ON QUARTIC FORMS ASSOCIATED WITH CUBIC TRANSFORMATIONS OF THE REAL PLANE.

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Abstract. A polynomial transformation of the real plane $\mathbb{R}^2$ is a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ given by two polynomials of two variables. Such a transformation is called cubic if the degrees of its polynomials are not greater than three. It turns out that cubic transformations are associated with some binary and quaternary quartic forms. In the present paper these forms are defined and studied.

1. Introduction.

Polynomial mappings $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ are of interests from many points of view (see [1–5]) most of which are associated with the Jacobian conjecture (see [6]). Polynomial mappings $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ are also considered (see [7], [8]) in connection with the real Jacobian conjecture. The rational real Jacobian conjecture is its generalization (see [9]). My interest is concentrated on cubic polynomial transformations of $\mathbb{R}^2$ due to their application to the perfect cuboid problem (see [10] and [11]).

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a cubic transformation of $\mathbb{R}^2$ and let $x = (x^1, x^2)$ be a point[1] of $\mathbb{R}^2$. Assume that $y = (y^1, y^2)$ is the image of x under this transformation. Then

$$y^i = \sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{p=1}^{2} F^i_{mnp}\, x^m\, x^n\, x^p + \ldots . \tag{1.1}$$

We use dots in order to designate lower degree terms, i. e. quadratic, linear and constant terms. The formula (1.1) is a coordinate presentation of the transformation $f\colon \mathbb{R}^2 \to \mathbb{R}^2$. The coefficients F^i_{mnp} in (1.1) are components of a tensor of the type $(1, 3)$ (see definition in [13]).

Along with (1.1), we consider linear transformations $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form

$$y^i = \sum_{m=1}^{2} T^i_m\, x^m + a^i. \tag{1.2}$$

Definition 1.1. Two cubic mappings $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\tilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ are called equivalent if there are two invertible linear transformations $\varphi_1 : \mathbb{R}^2 \to \mathbb{R}^2$ and $\varphi_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\varphi_1 \circ f = \tilde{f} \circ \varphi_2$.

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[1] We use upper and lower indices according to Einstein's tensorial notation (see [12])

Invertibility of a linear mapping $\varphi\colon\,\mathbb{R}^2\to\mathbb{R}^2$ means invertibility of its matrix T in (1.2). Assume that the mappings φ_1 and φ_2 in Definition 1.1 are given by the matrices $T[1]$ and $T[2]$ and assume that $S[1]=T[1]^{-1}$. Then the tensors F and $\tilde{F}$ associated with the cubic transformations f and $\tilde{f}$ are related to each other as

$$F^i_{mnp}=\sum^2_{\tilde{i}=1}\sum^2_{\tilde{m}=1}\sum^2_{\tilde{n}=1}\sum^2_{\tilde{p}=1}\tilde{F}^{\tilde{i}}_{\tilde{m}\tilde{n}\tilde{p}}\ S^i_{\tilde{i}}[1]\ T^{\tilde{m}}_m[2]\ T^{\tilde{n}}_n[2]\ T^{\tilde{p}}_p[2]. \tag{1.3}$$

The formula (1.3) is slightly different from the transformation formula for components of a tensor under a change of Cartesian coordinates in $\mathbb{R}^2$ (see [13]). The main goal of the present paper is to introduce some tensorial invariants associated with F and study their behavior under non-tensorial transformations given by the formula (1.3). These invariants are analogous to those considered in [14] in the case of quadratic transformations of the real plane $\mathbb{R}^2$.

2. Determinants and their collective behavior.

The tensor F in (1.1) is symmetric with respect to its lower indices. Taking into account this symmetry we can write (1.1) as

$$\begin{aligned}
y^1&=F^1_{111}\,(x^1)^3+3\,F^1_{112}\,(x^1)^2\,x^2+3\,F^1_{122}\,x^1\,(x^2)^2+F^1_{222}\,(x^2)^3,\\
y^2&=F^2_{111}\,(x^1)^3+3\,F^2_{112}\,(x^1)^2\,x^2+3\,F^2_{122}\,x^1\,(x^2)^2+F^2_{222}\,(x^2)^3.
\end{aligned}\tag{2.1}$$

Like in [14], using (2.1), we consider the following determinants

$$\begin{aligned}
&G_{1111}=\begin{vmatrix}F^1_{111}&F^1_{112}\\ F^2_{111}&F^2_{112}\end{vmatrix}, &&G_{1112}=\begin{vmatrix}F^1_{111}&F^1_{122}\\ F^2_{111}&F^2_{122}\end{vmatrix},\\
&G_{1122}=\begin{vmatrix}F^1_{111}&F^1_{222}\\ F^2_{111}&F^2_{222}\end{vmatrix}, &&G_{1212}=\begin{vmatrix}F^1_{112}&F^1_{122}\\ F^2_{112}&F^2_{122}\end{vmatrix},\\
&G_{1222}=\begin{vmatrix}F^1_{112}&F^1_{222}\\ F^2_{112}&F^2_{222}\end{vmatrix}, &&G_{2222}=\begin{vmatrix}F^1_{122}&F^1_{222}\\ F^2_{122}&F^2_{222}\end{vmatrix}.
\end{aligned}\tag{2.2}$$

Using (2.2), we define several quartic forms in $\mathbb{R}^2$ and in $\mathbb{R}^4$. Three of them are

$$\begin{gathered}
\omega[1]=G_{1111}\,(z^1)^4+2\,G_{1112}\,(z^1)^3\,z^2+\\
+(3\,G_{1212}+G_{1122})\,(z^1)^2\,(z^2)^2+2\,G_{1222}\,z^1\,(z^2)^3+G_{2222}\,(z^2)^4,\\[1ex]
\omega[2]=2\,G_{1111}\,(z^1)^3\,z^3+G_{1112}\,(z^1)^3\,z^4+3\,G_{1112}\,(z^1)^2\,z^2\,z^3+\\
+(3\,G_{1212}+G_{1122})\,(z^1)^2\,z^2\,z^4+(3\,G_{1212}+G_{1122})\,z^1\,(z^2)^2\,z^3+\\
+3\,G_{1222}\,z^1\,(z^2)^2\,z^4+G_{2222}\,(z^2)^3\,z^3+2\,G_{2222}\,(z^2)^3\,z^4,\\[1ex]
\omega[3]=3\,G_{1111}\,(z^1)^2\,(z^3)^2+3\,G_{1112}\,(z^1)^2\,z^3\,z^4+G_{1122}\,(z^1)^2\,(z^4)^2+\\
+3\,G_{1112}\,z^1\,z^2\,(z^3)^2+(9\,G_{1212}+G_{1122})\,z^1\,z^2\,z^3\,z^4+3\,G_{1222}\,z^1\,z^2\,(z^4)^2+\\
+G_{1122}\,(z^2)^2\,(z^3)^2+3\,G_{1222}\,(z^2)^2\,z^3\,z^4+3\,G_{2222}\,(z^2)^2\,(z^4)^2.
\end{gathered}\tag{2.3}$$

The other three quartic forms are

$$\begin{gathered}
\omega[4]=G_{1111}\,(z^1)^2\,(z^3)^2+G_{1112}\,(z^1)^2\,z^3\,z^4+G_{1212}\,(z^1)^2\,(z^4)^2+\\
+\,G_{1112}\,z^1\,z^2\,(z^3)^2+\,(G_{1212}+G_{1122})\,z^1\,z^2\,z^3\,z^4+G_{1222}\,z^1\,z^2\,(z^4)^2+\\
+\,G_{1212}\,(z^2)^2\,(z^3)^2+G_{1222}\,(z^2)^2\,z^3\,z^4+G_{1222}\,(z^2)^2\,(z^4)^2,
\end{gathered}$$

$$\begin{gathered}
\omega[5]=2\,G_{1111}\,z^1\,(z^3)^3+G_{1112}\,z^2\,(z^3)^3+3\,G_{1112}\,z^1\,(z^3)^2\,z^4+\\
+\,(3\,G_{1212}+G_{1122})\,z^2\,(z^3)^2\,z^4+(3\,G_{1212}+G_{1122})\,z^1\,z^3\,(z^4)^2+\\
+\,3\,G_{1222}\,z^2\,z^3\,(z^4)^2+G_{1222}\,z^1\,(z^4)^3+2\,G_{2222}\,z^2\,(z^4)^3,
\end{gathered}\tag{2.4}$$

$$\begin{gathered}
\omega[6]=G_{1111}\,(z^3)^4+2\,G_{1112}\,(z^3)^3\,z^4+\\
+\,(3\,G_{1212}+G_{1122})\,(z^3)^2\,(z^4)^2+2\,G_{1222}\,z^3\,(z^4)^3+G_{2222}\,(z^4)^4.
\end{gathered}$$

The quantities z^1, z^2, z^3, z^4 are interpreted as components of two vectors

$$\begin{Vmatrix} z^1\\ z^2 \end{Vmatrix}\in\mathbb{R}^2,\qquad\qquad \begin{Vmatrix} z^3\\ z^4 \end{Vmatrix}\in\mathbb{R}^2 \tag{2.5}$$

in $\omega[1]$ and $\omega[6]$ or as components of a single vector

$$\begin{Vmatrix} z^1\\ z^2\\ z^3\\ z^4 \end{Vmatrix}\in\mathbb{R}^2\oplus\mathbb{R}^2=\mathbb{R}^4 \tag{2.6}$$

in $\omega[2]$, $\omega[3]$, $\omega[4]$, and $\omega[5]$. Using the components of the vectors (2.5), we can write $\omega[1]$ and $\omega[6]$ from (2.3) and (2.4) in terms of their components:

$$\omega[1]=\sum_{i=1}^{2}\sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{p=1}^{2}\Omega[1]_{imnp}\,z^i\,z^m\,z^n\,z^p, \tag{2.7}$$

$$\omega[6]=\sum_{i=3}^{4}\sum_{m=3}^{4}\sum_{n=3}^{4}\sum_{p=3}^{4}\Omega[6]_{imnp}\,z^i\,z^m\,z^n\,z^p. \tag{2.8}$$

Similarly, using the components of the vector (2.6), we can write the quartic forms $\omega[2]$, $\omega[3]$, $\omega[4]$, and $\omega[5]$ in terms of their components:

$$\begin{aligned}
\omega[2]&=\sum_{i=1}^{4}\sum_{m=1}^{4}\sum_{n=1}^{4}\sum_{p=1}^{4}\Omega[2]_{imnp}\,z^i\,z^m\,z^n\,z^p,\\
\omega[3]&=\sum_{i=1}^{4}\sum_{m=1}^{4}\sum_{n=1}^{4}\sum_{p=1}^{4}\Omega[3]_{imnp}\,z^i\,z^m\,z^n\,z^p,\\
\omega[4]&=\sum_{i=1}^{4}\sum_{m=1}^{4}\sum_{n=1}^{4}\sum_{p=1}^{4}\Omega[4]_{imnp}\,z^i\,z^m\,z^n\,z^p,\\
\omega[5]&=\sum_{i=1}^{4}\sum_{m=1}^{4}\sum_{n=1}^{4}\sum_{p=1}^{4}\Omega[5]_{imnp}\,z^i\,z^m\,z^n\,z^p.
\end{aligned}\tag{2.9}$$

Now assume that we perform a linear change of coordinates in $\mathbb{R}^2$. It is expressed by the following matrix formulas for the vectors (2.5):

$$\begin{Vmatrix} z^1 \\ z^2 \end{Vmatrix} = \begin{Vmatrix} S^1_1 & S^1_2 \\ S^2_1 & S^2_2 \end{Vmatrix} \cdot \begin{Vmatrix} \tilde{z}^1 \\ \tilde{z}^2 \end{Vmatrix}, \qquad \begin{Vmatrix} z^3 \\ z^4 \end{Vmatrix} = \begin{Vmatrix} S^1_1 & S^1_2 \\ S^2_1 & S^2_2 \end{Vmatrix} \cdot \begin{Vmatrix} \tilde{z}^3 \\ \tilde{z}^4 \end{Vmatrix}. \tag{2.10}$$

For the vector (2.6) the formulas (2.10) imply

$$\begin{Vmatrix} z^1 \\ z^2 \\ z^3 \\ z^4 \end{Vmatrix} = \begin{Vmatrix} S^1_1 & S^1_2 & 0 & 0 \\ S^2_1 & S^2_2 & 0 & 0 \\ 0 & 0 & S^1_1 & S^1_2 \\ 0 & 0 & S^2_1 & S^2_2 \end{Vmatrix} \cdot \begin{Vmatrix} \tilde{z}^1 \\ \tilde{z}^2 \\ \tilde{z}^3 \\ \tilde{z}^4 \end{Vmatrix}. \tag{2.11}$$

The matrix S used in (2.10) is called a transition matrix (see [12] or [13]). Let's denote through $\hat{S}$ the block-diagonal matrix in (2.11). It plays the role of a transition matrix for the linear change of coordinates (2.11) in $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$.

Each linear change of coordinates implies some definite associated change of components for all tensors (see [13]). In the case of the tensor F in (1.1) we have

$$\tilde{F}^i_{mnp} = \sum_{\tilde{i}=1}^{2} \sum_{\tilde{m}=1}^{2} \sum_{\tilde{n}=1}^{2} \sum_{\tilde{p}=1}^{2} F^{\tilde{i}}_{\tilde{m}\tilde{n}\tilde{p}}\ T^i_{\tilde{i}}\ S^{\tilde{m}}_m\ S^{\tilde{n}}_n\ S^{\tilde{p}}_p. \tag{2.12}$$

Here $T = S^{-1}$. The formula (2.12) has its inverse formula

$$F^i_{mnp} = \sum_{\tilde{i}=1}^{2} \sum_{\tilde{m}=1}^{2} \sum_{\tilde{n}=1}^{2} \sum_{\tilde{p}=1}^{2} \tilde{F}^{\tilde{i}}_{\tilde{m}\tilde{n}\tilde{p}}\ S^i_{\tilde{i}}\ T^{\tilde{m}}_m\ T^{\tilde{n}}_n\ T^{\tilde{p}}_p, \tag{2.13}$$

which is very similar to the formula (1.3), though the meanings of the formulas (1.3) and (2.13) are quite different.

Now we can use (2.12) in (2.2) instead of F^i_{mnp} and obtain six determinants $\tilde{G}_{1111}$, $\tilde{G}_{1112}$, $\tilde{G}_{1122}$, $\tilde{G}_{1212}$, $\tilde{G}_{1222}$, $\tilde{G}_{2222}$. Then we can use these determinants in (2.3) and (2.4) instead of G_{1111}, G_{1112}, G_{1122}, G_{1212}, G_{1222}, G_{2222} simultaneously replacing z^1, z^2, z^3, z^4 by $\tilde{z}^1$, $\tilde{z}^2$, $\tilde{z}^3$, $\tilde{z}^4$. As a result we get some expressions for $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ through F, T, S, and $\tilde{z}$. On the other hand we can apply (2.10) or (2.11) directly to (2.3) and (2.4). It turns out that the results of these two ways of expressing $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ through F, T, S, and $\tilde{z}$ do always coincide. We write this fact as

$$\omega[q](\tilde{F}(F), \tilde{z}) = \omega[q](F, z(\tilde{z})),\ \ q = 1, \ldots, 6. \tag{2.14}$$

We can also express this fact using the component notations (2.7), (2.8) and (2.9):

$$\tilde{\Omega}[1]_{\tilde{i}\tilde{m}\tilde{n}\tilde{p}} = \sum_{i=1}^{2} \sum_{m=1}^{2} \sum_{n=1}^{2} \sum_{p=1}^{2} \Omega[1]_{imnp}\ S^i_{\tilde{i}}\ S^m_{\tilde{m}}\ S^n_{\tilde{n}}\ S^p_{\tilde{p}}, \tag{2.15}$$

$$\tilde{\Omega}[6]_{\tilde{i}\tilde{m}\tilde{n}\tilde{p}} = \sum_{i=3}^{4} \sum_{m=3}^{4} \sum_{n=3}^{4} \sum_{p=3}^{4} \Omega[6]_{imnp}\ S^i_{\tilde{i}}\ S^m_{\tilde{m}}\ S^n_{\tilde{n}}\ S^p_{\tilde{p}}, \tag{2.16}$$

$$\tilde{\Omega}[q]_{\tilde{i}\tilde{m}\tilde{n}\tilde{p}} = \sum_{i=1}^{4}\sum_{m=1}^{4}\sum_{n=1}^{4}\sum_{p=1}^{4}\Omega[q]_{imnp}\ \hat{S}^{i}_{\tilde{i}}\ \hat{S}^{m}_{\tilde{m}}\ \hat{S}^{n}_{\tilde{n}}\ \hat{S}^{p}_{\tilde{p}},\ q = 2,\ \dots\ ,\ 5. \tag{2.17}$$

The formula (2.14), as well as the formulas (2.15), (2.16), and (2.17), means that the components of the quartic forms $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ exhibit true tensorial behavior under the linear change of coordinates (2.10) and (2.11).

3. Explicit formulas for components of quartic forms.

In our case $\dim\mathbb{R}^2 = 2$. There is a fundamental pseudotensor $\mathbf{d}$ of the type $(0,2)$ and of the weight -1 in each two-dimensional linear vector space V. Its components are given by the following skew-symmetric matrix

$$d_{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{3.1}$$

The dual object for $\mathbf{d}$ is given by the same matrix (3.1):

$$d^{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{3.2}$$

This dual object is denoted by the same symbol $\mathbf{d}$ as the initial one. It is a pseudotensor of the type $(2,0)$, its weight is equal to 1. The following definition is provided for reference purposes only.

Definition 3.1. A pseudotensor of the type (r,s) and of the weight m in $\mathbb{R}^2$ is a geometrical and/or physical object presented by an array of quantities $F^{i_1\dots\,i_r}_{j_1\dots\,j_s}$ transformed as follows under any linear change of coordinates like (2.10):

$$F^{i_1\dots\,i_r}_{j_1\dots\,j_s} = (\det T)^m \sum_{\substack{p_1\dots p_r\\ q_1\dots q_s}} S^{i_1}_{p_1}\dots\, S^{i_r}_{p_r}\ T^{q_1}_{j_1}\dots\, T^{q_s}_{j_s}\ \tilde{F}^{p_1\dots\,p_r}_{q_1\dots\,q_s}. \tag{3.3}$$

The definition 3.1 can be easily modified for the case of the space $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ using the matrices $\hat{S}$ and $\hat{T} = \hat{S}^{-1}$ instead of S and $T = S^{-1}$.

Definition 3.2. A pseudotensor of the type (r,s) and the weight m in the space $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$ is a geometrical and/or physical object presented by an array $F^{i_1\dots\,i_r}_{j_1\dots\,j_s}$ transformed as follows under any linear change of coordinates like (2.11):

$$F^{i_1\dots\,i_r}_{j_1\dots\,j_s} = (\det T)^m \sum_{\substack{p_1\dots p_r\\ q_1\dots q_s}} \hat{S}^{i_1}_{p_1}\dots\, \hat{S}^{i_r}_{p_r}\ \hat{T}^{q_1}_{j_1}\dots\, \hat{T}^{q_s}_{j_s}\ \tilde{F}^{p_1\dots\,p_r}_{q_1\dots\,q_s}. \tag{3.4}$$

Pseudotensors of the weight $m = 0$ are known as tensors (see [13]), e.g. the formula (2.13) is a particular instance of the formula (3.3). The formulas (2.15) and (2.16) can be transformed to special instances of the formula (3.3), while the formula (2.17) can be transformed to a special instance of the formula (3.4).

The weights of the pseudotensors (3.1) and (3.2) are opposite to each other. Therefore they can be used in order to build a tensor. Using these two pseudotensors and the tensor F from (1.1), we define a new tensor with the components

$$\Omega_{imnp}=\frac{1}{2}\sum_{r_1=1}^{2}\sum_{r_2=1}^{2}\sum_{s_1=1}^{2}\sum_{s_2=1}^{2}F^{r_1}_{s_1im}\,F^{r_2}_{s_2np}\,d^{s_1s_2}\,d_{r_1r_2}. \tag{3.5}$$

Theorem 3.1. *The components of the quartic form $\omega[1]$ in (2.7) are produced from the components of the tensor (3.5) by symmetrizing them with respect to i, m, n, p:*

$$\Omega[1]_{imnp}=\frac{1}{3}\bigl(\Omega_{imnp}+\Omega_{mnip}+\Omega_{nimp}\bigr). \tag{3.6}$$

The proof of Theorem 3.1 is pure computations. In particular, one can easily verify that the right hand side of the formula (3.6) is fully symmetric with respect to the indices i, m, n, p.

The form $\omega[6]$ in (2.4) does coincide with the form $\omega[1]$ in (2.3) up to the substitution of z^3 for z^1 and z^4 for z^2. Its components in (2.8) are given by the formula

$$\Omega[6]_{imnp}=\Omega[1]_{(i-2)\,(m-2)\,(n-2)\,(p-2)}.$$

The forms $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$ are more complicated. In order to serve $\omega[2]$ we need to extend the tensor (3.5) from $\mathbb{R}^2$ to $\mathbb{R}^4=\mathbb{R}^2\oplus\mathbb{R}^2$. We do it as follows:

$$\hat{\Omega}_{imnp}=\begin{cases}\Omega_{imnp} & \text{if }\ i\leqslant 2,\ m\leqslant 2,\ n\leqslant 2,\ p\leqslant 2,\\ 0 & \text{in all other cases.}\end{cases} \tag{3.7}$$

Apart from (3.7) we define the exchange operator ε given by the matrix

$$\varepsilon^i_j=\begin{Vmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{Vmatrix}. \tag{3.8}$$

Then we combine the tensors (3.7) and (3.8) in the following way:

$$\Omega[A]_{imnp}=\sum_{\tilde{p}=1}^{4}2\,\hat{\Omega}_{imn\tilde{p}}\ \varepsilon^{\tilde{p}}_p\,. \tag{3.9}$$

Theorem 3.2. *The components of the form $\omega[2]$ in (2.9) are produced from the components of the tensor (3.9) by symmetrizing them with respect to i, m, n, p:*

$$\begin{aligned}\Omega[2]_{imnp}=\frac{1}{12}\bigl(&\Omega[A]_{imnp}+\Omega[A]_{ipmn}+\Omega[A]_{inpm}+\Omega[A]_{inmp}+\\&+\Omega[A]_{ipnm}+\Omega[A]_{impn}+\Omega[A]_{mpin}+\Omega[A]_{mnpi}+\Omega[A]_{pnim}+\\&+\Omega[A]_{pmni}+\Omega[A]_{nmip}+\Omega[A]_{npmi}\bigr).\end{aligned}$$

In order to serve the forms $\omega[3]$ and $\omega[4]$ we define the following two tensors:

$$\Omega[B]_{imnp}=\sum_{\tilde{m}=1}^{4}\sum_{\tilde{p}=1}^{4}\hat{\Omega}_{i\tilde{m}n\tilde{p}}\ \varepsilon^{\tilde{m}}_m\ \varepsilon^{\tilde{p}}_p+\sum_{\tilde{n}=1}^{4}\sum_{\tilde{p}=1}^{4}2\,\hat{\Omega}_{im\tilde{n}\tilde{p}}\ \varepsilon^{\tilde{n}}_n\ \varepsilon^{\tilde{p}}_p\,, \tag{3.10}$$

$$\Omega[C]_{imnp}=\sum_{\tilde{m}=1}^{4}\sum_{\tilde{p}=1}^{4}\hat{\Omega}_{i\tilde{m}n\tilde{p}}\;\varepsilon^{\tilde{m}}_{m}\;\varepsilon^{\tilde{p}}_{p}\,. \tag{3.11}$$

Theorem 3.3. *The components of the form $\omega[3]$ in (2.9) are produced from the components of the tensor (3.10) by symmetrizing them with respect to i, m, n, p:*

$$\begin{aligned}\Omega[3]_{imnp}=\frac{1}{24}\bigl(&\Omega[B]_{imnp}+\Omega[B]_{ipmn}+\Omega[B]_{inpm}+\Omega[B]_{inmp}+\\+&\Omega[B]_{ipnm}+\Omega[B]_{impn}+\Omega[B]_{minp}+\Omega[B]_{mpin}+\Omega[B]_{mnpi}+\\+&\Omega[B]_{mnip}+\Omega[B]_{mpni}+\Omega[B]_{mipn}+\Omega[B]_{pimn}+\Omega[B]_{pnim}+\\+&\Omega[B]_{pmni}+\Omega[B]_{pmin}+\Omega[B]_{pnmi}+\Omega[B]_{pinm}+\Omega[B]_{nipm}+\\+&\Omega[B]_{nmip}+\Omega[B]_{npmi}+\Omega[B]_{npim}+\Omega[B]_{nmpi}+\Omega[B]_{nimp}\bigr).\end{aligned}$$

Theorem 3.4. *The components of the form $\omega[4]$ in (2.9) are produced from the components of the tensor (3.11) by symmetrizing them with respect to i, m, n, p:*

$$\begin{aligned}\Omega[4]_{imnp}=\frac{1}{12}\bigl(&\Omega[C]_{imnp}+\Omega[C]_{ipmn}+\Omega[C]_{inpm}+\Omega[C]_{inmp}+\\+&\Omega[C]_{ipnm}+\Omega[C]_{impn}+\Omega[C]_{minp}+\Omega[C]_{mnpi}+\Omega[C]_{mpni}+\\&+\Omega[C]_{mipn}+\Omega[C]_{pmni}+\Omega[C]_{pinm}\bigr).\end{aligned}$$

The form $\omega[5]$ is similar to $\omega[2]$. It differs from $\omega[2]$ by exchanging $z^1\longleftrightarrow z^3$ and $z^2\longleftrightarrow z^4$. Such an exchange is performed by means of the operator (3.8). Therefore, looking at (3.9), by analogy we define the tensor

$$\Omega[D]_{imnp}=\sum_{\tilde{m}=1}^{4}\sum_{\tilde{n}=1}^{4}\sum_{\tilde{p}=1}^{4}2\,\hat{\Omega}_{i\tilde{m}\tilde{n}\tilde{p}}\;\varepsilon^{\tilde{m}}_{m}\;\varepsilon^{\tilde{n}}_{n}\;\varepsilon^{\tilde{p}}_{p}\,. \tag{3.12}$$

Theorem 3.5. *The components of the form $\omega[5]$ in (2.9) are produced from the components of the tensor (3.12) by symmetrizing them with respect to i, m, n, p:*

$$\begin{aligned}\Omega[5]_{imnp}=\frac{1}{12}\bigl(&\Omega[D]_{imnp}+\Omega[D]_{nimp}+\Omega[D]_{mnip}+\Omega[D]_{inmp}\\+&\Omega[D]_{minp}+\Omega[D]_{nmip}+\Omega[D]_{pimn}+\Omega[D]_{mpin}+\Omega[D]_{pnim}+\\&+\Omega[D]_{ipnm}+\Omega[D]_{pmni}+\Omega[D]_{npmi}\bigr).\end{aligned}$$

Theorems 3.2, 3.3, 3.4, 3.5 are similar to each other. All of them are proved by means of direct calculations.

4. Behavior under left and right compositions with linear transformations.

The tensorial behavior of the quartic forms $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ revealed in Section 2 is essential for understanding their nature. However, it is inessential for their prospective applications. In the present section we study their behavior under left and right compositions with linear transformations of the form (1.2) introduced in Definition 1.1.

Assume that we removed the right composition in Definition 1.1 so that we have the left composition only, i.e. assume that two cubic mappings $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\tilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form (1.1) are related to each other as

$$f = \varphi^{-1} \circ \tilde{f}, \tag{4.1}$$

where $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation of the form (1.2). In the case of (4.1) the formula (1.3) reduces to the following one:

$$F^i_{mnp} = \sum_{\tilde{i}=1}^{2} \tilde{F}^{\tilde{i}}_{mnp}\ S^i_{\tilde{i}}. \tag{4.2}$$

Here $S = T^{-1}$ and T is the matrix from (1.2). Applying (4.2) to (2.2) we derive

$$\begin{aligned} &G_{1111} = \det S \cdot \tilde{G}_{1111}, && G_{1112} = \det S \cdot \tilde{G}_{1112},\\ &G_{1122} = \det S \cdot \tilde{G}_{1122}, && G_{1212} = \det S \cdot \tilde{G}_{1212},\\ &G_{1222} = \det S \cdot \tilde{G}_{1222}, && G_{2222} = \det S \cdot \tilde{G}_{2222}. \end{aligned} \tag{4.3}$$

Then, applying (4.3) to the components of the quartic forms $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ in (2.7), (2.8), and (2.9), we derive

$$\Omega[q]_{imnp} = \det S \cdot \tilde{\Omega}[q]_{imnp}, \ \text{ where } \ q = 1,], \ldots, 6. \tag{4.4}$$

This result is summarized in the following theorem.

Theorem 4.1. *Under the left composition (4.1) of a cubic transformation $\tilde{f}$ with the inverse of a liner transformation φ in (1.2) its associated quartic forms $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ are transformed according to the formulas (4.4).*

Now assume that we removed the left composition in Definition 1.1 so that we have the right composition only, i.e. assume that two cubic mappings $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\tilde{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the form (1.1) are related to each other as

$$f = \tilde{f} \circ \varphi, \tag{4.5}$$

where $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation of the form (1.2). In the case of (4.5) the formula (1.3) reduces to the following one:

$$F^i_{mnp} = \sum_{\tilde{m}=1}^{2} \sum_{\tilde{n}=1}^{2} \sum_{\tilde{p}=1}^{2} \tilde{F}^{\,i}_{\tilde{m}\tilde{n}\tilde{p}}\ T^{\tilde{m}}_m\ T^{\tilde{n}}_n\ T^{\tilde{p}}_p. \tag{4.6}$$

Applying (4.6) to (2.2) and then to (2.3), one can derive the formulas

$$\Omega[1]_{imnp} = \det T \cdot \sum_{\tilde{i}=1}^{2} \sum_{\tilde{m}=1}^{2} \sum_{\tilde{n}=1}^{2} \sum_{\tilde{p}=1}^{2} \tilde{\Omega}[1]_{\tilde{i}\tilde{m}\tilde{n}\tilde{p}}\ T^{\tilde{i}}_i\ T^{\tilde{m}}_m\ T^{\tilde{n}}_n\ T^{\tilde{p}}_p, \tag{4.7}$$

$$\Omega[6]_{imnp} = \det T \cdot \sum_{\tilde{i}=3}^{4} \sum_{\tilde{m}=3}^{4} \sum_{\tilde{n}=3}^{4} \sum_{\tilde{p}=3}^{4} \tilde{\Omega}[6]_{\tilde{i}\tilde{m}\tilde{n}\tilde{p}}\ \hat{T}^{\tilde{i}}_i\ \hat{T}^{\tilde{m}}_m\ \hat{T}^{\tilde{n}}_n\ \hat{T}^{\tilde{p}}_p, \tag{4.8}$$

$$\Omega[q]_{imnp} = \det T\,\cdot \sum_{\tilde{i}=1}^{4}\sum_{\tilde{m}=1}^{4}\sum_{\tilde{n}=1}^{4}\sum_{\tilde{p}=1}^{4}\tilde{\Omega}[q]_{\tilde{i}\tilde{m}\tilde{n}\tilde{p}}\ \hat{T}^{\tilde{i}}_{i}\ \hat{T}^{\tilde{m}}_{m}\ \hat{T}^{\tilde{n}}_{n}\ \hat{T}^{\tilde{p}}_{p},\ q = 2,\ldots,5. \tag{4.9}$$

This result is summarized in the following theorem.

Theorem 4.2. *Under the right composition* (4.5) *of a cubic transformation* $\tilde{f}$ *with a liner transformation* φ *in* (1.2) *its associated quartic forms* $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ *are transformed according to the formulas* (4.7), (4.8), *and* (4.9), *where* $\hat{T}$ *is the block-diagonal matrix similar to* $\hat{S}$ *in* (2.11) *and built with the use of* T.

The formulas (4.7), (4.8), (4.9) resemble the formula (3.3) in Definition 3.1. For this reason the analogs of the forms $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ in [14] were called pseudotensors. However, this is not correct with respect to changes of coordinates of the form (2.10) and (2.11).

The origin and the prospective applications of the quartic forms $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ are associated with the following theorem.

Theorem 4.3. *If a cubic transformation* f *is produced as the right composition* (4.5) *of another cubic transformation* $\tilde{f}$ *with a liner transformation* φ *of the form* (1.2), *then the determinants* (2.2) *associated with* f *are expressed through the values of the quartic forms* $\tilde{\omega}[1]$, $\tilde{\omega}[2]$, $\tilde{\omega}[3]$, $\tilde{\omega}[4]$, $\tilde{\omega}[5]$, $\tilde{\omega}[6]$ *associated with the second cubic transformation* $\tilde{f}$ *according to the formulas*

$$\begin{aligned}
G_{1111}(f) &= \det T\cdot\tilde{\omega}[1](z^1,z^2),\\
G_{1112}(f) &= \det T\cdot\tilde{\omega}[2](z^1,z^2,z^3,z^4),\\
G_{1122}(f) &= \det T\cdot\tilde{\omega}[3](z^1,z^2,z^3,z^4),\\
G_{1212}(f) &= \det T\cdot\tilde{\omega}[4](z^1,z^2,z^3,z^4),\\
G_{1222}(f) &= \det T\cdot\tilde{\omega}[5](z^1,z^2,z^3,z^4),\\
G_{2222}(f) &= \det T\cdot\tilde{\omega}[6](z^3,z^4),
\end{aligned} \tag{4.10}$$

where z^1, z^2, z^3, z^4 *are given by the components of the matrix* T *in* (1.2):

$$\begin{aligned}
z^1 &= T^1_1, & z^3 &= T^1_2,\\
z^2 &= T^2_1, & z^6 &= T^2_2.
\end{aligned} \tag{4.11}$$

Theorem 4.3 is proved by means of direct calculations which consist in deriving the formulas (4.10) with the use of (4.11).

5. Conclusions.

Theorem 4.3 is the most important result of the present paper in view of its prospective application to classification of potentially invertible cubic transformations of the real plane $\mathbb{R}^2$. Such a classification of potentially invertible quadratic transformations can be found in [14].

References

1. Vitushkin A. G., *On polynomial transformations of* $\mathbb{C}^2$, Proceedings of the International Conference on Manifolds and Related Topics in Topology, Tokyo, 1973, University of Tokyo Press, 1975, pp. 415–417.
2. Vitushkin A. G., *Some examples in connection with problems about polynomial transformations of* $\mathbb{C}^2$, Izv. Akad. Nauk SSSR, Ser. Mat. **35** (1971), no. 2, 269–279.
3. Orevkov S. Yu., *On three-sheeted polynomial mappings of* $\mathbb{C}^2$, Izv. Akad. Nauk SSSR, Ser. Mat. **50** (1986), no. 6, 1231–1240.
4. Domrina A. V., Orevkov S. Yu., *On four-sheeted polynomial mappings of* $\mathbb{C}^2$*. I. The case of an irreducible ramification curve*, Mat. Zamretki **64** (1998), no. 6, 847–862.
5. Guedj V., *Dynamics of quadratic polynomial mappings of* $\mathbb{C}^2$, Michigan Math. Journal **52** (2004), no. 3, 627–648.
6. *Jacobian conjecture*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
7. Pinchuk S. I., *A counterexample to the strong real Jacobian conjecture*, Mathematische Zeitschrift **217** (1994), 1–4.
8. Jelonek J. A., *A geometry of polynomial transformations of the real plane*, Bull. Polish Academy of Sciences, Mathematics **48** (2000), no. 1, 57–62.
9. Campbell L. A., *On the rational real Jacobian conjecture*, e-print arXiv:1210.0251 in Electronic Archive http://arXiv.org.
10. *Euler brick*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
11. Sharipov R. A., *Asymptotic estimates for roots of the cuboid characteristic equation in the nonlinear region*, e-print arXiv:1506.04705 in http://arXiv.org.
12. Sharipov R. A., *Course of analytical geometry*, Bashkir State University, Ufa, 2011; see also arXiv:1111.6521 in Electronic Archive http://arXiv.org.
13. Sharipov R. A., *Quick introduction to tensor analysis*, free on-line textbook arXiv:math/0403252 in Electronic Archive http://arXiv.org.
14. Sharipov R. A., *A note on invertible quadratic transformations of the real plane*, e-print arXiv:1507.01861 in Electronic Archive http://arXiv.org.

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