A ROUGH CLASSIFICATION OF POTENTIALLY INVERTIBLE CUBIC TRANSFORMATIONS OF THE REAL PLANE.

Ruslan Sharipov

Abstract. A polynomial transformation of the real plane $\mathbb{R}^2$ is a mapping $\mathbb{R}^2 \to \mathbb{R}^2$ given by two polynomials of two variables. Such a transformation is called cubic if the degrees of its polynomials are not greater than three. In the present paper a rough classification scheme for cubic transformations of $\mathbb{R}^2$ is suggested. It is based on quartic forms associated with these transformations.

1. Introduction.

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a cubic transformation of $\mathbb{R}^2$. In the coordinate presentation it is given by the following formula:

$$y^i = \sum_{m=1}^{2}\sum_{n=1}^{2}\sum_{p=1}^{2} F^i_{mnp}\, x^m\, x^n\, x^p + \ldots\,. \tag{1.1}$$

We use upper and lower indices in (1.1) according to Einstein's tensorial notation (see [1]). Polynomial mappings $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ were considered in [2], [3] in connection with the real Jacobian conjecture. The rational real Jacobian conjecture is its generalization (see [4]). Polynomial mappings $f\colon \mathbb{C}^2 \to \mathbb{C}^2$ of the complex plane $\mathbb{C}^2$ were considered in [5–9].

My interest to cubic polynomial transformations of $\mathbb{R}^2$ is due to their application to the perfect cuboid problem (see [10] and [11]). However, in the present paper we study these transformations by themselves. Cubic polynomial transformations of the form (1.1) are associated with some definite quartic forms (see [12]). They are produced using components of the tensor F in (1.1). In the present paper we use them as a basis for building a rough classification of potentially invertible cubic transformations of $\mathbb{R}^2$. This classification can further be refined using other invariants and/or methods. The classification of potentially invertible quadratic transformations of $\mathbb{R}^2$ can be found in [13].

2. Determinants and quartic forms.

The tensor F determines the leading terms of the cubic transformation (1.1). They are of the most attention below. Terms of lower degrees are denoted by dots. They are not used in our rough classification scheme.

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Along with (1.1), we consider linear transformations $\varphi\colon\,\mathbb{R}^2\to\mathbb{R}^2$ of the form

$$y^i=\sum_{m=1}^{2}T^i_m\,x^m+a^i. \tag{2.1}$$

Definition 1.1. Two cubic mappings $f:\,\mathbb{R}^2\to\mathbb{R}^2$ and $\tilde f:\,\mathbb{R}^2\to\mathbb{R}^2$ are called equivalent if there are two invertible linear transformations $\varphi_1:\,\mathbb{R}^2\to\mathbb{R}^2$ and $\varphi_2\colon\,\mathbb{R}^2\to\mathbb{R}^2$ such that $\varphi_1\circ f=\tilde f\circ\varphi_2$.

It is natural to classify cubic transformations up to the equivalence introduced in Definition 1.1. For this purpose we need some parameters which are relatively stable when passing from a given cubic transformations to an equivalent one. These parameters are constructed through the tensor F in (1.1).

The tensor F in (1.1) is symmetric with respect to its lower indices. Taking into account this symmetry we can write (1.1) as

$$\begin{aligned}y^1&=F^1_{111}\,(x^1)^3+3\,F^1_{112}\,(x^1)^2\,x^2+3\,F^1_{122}\,x^1\,(x^2)^2+F^1_{222}\,(x^2)^3+\ldots,\\ y^2&=F^2_{111}\,(x^1)^3+3\,F^2_{112}\,(x^1)^2\,x^2+3\,F^2_{122}\,x^1\,(x^2)^2+F^2_{222}\,(x^2)^3+\ldots.\end{aligned} \tag{2.2}$$

According to [12] and [13], we consider the following determinants:

$$\begin{aligned}&G_{1111}=\begin{vmatrix}F^1_{111}&F^1_{112}\\ F^2_{111}&F^2_{112}\end{vmatrix}, &&G_{1112}=\begin{vmatrix}F^1_{111}&F^1_{122}\\ F^2_{111}&F^2_{122}\end{vmatrix},\\ &G_{1122}=\begin{vmatrix}F^1_{111}&F^1_{222}\\ F^2_{111}&F^2_{222}\end{vmatrix}, &&G_{1212}=\begin{vmatrix}F^1_{112}&F^1_{122}\\ F^2_{112}&F^2_{122}\end{vmatrix},\\ &G_{1222}=\begin{vmatrix}F^1_{112}&F^1_{222}\\ F^2_{112}&F^2_{222}\end{vmatrix}, &&G_{2222}=\begin{vmatrix}F^1_{122}&F^1_{222}\\ F^2_{122}&F^2_{222}\end{vmatrix}.\end{aligned} \tag{2.3}$$

The determinants (2.3) are related to six quartic forms $\omega[1]$, $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$, $\omega[6]$ introduced in [12]. They are given by the following formulas:

$$\begin{gathered}\omega[1]=G_{1111}\,(z^1)^4+2\,G_{1112}\,(z^1)^3\,z^2+\\ +(3\,G_{1212}+G_{1122})\,(z^1)^2\,(z^2)^2+2\,G_{1222}\,z^1\,(z^2)^3+G_{2222}\,(z^2)^4,\end{gathered} \tag{2.4}$$

$$\begin{gathered}\omega[2]=2\,G_{1111}\,(z^1)^3\,z^3+G_{1112}\,(z^1)^3\,z^4+3\,G_{1112}\,(z^1)^2\,z^2\,z^3+\\ +(3\,G_{1212}+G_{1122})\,(z^1)^2\,z^2\,z^4+(3\,G_{1212}+G_{1122})\,z^1\,(z^2)^2\,z^3+\\ +3\,G_{1222}\,z^1\,(z^2)^2\,z^4+G_{2222}\,(z^2)^3\,z^3+2\,G_{2222}\,(z^2)^3\,z^4,\end{gathered} \tag{2.5}$$

$$\begin{gathered}\omega[3]=3\,G_{1111}\,(z^1)^2\,(z^3)^2+3\,G_{1112}\,(z^1)^2\,z^3\,z^4+G_{1122}\,(z^1)^2\,(z^4)^2+\\ +3\,G_{1112}\,z^1\,z^2\,(z^3)^2+(9\,G_{1212}+G_{1122})\,z^1\,z^2\,z^3\,z^4+3\,G_{1222}\,z^1\,z^2\,(z^4)^2+\\ +G_{1122}\,(z^2)^2\,(z^3)^2+3\,G_{1222}\,(z^2)^2\,z^3\,z^4+3\,G_{2222}\,(z^2)^2\,(z^4)^2,\end{gathered} \tag{2.6}$$

$$\begin{aligned}\omega[4] = G_{1111}\,(z^1)^2\,(z^3)^2 + G_{1112}\,(z^1)^2\,z^3\,z^4 + G_{1212}\,(z^1)^2\,(z^4)^2 +\\ + G_{1112}\,z^1\,z^2\,(z^3)^2 + \,(G_{1212} + G_{1122})\,z^1\,z^2\,z^3\,z^4 + G_{1222}\,z^1\,z^2\,(z^4)^2 +\\ + G_{1212}\,(z^2)^2\,(z^3)^2 + G_{1222}\,(z^2)^2\,z^3\,z^4 + G_{1222}\,(z^2)^2\,(z^4)^2,\end{aligned} \tag{2.7}$$

$$\begin{aligned}\omega[5] = 2\,G_{1111}\,z^1\,(z^3)^3 + G_{1112}\,z^2\,(z^3)^3 + 3\,G_{1112}\,z^1\,(z^3)^2\,z^4 +\\ + (3\,G_{1212} + G_{1122})\,z^2\,(z^3)^2\,z^4 + (3\,G_{1212} + G_{1122})\,z^1\,z^3\,(z^4)^2 +\\ + 3\,G_{1222}\,z^2\,z^3\,(z^4)^2 + G_{1222}\,z^1\,(z^4)^3 + 2\,G_{2222}\,z^2\,(z^4)^3,\end{aligned} \tag{2.8}$$

$$\begin{aligned}\omega[6] = G_{1111}\,(z^3)^4 + 2\,G_{1112}\,(z^3)^3\,z^4 +\\ + (3\,G_{1212} + G_{1122})\,(z^3)^2\,(z^4)^2 + 2\,G_{1222}\,z^3\,(z^4)^3 + G_{2222}\,(z^4)^4.\end{aligned} \tag{2.9}$$

The determinants (2.3) arise not only as coefficients in the forms (2.4), (2.5), (2.6), (2.7), (2.8), (2.9). They generate these forms according to the following theorem.

Theorem 2.1. *If a cubic transformation f is produced as the right composition $f = \tilde{f} \circ \varphi$ of another cubic transformation $\tilde{f}$ with a liner transformation φ of the form (2.1), then the determinants (2.3) associated with f are expressed through the values of the quartic forms $\tilde{\omega}[1]$, $\tilde{\omega}[2]$, $\tilde{\omega}[3]$, $\tilde{\omega}[4]$, $\tilde{\omega}[5]$, $\tilde{\omega}[6]$ associated with the second cubic transformation $\tilde{f}$ according to the formulas*

$$\begin{aligned}
&G_{1111}(f) = \det T \cdot \tilde{\omega}[1](z^1, z^2),\\
&G_{1112}(f) = \det T \cdot \tilde{\omega}[2](z^1, z^2, z^3, z^4),\\
&G_{1122}(f) = \det T \cdot \tilde{\omega}[3](z^1, z^2, z^3, z^4),\\
&G_{1212}(f) = \det T \cdot \tilde{\omega}[4](z^1, z^2, z^3, z^4),\\
&G_{1222}(f) = \det T \cdot \tilde{\omega}[5](z^1, z^2, z^3, z^4),\\
&G_{2222}(f) = \det T \cdot \tilde{\omega}[6](z^3, z^4),
\end{aligned}$$

where z^1, z^2, z^3, z^4 are given by the components of the matrix T in (2.1):

$$\begin{aligned}
&z^1 = T^1_1, &&z^3 = T^1_2,\\
&z^2 = T^2_1, &&z^4 = T^2_2.
\end{aligned} \tag{2.10}$$

The transformation of the determinants (2.3) under left compositions is more simple. It is described by another theorem.

Theorem 2.2. *If a cubic transformation f is produced as the left composition $f = \varphi^{-1} \circ \tilde{f}$ of another cubic transformation $\tilde{f}$ with the inverse of a liner transformation φ in (2.1), then the associated determinants of the transformations f and $\tilde{f}$ are related to each other according to the formulas*

$$\begin{aligned}
&G_{1111} = \det S \cdot \tilde{G}_{1111}, &&G_{1112} = \det S \cdot \tilde{G}_{1112},\\
&G_{1122} = \det S \cdot \tilde{G}_{1122}, &&G_{1212} = \det S \cdot \tilde{G}_{1212},\\
&G_{1222} = \det S \cdot \tilde{G}_{1222}, &&G_{2222} = \det S \cdot \tilde{G}_{2222},
\end{aligned} \tag{2.11}$$

where $S = T^{-1}$ is the inverse of the matrix T whose components are used in (2.1).

Theorems 2.1 and 2.2 are proved by means of direct calculations. Theorem 2.1 was formulated in [12]. The formulas (2.11) were also written in [12], though they were not formulated as a theorem.

3. The case where $\omega[1]$ is zero.

If the quartic form $\omega[1]$ in (2.4) is identically zero, this means that the determinants G_{1111}, G_{1112}, G_{1222}, G_{2222} in (2.3) are zero:

$$\begin{aligned} &G_{1111}=0, && G_{1112}=0,\\ &G_{1222}=0, && G_{2222}=0. \end{aligned} \tag{3.1}$$

Apart from (3.1), this means that the following equality is fulfilled:

$$3\,G_{1212}+G_{1122}=0. \tag{3.2}$$

The matrices producing the determinants G_{1111}, G_{1112}, G_{1122}, G_{1212}, G_{1222}, G_{2222} in (2.3) are composed with the use of the following four vector-columns:

$$\begin{Vmatrix} F^1_{111}\\ F^2_{111} \end{Vmatrix}, \qquad \begin{Vmatrix} F^1_{112}\\ F^2_{112} \end{Vmatrix}, \qquad \begin{Vmatrix} F^1_{122}\\ F^2_{122} \end{Vmatrix}, \qquad \begin{Vmatrix} F^1_{222}\\ F^2_{222} \end{Vmatrix}. \tag{3.3}$$

At least one of the vector-columns (3.3) is nonzero (since f is cubic).

Assume that the first column in (3.3) is nonzero, i. e. assume that

$$\begin{Vmatrix} F^1_{111}\\ F^2_{111} \end{Vmatrix}\neq 0. \tag{3.4}$$

Then due to $G_{1111}=0$ and $G_{1112}=0$ in (3.1) the second and the third columns in (3.3) are expressed through the first one as follows:

$$\begin{Vmatrix} F^1_{112}\\ F^2_{112} \end{Vmatrix}=\alpha\begin{Vmatrix} F^1_{111}\\ F^2_{111} \end{Vmatrix}, \qquad \begin{Vmatrix} F^1_{122}\\ F^2_{122} \end{Vmatrix}=\beta\begin{Vmatrix} F^1_{111}\\ F^2_{111} \end{Vmatrix}. \tag{3.5}$$

Substituting (3.5) into the determinant G_{1212} in (2.3) we derive $G_{1212}=0$. Applying $G_{1212}=0$ to (3.2), we derive the equality

$$G_{1122}=0. \tag{3.6}$$

The equality (3.6) combined with the inequality (3.4) means that the fourth column in (3.3) is expressed through the first one as follows:

$$\begin{Vmatrix} F^1_{222}\\ F^2_{222} \end{Vmatrix}=\gamma\begin{Vmatrix} F^1_{111}\\ F^2_{111} \end{Vmatrix}. \tag{3.7}$$

The equalities (3.5) and (3.7) mean that the cubic terms in (2.2) are proportional to each other. Therefore, applying the left composition with some properly chosen linear transformation of the form (2.1), we can pass to an equivalent cubic transformation such that $F^2_{111} = F^2_{112} = F^2_{122} = F^2_{222} = 0$.

If the inequality (3.4) is not fulfilled, then at least one of the last three columns in (3.3) is nonzero. Assume that the second column in (3.3) is nonzero:

$$\left\|\begin{matrix} F^1_{111} \\ \\ F^2_{111} \end{matrix}\right\| = 0, \qquad\qquad \left\|\begin{matrix} F^1_{112} \\ \\ F^2_{112} \end{matrix}\right\| \neq 0. \tag{3.8}$$

From (3.8) we derive $G_{1122} = 0$. Combining $G_{1122} = 0$ with (3.2), we derive $G_{1212} = 0$. The equality $G_{1212} = 0$ combined with $G_{1222} = 0$ in (3.1) and with (3.8) means that the last two columns (3.3) are expressed through the second one:

$$\left\|\begin{matrix} F^1_{122} \\ \\ F^2_{122} \end{matrix}\right\| = \alpha \left\|\begin{matrix} F^1_{112} \\ \\ F^2_{112} \end{matrix}\right\|, \qquad\qquad \left\|\begin{matrix} F^1_{222} \\ \\ F^2_{222} \end{matrix}\right\| = \beta \left\|\begin{matrix} F^1_{112} \\ \\ F^2_{112} \end{matrix}\right\|. \tag{3.9}$$

The relationships (3.8) and (3.9) again mean that the cubic terms in (2.2) are proportional to each other. Therefore, applying the left composition with some properly chosen linear transformation of the form (2.1), we can pass to an equivalent cubic transformation such that $F^2_{111} = F^2_{112} = F^2_{122} = F^2_{222} = 0$.

If the inequality in (3.8) is not fulfilled, then the first two columns (3.3) are zero. In this case at least one of the last two columns (3.3) is nonzero. Assume that

$$\left\|\begin{matrix} F^1_{111} \\ \\ F^2_{111} \end{matrix}\right\| = 0, \qquad\qquad \left\|\begin{matrix} F^1_{112} \\ \\ F^2_{112} \end{matrix}\right\| = 0, \qquad\qquad \left\|\begin{matrix} F^1_{122} \\ \\ F^2_{122} \end{matrix}\right\| \neq 0. \tag{3.10}$$

Due to (3.10) from $G_{2222} = 0$ in (3.1) we derive

$$\left\|\begin{matrix} F^1_{222} \\ \\ F^2_{222} \end{matrix}\right\| = \alpha \left\|\begin{matrix} F^1_{122} \\ \\ F^2_{122} \end{matrix}\right\|. \tag{3.11}$$

The equality (3.11) along with (3.10) means that the cubic terms in (2.2) are proportional to each other. Therefore, applying the left composition with some properly chosen linear transformation of the form (2.1), we can pass to an equivalent cubic transformation such that $F^2_{111} = F^2_{112} = F^2_{122} = F^2_{222} = 0$.

And finally, if the inequality in (3.10) is not fulfilled, then the first three columns in (3.3) are zero, while the last column is nonzero. In this case applying the left composition with some properly chosen linear transformation of the form (2.1), we can pass to an equivalent cubic transformation such that

$$F^2_{111} = F^2_{112} = F^2_{122} = F^2_{222} = 0. \tag{3.12}$$

The above results are summarized in the following theorem.

Theorem 3.1. *In the case where the associated quartic form* (2.4) *is zero any cubic transformation* (1.1) *is equivalent to a cubic transformation whose leading coefficients in* (2.2) *obey the relationships* (3.12).

4. The case where $\omega[1]$ is indefinite.

An indefinite form takes both positive and negative values. Therefore it takes zero values either. The form $\omega[1]$ is a quartic form in $\mathbb{R}^2$. It is given by the formula (2.4). Without loss of generality we can assume that $G_{1111} \neq 0$. Indeed, otherwise we can apply Theorem 2.1 and pass to an equivalent cubic transformation whose determinant $G_{1111} \neq 0$. The property of the associated quartic form $\omega[1]$ being indefinite is invariant under passing to an equivalent cubic transformation. This fact follows from Theorems 4.1 and 4.2 in [12].

Thus we have an indefinite quartic form $\omega[1]$ in (2.4) with $G_{1111} \neq 0$. Setting $z^2 = 1$, we can treat $\omega[1]$ as a quartic polynomial with respect to z^1. This polynomial takes both positive and negative values. It is an elementary fact that a polynomial of even degree taking values of both signs has at least two different roots $z^1 \neq \tilde{z}^1$. Due to $z^1 \neq \tilde{z}^1$ two vector-columns

$$\left\|\begin{array}{c} z^1 \\ \\ 1 \end{array}\right\|, \qquad\qquad \left\|\begin{array}{c} \tilde{z}^1 \\ \\ 1 \end{array}\right\| \tag{4.1}$$

are linearly independent. Applying Theorem 2.1 and the formulas (2.10) to the vector-columns (4.1), we define the matrix T by setting

$$\begin{aligned} &T^1_1 = z^1, && T^1_2 = \tilde{z}^1,\\ &T^2_1 = 1, && T^2_2 = 1. \end{aligned} \tag{4.2}$$

The matrix T defined through (4.2) is non-degenerate. Using it as the matrix of a linear transformation in (2.1), we can pass to an equivalent cubic transformation such that two of the six determinants in (2.3) do vanish simultaneously

$$G_{1111} = 0, \qquad\qquad G_{2222} = 0. \tag{4.3}$$

Along with (4.3), some of the other four determinants can vanish. Vanishing of them specify refinement subcases within the case of an indefinite form $\omega[1]$.

Definition 4.1. A cubic transformation f of $\mathbb{R}^2$ with indefinite form $\omega[1]$ is called obeying the first refinement condition (or the condition R1) if there is an equivalent cubic transformation $\tilde{f}$ such that the equalities (4.3) are fulfilled along with

$$G_{1112} = 0. \tag{4.4}$$

Definition 4.2. A cubic transformation f of $\mathbb{R}^2$ with indefinite form $\omega[1]$ is called obeying the second refinement condition (or the condition R2) if there is an equivalent cubic transformation $\tilde{f}$ such that the equalities (4.3) are fulfilled along with

$$G_{1122} = 0. \tag{4.5}$$

Definition 4.3. A cubic transformation f of $\mathbb{R}^2$ with indefinite form $\omega[1]$ is called obeying the third refinement condition (or the condition R3) if there is an equivalent cubic transformation $\tilde{f}$ such that the equalities (4.3) are fulfilled along with

$$G_{1212} = 0. \tag{4.6}$$

Definition 4.4. A cubic transformation f of $\mathbb{R}^2$ with indefinite form $\omega[1]$ is called obeying the fourth refinement condition (or the condition R4) if there is an equivalent cubic transformation $\tilde{f}$ such that the equalities (4.3) are fulfilled along with

$$G_{1222} = 0. \tag{4.7}$$

The equalities (4.4), (4.5), (4.6), (4.7) can be fulfilled either separately or simultaneously in various combinations. Therefore one can speak of the conditions R1.2, R1.3, R1.4, R1.2.3 etc. Studying all of them is far beyond the scope of the present paper. We note only that the condition R1.2.3.4 is incompatible with the case of an indefinite form $\omega[1]$ since it implies $\omega[1] = 0$.

5. The case where $\omega[1]$ is semi-definite.

A semi-definite form takes values of some definite sign and vanishes for some values of its arguments not all zero. The property of $\omega[1]$ being semi-definite is invariant under passing to an equivalent cubic transformation. Therefore, like in the previous section, without loss of generality we can initially assume that $G_{1111} \neq 0$. Then we set $z^2 = 1$ and treat $\omega[1]$ as a quartic polynomial with respect to z^1. This polynomial takes values of the same sign and has real roots since $\omega[1]$ is semi-definite. The nature of its roots defines two subcases of the present case:

1) the case where $\omega[1](z^1, 1)$ has two distinct roots each of multiplicity two;
2) the case where $\omega[1](z^1, 1)$ has one root of multiplicity four.

Actually the subcases 1) and 2) can be defined in other words. Indeed, $\omega[1](z^1, z^2)$ in (2.4) is a homogeneous quartic polynomial of two variables z^1 and z^2. It can be understood as a function in the real projective space $\mathbb{R}\mathrm{P}^1$. Then the above two subcases are formulated as follows:

1) the case where $\omega[1]$ has two distinct roots in $\mathbb{R}\mathrm{P}^1$ each of multiplicity two;
2) the case where $\omega[1]$ has one root of multiplicity four in $\mathbb{R}\mathrm{P}^1$.

The subcase 1) of the present case is similar to the case considered in the previous section. Indeed, if (z^1, z^2) and $(\tilde{z}^1, \tilde{z}^2)$ are two distinct roots of the form $\omega[1]$ in $\mathbb{R}\mathrm{P}^1$, then we can use them for defining a non-degenerate matrix T:

$$\begin{aligned} &T^1_1 = z^1, && T^1_2 = \tilde{z}^1,\\ &T^2_1 = z^2, && T^2_2 = \tilde{z}^2. \end{aligned} \tag{5.1}$$

Applying the matrix T with the components (5.1) as the matrix of a linear transformation in (2.1), we can pass to an equivalent cubic transformation such that

$$G_{1111} = 0 \qquad\qquad G_{2222} = 0. \tag{5.2}$$

The equalities (5.2) do coincide with (4.3). Therefore we can complement them with the refinement conditions R1, R2, R3, R4 using definitions similar to Definitions 4.1, 4.2, 4.3, and 4.4. These refinement conditions can be combined into groups forming the conditions R1.2, R1.3, R1.4, R1.2.3 etc as described above. The condition R1.2.3.4 is incompatible with the present case of a semi-definite form $\omega[1]$ since it implies $\omega[1] = 0$.

The subcase 2) is more complicated. In this subcase the root (z^1, z^2) of the form $\omega[1]$ in $\mathbb{R}\mathrm{P}^1$ defines only the first column of the matrix T in (5.1). The second column is deliberate or possibly it can be specified using the other forms $\omega[2]$, $\omega[3]$, $\omega[4]$, $\omega[5]$. More details are to be elaborated in forthcoming papers.

5. The case where $\omega[1]$ is definite.

A definite form takes values of some definite sign and does not vanish if its arguments are not all zero. The property of the form $\omega[1]$ being definite is invariant under passing to an equivalent cubic transformation. This fact follows from Theorems 4.1 and 4.2 in [12]. Other details of the case of a definite form $\omega[1]$ are yet uncertain. They should be elaborated in future within a refinement procedure.

5. Conclusions.

The rough classification scheme suggested in the present paper is only a framework that does not contain all possible subcases and canonical presentations of cubic transformations within each particular subcase. These subcases and canonical presentations should be discovered and cataloged in forthcoming papers within a refinement procedure.

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Bashkir State University, 32 Zaki Validi street, 450074 Ufa, Russia
E-mail address: r-sharipov@mail.ru