=0 \tag{5.9}
\end{equation*}
$$

Comparing (5.9) with (5.5), we see that (5.3) implies

$$
\begin{equation*}
D^{\prime}\left(y_{0}\right)=0 \tag{5.10}
\end{equation*}
$$

The rest is to note that (5.10) and (5.8) coincide with (4.3) and conclude that (4.2) implies (4.3) in the case of the polynomial (5.1). Thus, Theorem 4.2 is proved for the case of quadratic polynomials.

## 6. Proof of Theorem 4.2 in the case of a double root polynomial.

If $y=y_{0}$ is fixed, the equations (4.2) mean that the polynomial $p$ vanishes along with its derivative $p_{x}^{\prime}$ at the point $x=x_{0}$, i. e. it has a multiple root at this point. The case of a double root is the most simple in this situation.

Let $x_{1} \ldots, x_{n}$ be roots of the polynomial (4.6). They depend on $y$, i. e.

$$
\begin{equation*}
x_{i}=x_{i}(y), \quad i=1, \ldots, n \tag{6.1}
\end{equation*}
$$

It is known that roots of a univariate polynomial are continuous functions of its coefficients (see [8]). Hence in our case the roots (6.1) are continuous functions of $y$. Two of them tend to $x_{0}$ as $y \rightarrow y_{0}$. Without loss of generality we can set

$$
\begin{equation*}
x_{1}(y) \rightarrow x_{0} \text { and } x_{2}(y) \rightarrow x_{0} \quad \text { as } y \rightarrow y_{0} . \tag{6.2}
\end{equation*}
$$

Assume that the roots $x_{3}\left(y_{0}\right), \ldots, x_{n}\left(y_{0}\right)$ are distinct and different from the double root $x_{1}\left(y_{0}\right)=x_{2}\left(y_{0}\right)=x_{0}$. Under this assumption the roots

$$
\begin{equation*}
x_{3}(y), \ldots, x_{n}(y) \tag{6.3}
\end{equation*}
$$

are smooth functions of $y$ in some neighborhood of the point $y=y_{0}$ (see [8]). Unlike them the roots $x_{1}(y)$ and $x_{2}(y)$ in (6.2) are not necessarily smooth, though they are continuous functions of $y$ in this neighborhood of the point $y=y_{0}$.

Using the roots (6.2) and (6.3), we define two polynomials which are two complementary factors of the initial polynomial $p(x, y)$ in (4.6):

$$
\begin{equation*}
q(x, y)=\prod_{i=1}^{2}\left(x-x_{i}(y)\right), \quad r(x, y)=\prod_{i=3}^{n}\left(x-x_{i}(y)\right) \tag{6.4}
\end{equation*}
$$

Indeed, from (6.4) we easily derive the equality

$$
\begin{equation*}
p(x, y)=q(x, y) r(x, y) \tag{6.5}
\end{equation*}
$$

The coefficients of the polynomial $r(x, y)$ in (6.4) are smooth functions of $y$ since they are elementary symmetric functions of the smooth roots (6.3) (see [1]). The polynomial $q(x, y)$ in (6.4) is quadratic:

$$
\begin{equation*}
q(x, y)=x^{2}+\beta_{1}(y) x+\beta_{0}(y) \tag{6.6}
\end{equation*}
$$

Its coefficients are also smooth functions of $y$ since due to (6.5) the polynomial (6.6) can be produced by means of the polynomial long division algorithm (see [9]) with the monic polynomial $r(x, y)$ as a divisor:

$$
q(x, y)=p(x, y) \div r(x, y)
$$

Let's recall that the roots $x_{3}\left(y_{0}\right), \ldots, x_{n}\left(y_{0}\right)$ are different from the double root $x_{0}$. Therefore from (6.4) we derive the inequality

$$
\begin{equation*}
r\left(x_{0}, y_{0}\right)=\prod_{i=3}^{n}\left(x_{0}-x_{i}\left(y_{0}\right)\right) \neq 0 \tag{6.7}
\end{equation*}
$$

Similarly, applying (6.2) to $q(x, y)$ in (6.4), we derive the equality

$$
\begin{equation*}
q\left(x_{0}, y_{0}\right)=0 \tag{6.8}
\end{equation*}
$$

Differentiating the equality (6.5), we find that

$$
\begin{align*}
& p_{x}^{\prime}\left(x_{0}, y_{0}\right)=q_{x}^{\prime}\left(x_{0}, y_{0}\right) r\left(x_{0}, y_{0}\right)+q\left(x_{0}, y_{0}\right) r_{x}^{\prime}\left(x_{0}, y_{0}\right) \\
& p_{y}^{\prime}\left(x_{0}, y_{0}\right)=q_{y}^{\prime}\left(x_{0}, y_{0}\right) r\left(x_{0}, y_{0}\right)+q\left(x_{0}, y_{0}\right) r_{y}^{\prime}\left(x_{0}, y_{0}\right) \tag{6.9}
\end{align*}
$$

Now, applying (6.8) to (6.9), we obtain the equalities

$$
\begin{align*}
& p_{x}^{\prime}\left(x_{0}, y_{0}\right)=q_{x}^{\prime}\left(x_{0}, y_{0}\right) r\left(x_{0}, y_{0}\right),  \tag{6.10}\\
& p_{y}^{\prime}\left(x_{0}, y_{0}\right)=q_{y}^{\prime}\left(x_{0}, y_{0}\right) r\left(x_{0}, y_{0}\right)
\end{align*}
$$

If the equations (4.2) are fulfilled, then, using (6.7), from (6.10) we derive

$$
\begin{equation*}
q_{x}^{\prime}\left(x_{0}, y_{0}\right)=0, \quad q_{y}^{\prime}\left(x_{0}, y_{0}\right)=0 \tag{6.11}
\end{equation*}
$$

The equalities (6.11) combined with (6.8) mean that the equations (4.2) for $p$, once they are fulfilled, imply similar equations

$$
\begin{equation*}
q_{x}^{\prime}\left(x_{0}, y_{0}\right)=0, \quad q_{y}^{\prime}\left(x_{0}, y_{0}\right)=0, \quad q\left(x_{0}, y_{0}\right)=0 \tag{6.12}
\end{equation*}
$$

for the quadratic polynomial (6.6) whose coefficients are smooth functions of $y$. Theorem 4.2 is already proved for quadratic polynomials. Therefore, applying this theorem to $q(x, y)$ and (6.12), we derive the equalities

$$
\begin{equation*}
D_{q}^{\prime}\left(y_{0}\right)=0, \quad D_{q}\left(y_{0}\right)=0 \tag{6.13}
\end{equation*}
$$

where $D_{q}(y)$ is the discriminant of the polynomial $q(x, y)$ in (6.6).

Let's proceed to the discriminant $D(y)$ of the polynomial $p(x, y)$ in (4.6). In terms of its roots the discriminant $D(y)$ is given by the formula

$$
\begin{equation*}
D(y)=\prod_{i<j}^{n}\left(x_{i}(y)-x_{j}(y)\right)^{2} \tag{6.14}
\end{equation*}
$$

(see [1]). If we denote through $D_{r}(y)$ the discriminant of the polynomial $r(x, y)$ in (6.4), then we can factorize the discriminant (6.14) as follows:

$$
\begin{equation*}
D(y)=D_{q}(y) D_{r}(y) \prod_{i=1}^{2} \prod_{j=3}^{n}\left(x_{i}(y)-x_{j}(y)\right)^{2} \tag{6.15}
\end{equation*}
$$

Taking into account the formula for $q(x, y)$ in (6.4), we can transform (6.15) as

$$
\begin{equation*}
D(y)=D_{q}(y) D_{r}(y)\left(\prod_{j=3}^{n} q\left(x_{j}(y), y\right)\right)^{2} \tag{6.16}
\end{equation*}
$$

Relying on (6.16), lets introduce the following notation:

$$
\begin{equation*}
\alpha(y)=D_{r}(y)\left(\prod_{j=3}^{n} q\left(x_{j}(y), y\right)\right)^{2} . \tag{6.17}
\end{equation*}
$$

The function $\alpha(y)$ in (6.17) is a smooth function of $y$ since the coefficients of the polynomial $q(x, y)$ in (6.6) and the roots $x_{3}(y), \ldots, x_{n}(y)$ of the polynomial $r(x, y)$ in (6.3) are smooth functions of $y$. Applying (6.17) to (6.16), we get

$$
\begin{equation*}
D(y)=D_{q}(y) \alpha(y) \tag{6.18}
\end{equation*}
$$

Both multiplicands $D_{q}(y)$ and $\alpha(y)$ in (6.18) are smooth functions of $y$. Therefore, differentiating (6.18), we derive the following formula:

$$
\begin{equation*}
D^{\prime}\left(y_{0}\right)=D_{q}^{\prime}\left(y_{0}\right) \alpha\left(y_{0}\right)+D_{q}\left(y_{0}\right) \alpha^{\prime}\left(y_{0}\right) \tag{6.19}
\end{equation*}
$$

The rest is to apply (6.13) to (6.18) and (6.19) and derive (4.3). Thus we have proved that (4.2) implies (4.3) in our present case. Theorem 4.2 is proved in the case of a polynomial (4.6) with exactly one double root and the other simple roots.

## 7. Proof of Theorem 4.2 in general case.

Note that the polynomial $p(x, y)$ in (4.6) depends on $y$ through its coefficients. Therefore the derivative $D^{\prime}(y)$ of its discriminant is calculated is follows:

$$
\begin{equation*}
D^{\prime}(y)=\sum_{i=0}^{n-1} \frac{\partial D}{\partial a_{i}} a_{i}^{\prime}(y) \tag{7.1}
\end{equation*}
$$

Let's denote $\delta p(x, y)=p_{y}^{\prime}(x, y)$. Then, differentiating (4.6), we get

$$
\begin{equation*}
\delta p(x, y)=a_{n-1}^{\prime}(y) x^{n-1}+\ldots+a_{1}^{\prime}(y) x+a_{0}^{\prime}(y) . \tag{7.2}
\end{equation*}
$$

Let's denote $a_{i}=a_{i}^{\prime}\left(y_{0}\right), b_{i}=a_{i}^{\prime}\left(y_{0}\right)$ and $\delta D=D^{\prime}\left(y_{0}\right)$. Then, substituting $y=y_{0}$ into (4.6), (7.2) and (7.1), we obtain two polynomials

$$
\begin{align*}
p(x) & =x^{n}+a_{n-1} x^{n-1}+\ldots+a_{1} x+a_{0}, \\
\delta p(x) & =b_{n-1} x^{n-1}+\ldots+b_{1} x+b_{0} \tag{7.3}
\end{align*}
$$

with purely numeric coefficients and the purely numeric quantity

$$
\begin{equation*}
\delta D=\sum_{i=0}^{n-1} \frac{\partial D}{\partial a_{i}} b_{i} \tag{7.4}
\end{equation*}
$$

associated with the polynomials (7.3). The following definition is terminological. It is designed for the sake of beauty.

Definition 7.1. The polynomial $\delta p(x)$ in (7.3) is called a first variation of the polynomial $p(x)$, while the numeric quantity (7.4) is called the first variation of the discriminant $D$ of $p$ associated with $\delta p(x)$.

It turns out that the functional nature of the coefficients of $p(x, y)$ in (4.6) is inessential in Theorem 4.2. This theorem can be reformulated as follows.

Theorem 7.1. If a monic polynomial $p(x)$ in (7.3) has a multiple root of multiplicity $m \geqslant 2$ and if it shares this root with its first variation $\delta p(x)$, then the discriminant $D$ of $p(x)$ vanishes along with its first variation $\delta D$ in (7.4).

Theorem 7.1 is equivalent to Theorem 4.2. The equivalence can be established using the linear functions $a_{i}(y)=a_{i}+b_{i}\left(y-y_{0}\right)$ in (4.6). The result of the previous section means that we have already proved Theorem 7.1 for any monic polynomial $p(x)$ with exactly one double root and the other simple roots.

Let $p(x)$ be a monic polynomial with at least one multiple root $x_{0}$. Then it can be produced as a limit of some sequence of monic polynomials $p_{s}(x)$ with exactly one double root $x=x_{0}$ and the other simple roots. Assume that $p(x)$ shares its multiple root $x=x_{0}$ with its first variation $\delta p(x)$ in (7.3). Then

$$
\begin{equation*}
\delta p\left(x_{0}\right)=0 \tag{7.5}
\end{equation*}
$$

The equality (7.5) is written as a linear relationships with respect to the coefficients $b_{0}, \ldots, b_{n-1}$ of the polynomial $\delta p(x)$ in (7.3):

$$
\begin{equation*}
x_{0}^{n-1} b_{n-1}+\ldots+x_{0} b_{1}+b_{0}=0 . \tag{7.6}
\end{equation*}
$$

The coefficients of the linear combination (7.6) depend only on the root $x_{0}$, which is common for $p(x)$ and for any polynomial in the sequence $p_{s}(x)$. This means that $p(x)$ and the polynomials $p_{s}(x)$ share the same first variation $\delta p(x)$ obeying the relationship (7.5). This first variation $\delta p(x)$ combined with each $p_{s}(x)$ according to (7.4) produces a numeric sequence $\delta D_{s}$. The discriminant $D$ and its partial derivatives in (7.4) are smooth functions of $a_{0}, \ldots, a_{n-1}$. Therefore we have

$$
\begin{equation*}
\delta D=\lim _{s \rightarrow \infty} \delta D_{s} \tag{7.7}
\end{equation*}
$$

The rest is to apply Theorem 7.1 to each $p_{s}(x)$ combined with $\delta p(x)$. This yields
$\delta D_{s}=0$. Substituting $\delta D_{s}=0$ into (7.7), we derive the required result $\delta D=0$. Thus, Theorem 7.1 is proved in its full generality. The same is true for Theorems 4.2 and 4.1, which are equivalent to Theorem 7.1.

## 8. Application to critical values.

Having been equipped with Theorem 4.1, now we return to our initial problem of deriving an equation of the form (1.3) from the equations (1.2) in the case of a multivariate polynomial $f\left(x_{1}, \ldots, x_{n}\right)$. Passing from $f\left(x_{1}, \ldots, x_{n}\right)$ to the polynomial $p\left(x_{1}, \ldots, x_{n}\right)$ in (3.1), we treat $p\left(x_{1}, \ldots, x_{n}\right)$ as a univariate polynomial with respect to the variable $x=x_{n}$ and treat the other variables $x_{1}, \ldots, x_{n-1}$ and $v$ as parameters. Then we calculate the univariate discriminant $D$ of the polynomial $p\left(x_{1}, \ldots, x_{n}\right)$ with respect to the last variable $x=x_{n}$ :

$$
\begin{equation*}
\tilde{p}\left(x_{1}, \ldots, x_{n-1}\right)=D\left(x_{n}, p\left(x_{1}, \ldots, x_{n}\right)\right) . \tag{8.1}
\end{equation*}
$$

The univariate discriminant $D$ in (8.1) acts as a nonlinear operator sending the $n$ variate polynomial $p\left(x_{1}, \ldots, x_{n}\right)$ to the $(n-1)$-variate polynomial $\tilde{p}\left(x_{1}, \ldots, x_{n-1}\right)$. Setting $y=x_{i}$ and applying Theorem 4.1 for each $i=1, \ldots, n-1$, we derive the following system of equations from the equations (3.2):

$$
\left\{\begin{array}{l}
\frac{\partial \tilde{p}\left(x_{1}, \ldots, x_{n-1}\right)}{\partial x_{1}}=0  \tag{8.2}\\
\dot{\partial \tilde{p}\left(x_{1}, \ldots, x_{n-1}\right)} \\
\partial x_{n-1}
\end{array}=0, ~=0 .\right.
$$

The structure of the equations (8.2) is the same as the structure of the equations (3.2). Therefore we can apply the operator (8.1) repeatedly:

$$
\begin{equation*}
D D_{p}=D\left(x_{1}, D\left(x_{2}, \ldots, D\left(x_{n}, p\left(x_{1}, \ldots, x_{n}\right)\right) \ldots\right)\right) \tag{8.3}
\end{equation*}
$$

Definition 8.1. The numeric quantity $D D_{p}$ introduced through the formula (8.3) is called the multiple discriminant of a multivariate polynomial $p\left(x_{1}, \ldots, x_{n}\right)$.

Generally speaking, the multiple discriminant $D D_{p}$ is different from the discriminant $D_{p}$ of a multivariate polynomial introduced in Definition 3.1. Unlike $D_{p}$, the formula (8.3) provides a clear algorithm for calculating $D D_{p}$.

The coefficients of the polynomial $p\left(x_{1}, \ldots, x_{n}\right)$ in (3.1) depend on $v$. Therefore the multiple discriminant $D D_{p}$ depends on $v$. Hence we can write the equation

$$
\begin{equation*}
D D_{p}(v)=0 \tag{8.4}
\end{equation*}
$$

where $\left.D D_{p}(v)=D\left(x_{1}, D\left(x_{2}, \ldots, D\left(x_{n}, f\left(x_{1}, \ldots, x_{n}\right)-v\right)\right) \ldots\right)\right)$.
Theorem 8.1. If a complex number $v$ is a critical value of a multivariate polynomial $f\left(x_{1}, \ldots, x_{n}\right)$, then it is a root of the equation (8.4).

Theorem 8.1 is proved by applying Theorem 4.1 repeatedly. Theorem 8.1 is similar to Theorem 3.1, but it is somewhat weaker, i. e. each critical value of a
multivariate polynomial $f\left(x_{1}, \ldots, x_{n}\right)$ is a root of the equation (8.4), but not each root of the equation (8.4) is a critical value of $f\left(x_{1}, \ldots, x_{n}\right)$.

## 9. Conclusions.

Theorem 8.1 along with the equation (8.4) and the formula (8.3) constitutes the main result of the present paper. It can be applied in testing positivity of multivariate quartic forms and forms of higher degrees.

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