COMPARISON OF TWO CLASSIFICATIONS OF A CLASS OF ODE'S IN THE CASE OF GENERAL POSITION.

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ABSTRACT. Two classifications of second order ODE's cubic with respect to the first order derivative are compared in the case of general position, which is common for both classifications. The correspondence of vectorial, pseudovectorial, scalar, and pseudoscalar invariants is established.

1. INTRODUCTION.

The main object for the present research is the following class of ordinary differential equations cubic with respect to the first order derivative:

$$y'' = P(x, y) + 3\, Q(x, y)\, y' + 3\, R(x, y)\, (y')^2 + S(x, y)\, (y')^3. \qquad (1.1)$$

Differential equations of the form (1.1) attracted the attention since the epoch of classical papers (see [1] and [2]). Nowadays they are studied in a large variety of papers so that one cannot cite all of them without risks to unintentionally miss some names. With my apologies I proceed to the papers [3] and [4], where the equations (1.1) were classified. They were subdivided into **nine subclasses** invariant with respect to transformations of the form

$$\begin{cases} \tilde{x} = \tilde{x}(x, y), \\ \tilde{y} = \tilde{y}(x, y). \end{cases} \qquad (1.2)$$

This subdivision is based on scalar invariants of the equations (1.1) (see definition below) and on their symmetry groups, which are nontrivial in some cases.

Recently Yu. Yu. Bagderina in [5] presented another classification of the equations (1.1) again subdividing them into **nine subclasses** invariant with respect to transformations of the form (1.2) (see Theorem 2 in [5]). Her approach is based on Sophus Lie's method of infinitesimal transformations adapted to equations of the form (1.1) by N. H. Ibragimov in [6].

Unfortunately in [5] Yu. Yu. Bagderina does not mention the previously existing classification from [3] and [4]. Though she cites the paper [4] in [5], she references this paper only as a source of invariants and for criticism of its method.

In the present paper I start a research intended to examine the results of paper [5] and compare them with the prior results from [3, 4] and [7].

2. Some notations and definitions.

Transformations (1.2) are called point transformations. They are assumed to be locally invertible. For their inverse transformations we write

$$\begin{cases} x = \tilde{x}(\tilde{x}, \tilde{y}), \\ y = \tilde{y}(\tilde{x}, \tilde{y}). \end{cases} \tag{2.1}$$

Then, according to [3, 4] and [7], here we introduce the following notations for partial derivatives of the functions in (1.2) and (2.1):

$$\tilde{x}_{1.0} = \frac{\partial \tilde{x}}{\partial x}, \qquad \tilde{x}_{0.1} = \frac{\partial \tilde{x}}{\partial y}, \qquad \tilde{y}_{0.1} = \frac{\partial \tilde{y}}{\partial x}, \qquad \tilde{y}_{0.1} = \frac{\partial \tilde{y}}{\partial y}, \tag{2.2}$$

$$x_{1.0} = \frac{\partial x}{\partial \tilde{x}}, \qquad x_{0.1} = \frac{\partial x}{\partial \tilde{y}}, \qquad y_{0.1} = \frac{\partial y}{\partial \tilde{x}}, \qquad y_{0.1} = \frac{\partial y}{\partial \tilde{y}}. \tag{2.3}$$

In terms of the notations (2.2) and (2.3) the Jacoby matrices of the point transformations (1.2) and (2.1) are written as follows

$$S = \left\| \begin{matrix} x_{1.0} & x_{0.1} \\ y_{1.0} & y_{0.1} \end{matrix} \right\|, \qquad\qquad T = \left\| \begin{matrix} \tilde{x}_{1.0} & \tilde{x}_{0.1} \\ \tilde{y}_{1.0} & \tilde{y}_{0.1} \end{matrix} \right\|. \tag{2.4}$$

In differential geometry the matrices (2.1) are called the direct and inverse transition matrices (see [8]). As for the transformations (1.2) and (2.1), geometrically they are interpreted as changes of local curvilinear coordinates on the plane $\mathbb{R}^2$ or on some two-dimensional manifold.

Tensorial and pseudotensorial fields in local coordinates are presented by their components forming arrays of functions whose arguments are x, y or $\tilde{x}$, $\tilde{y}$ respectively. In the coordinate form they can be defined as follows.

Definition 2.1. A pseudotensorial field of the type (r, s) and weight m is an array of quantities $F^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ which under the change of coordinates (1.2) transforms as

$$F^{i_1 \ldots i_r}_{j_1 \ldots j_s} = (\det T)^m \sum_{\substack{p_1 \ldots p_r \\ q_1 \ldots q_s}} S^{i_1}_{p_1} \ldots S^{i_r}_{p_r} \, T^{q_1}_{j_1} \ldots T^{q_s}_{j_s} \, \tilde{F}^{p_1 \ldots p_r}_{q_1 \ldots q_s}. \tag{2.5}$$

Tensorial fields are those pseudotensorial fields whose weight is zero, i. e. $m = 0$ in (2.5) for a tensorial field.

Definition 2.2. Vectorial and pseudovectorial fields are those fields in Definition 2.1 whose type is $(1, 0)$. Covectorial and pseudocovectorial fields are those fields in Definition 2.1 whose type is $(0, 1)$. Scalar and pseudoscalar fields are those fields whose type is $(0, 0)$.

Definition 2.3. Tensorial and pseudotensorial fields whose components are expressed through y', through the coefficients P, Q, R, S of the equation (1.1), and through their partial derivatives are called tensorial and pseudotensorial invariants of this equation respectively.

Invariants of the equation (1.1) are subdivided into absolute and relative ones. Tensorial invariants are absolute invariants, while pseudotensorial invariants are

relative ones. This definition of relative invariants is close to that of [9]. In some papers (e. g. in [5]) the term absolute invariant is applied to scalar invariants only. The definition of relative invariants in [5] is rather loose. Therefore the usage of this term in [5] is different from the above definition.

3. Comparison of some relative invariants.

In [5] Yu. Yu. Bagderina introduces a long list of special notations. She call them relative invariants of various orders from one to six. The first order expressions are given by the formulas (2.1) in [5]. Here are they:

$$
\begin{aligned}
\alpha_0^{\mathrm{Bgd}} &= Q_{1.0} - P_{0.1} + 2\,P\,R - 2\,Q^2, \\
\alpha_1^{\mathrm{Bgd}} &= R_{1.0} - Q_{0.1} + P\,S - Q\,R, \\
\alpha_2^{\mathrm{Bgd}} &= S_{1.0} - R_{0.1} + 2\,Q\,S - 2\,R^2.
\end{aligned}
\tag{3.1}
$$

Some quantities by Yu. Yu. Bagderina from [5] share the same symbol with absolutely different quantities from [3, 4] and [7]. Therefore in (3.1) and in other formulas I use the upper mark «Bgd» in order to distinguish Bagderina's quantities from those of me and Vera V. Kartak (Dmitrieva).

The second order expressions are given by the formulas (2.2) in [5]:

$$
\begin{aligned}
\beta_1^{\mathrm{Bgd}} &= \partial_x\alpha_1^{\mathrm{Bgd}} - \partial_y\alpha_0^{\mathrm{Bgd}} + R\,\alpha_0^{\mathrm{Bgd}} - 2\,Q\,\alpha_1^{\mathrm{Bgd}} + P\,\alpha_2^{\mathrm{Bgd}}, \\
\beta_2^{\mathrm{Bgd}} &= \partial_x\alpha_2^{\mathrm{Bgd}} - \partial_y\alpha_1^{\mathrm{Bgd}} + S\,\alpha_0^{\mathrm{Bgd}} - 2\,R\,\alpha_1^{\mathrm{Bgd}} + Q\,\alpha_2^{\mathrm{Bgd}}.
\end{aligned}
\tag{3.2}
$$

The third order expressions are given by the formulas (2.3) in [5]:

$$
\begin{aligned}
\gamma_{10}^{\mathrm{Bgd}} &= \partial_x\beta_1^{\mathrm{Bgd}} - Q\,\beta_1^{\mathrm{Bgd}} + P\,\beta_2^{\mathrm{Bgd}}, \\
\gamma_{11}^{\mathrm{Bgd}} &= \partial_x\beta_2^{\mathrm{Bgd}} - R\,\beta_1^{\mathrm{Bgd}} + Q\,\beta_2^{\mathrm{Bgd}}, \\
\gamma_{20}^{\mathrm{Bgd}} &= \partial_y\beta_1^{\mathrm{Bgd}} - R\,\beta_1^{\mathrm{Bgd}} + Q\,\beta_2^{\mathrm{Bgd}}, \\
\gamma_{21}^{\mathrm{Bgd}} &= \partial_y\beta_2^{\mathrm{Bgd}} - S\,\beta_1^{\mathrm{Bgd}} + R\,\beta_2^{\mathrm{Bgd}}.
\end{aligned}
\tag{3.3}
$$

The fourth order expressions are given by the formulas (2.4) in [5]:

$$
\begin{aligned}
\delta_{10}^{\mathrm{Bgd}} &= \partial_x\gamma_{10}^{\mathrm{Bgd}} - 2\,Q\,\gamma_{10}^{\mathrm{Bgd}} + P\,(\gamma_{20}^{\mathrm{Bgd}} + \gamma_{11}^{\mathrm{Bgd}}) - 5\,\alpha_0^{\mathrm{Bgd}}\,\beta_1^{\mathrm{Bgd}}, \\
\delta_{20}^{\mathrm{Bgd}} &= \partial_x\gamma_{20}^{\mathrm{Bgd}} - R\,\gamma_{10}^{\mathrm{Bgd}} + P\,\gamma_{21}^{\mathrm{Bgd}} - 4\,\alpha_1^{\mathrm{Bgd}}\,\beta_1^{\mathrm{Bgd}} - \alpha_0^{\mathrm{Bgd}}\,\beta_2^{\mathrm{Bgd}}, \\
\delta_{30}^{\mathrm{Bgd}} &= \partial_y\gamma_{20}^{\mathrm{Bgd}} - S\,\gamma_{10}^{\mathrm{Bgd}} + Q\,\gamma_{21}^{\mathrm{Bgd}} - 4\,\alpha_2^{\mathrm{Bgd}}\,\beta_1^{\mathrm{Bgd}} - \alpha_1^{\mathrm{Bgd}}\,\beta_2^{\mathrm{Bgd}}, \\
\delta_{11}^{\mathrm{Bgd}} &= \partial_x\gamma_{11}^{\mathrm{Bgd}} - R\,\gamma_{10}^{\mathrm{Bgd}} + P\,\gamma_{21}^{\mathrm{Bgd}} - \alpha_1^{\mathrm{Bgd}}\,\beta_1^{\mathrm{Bgd}} - 4\,\alpha_0^{\mathrm{Bgd}}\,\beta_2^{\mathrm{Bgd}}, \\
\delta_{21}^{\mathrm{Bgd}} &= \partial_x\gamma_{21}^{\mathrm{Bgd}} - R\,(\gamma_{20}^{\mathrm{Bgd}} + \gamma_{11}^{\mathrm{Bgd}}) + 2\,Q\,\gamma_{21}^{\mathrm{Bgd}} - 5\,\alpha_1^{\mathrm{Bgd}}\,\beta_2^{\mathrm{Bgd}}, \\
\delta_{31}^{\mathrm{Bgd}} &= \partial_y\gamma_{21}^{\mathrm{Bgd}} - S\,(\gamma_{20}^{\mathrm{Bgd}} + \gamma_{11}^{\mathrm{Bgd}}) + 2\,R\,\gamma_{21}^{\mathrm{Bgd}} - 5\,\alpha_2^{\mathrm{Bgd}}\,\beta_2^{\mathrm{Bgd}}.
\end{aligned}
\tag{3.4}
$$

The fifth order expressions are given by the formulas (2.5) in [5]:

$$
\begin{aligned}
\epsilon_{10}^{\mathrm{Bgd}} &= \partial_x\delta_{10}^{\mathrm{Bgd}} - 3\,Q\,\delta_{10}^{\mathrm{Bgd}} + P\,(2\,\delta_{20}^{\mathrm{Bgd}} + \delta_{11}^{\mathrm{Bgd}}) - 12\,\alpha_0^{\mathrm{Bgd}}\,\gamma_{10}^{\mathrm{Bgd}}, \\
\epsilon_{20}^{\mathrm{Bgd}} &= \partial_y\delta_{10}^{\mathrm{Bgd}} - 3\,R\,\delta_{10}^{\mathrm{Bgd}} + Q\,(2\,\delta_{20}^{\mathrm{Bgd}} + \delta_{11}^{\mathrm{Bgd}}) - 12\,\alpha_1^{\mathrm{Bgd}}\,\gamma_{10}^{\mathrm{Bgd}}, \\
\epsilon_{11}^{\mathrm{Bgd}} &= \partial_x\delta_{11}^{\mathrm{Bgd}} - R\,\delta_{10}^{\mathrm{Bgd}} - Q\,\delta_{11}^{\mathrm{Bgd}} + 2\,P\,\delta_{21}^{\mathrm{Bgd}} - 2\,\alpha_1^{\mathrm{Bgd}}\,\gamma_{10}^{\mathrm{Bgd}} - \\
&\quad - 10\,\alpha_0^{\mathrm{Bgd}}\,\gamma_{11}^{\mathrm{Bgd}} - 10\,(\beta_1^{\mathrm{Bgd}})^2.
\end{aligned}
\tag{3.5}
$$

And finally, the sixth order expression is given by the formula (2.6) in [5]:

$$\lambda_{10}^{\mathrm{Bgd}} = \partial_x \epsilon_{10}^{\mathrm{Bgd}} - 4\,Q\,\epsilon_{10}^{\mathrm{Bgd}} + P\,(3\,\epsilon_{20}^{\mathrm{Bgd}} + \epsilon_{11}^{\mathrm{Bgd}}) - 21\,\alpha_0^{\mathrm{Bgd}}\,\delta_{10}^{\mathrm{Bgd}}. \tag{3.6}$$

Comparing (3.1) with the formulas (2.15) in [7], one can easily formulate and prove the following lemma.

Lemma 3.1. *Bagderina's alpha quantities* (3.1) *coincide with the components of the symmetric two-dimensional array* Ω *constructed in* [7]:

$$\alpha_0^{\mathrm{Bgd}} = \Omega_{11}, \qquad\qquad \alpha_1^{\mathrm{Bgd}} = \Omega_{12} = \Omega_{21}, \qquad\qquad \alpha_2^{\mathrm{Bgd}} = \Omega_{22}. \tag{3.7}$$

From [7] it is known, that the quantities Ω_{ij} in (3.7) constitute nether a tensor invariant nor a pseudotensor invariant. The transformation rule for them is similar to (2.5), but a little bit different. It is given by the formula (2.13) in [7].

Lemma 3.2. *Bagderina's beta quantities* (3.2) *coincide with the components of the pseudocovectorial field* $\boldsymbol{\alpha}$ *of the weight* 1 *constructed in* [7]:

$$\beta_1^{\mathrm{Bgd}} = \alpha_1 = A, \qquad\qquad\qquad \beta_2^{\mathrm{Bgd}} = \alpha_2 = B. \tag{3.8}$$

Lemma 3.2 is proved by comparing the formulas (1.6) from [5] with (2.19) in [7].

Apart from $\boldsymbol{\alpha}$, another pseudotensorial field $\mathbf{d}$ was introduced in [7]. Its components are given by the following skew-symmetric matrix in any local coordinates:

$$d_{ij} = \left\| \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right\|. \tag{3.9}$$

The type of the field $\mathbf{d}$ with the components (3.9) is $(0,2)$, its weight is -1. The same matrix (3.9) provide the components of a pseudotensor of the type $(2,0)$:

$$d^{ij} = \left\| \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right\|. \tag{3.10}$$

This field is denoted by the same symbol $\mathbf{d}$, its weight is 1. The fields (3.9) and (3.10) are used for raising in lowering indices of pseudotensorial invariants of the equation (1.1). They pay the same role as the metric tensors in metric geometry. In particular, the formulas (3.8) can be written as

$$\beta_1^{\mathrm{Bgd}} = A = -\alpha^2, \qquad\qquad\qquad \beta_2^{\mathrm{Bgd}} = B = \alpha^1, \tag{3.11}$$

where α^1 and α^2 are the components of a pseudovectorial field of the weight 2:

$$\alpha^i = \sum_{k=1}^{2} d^{ik}\,\alpha_k. \tag{3.12}$$

This field with the components (3.12) in [7] is denoted by the same symbol $\boldsymbol{\alpha}$ as the previous field with the components (3.8).

Theorem 3.1. *The case of the equations of the ninth type of Bagderina's classification in* [5] *coincide with the case of* **maximal degeneration** *in* [3, 4].

According to [3] (see page 7), the simultaneous vanishing condition $A = 0$ and $B = 0$ determines the case of **maximal degeneration**. In order to prove Theorem 3.1 now it is sufficient to compare this condition with the condition $\beta_1^{\mathrm{Bgd}} = 0$ and $\beta_2^{\mathrm{Bgd}} = 0$ in item 9 of Theorem 2 in [5] and apply the above formulas (3.8).

Apart from (3.1), (3.2), (3.3), (3.4), (3.5), and (3.6), Yu. Yu. Bagderina uses another series of huge special notations in [5]. The formulas (2.16) from [5] yield

$$
\begin{aligned}
J_0^{\mathrm{Bgd}} &= (\beta_2^{\mathrm{Bgd}})^2\,\gamma_{10}^{\mathrm{Bgd}} - \beta_1^{\mathrm{Bgd}}\,\beta_2^{\mathrm{Bgd}}\,(\gamma_{20}^{\mathrm{Bgd}} + \gamma_{11}^{\mathrm{Bgd}}) + (\beta_1^{\mathrm{Bgd}})^2\,\gamma_{21}^{\mathrm{Bgd}}, \\
J_1^{\mathrm{Bgd}} &= \beta_2^{\mathrm{Bgd}}\,(\delta_{20}^{\mathrm{Bgd}} - \delta_{11}^{\mathrm{Bgd}}) + \beta_1^{\mathrm{Bgd}}\,(\delta_{21}^{\mathrm{Bgd}} - \delta_{30}^{\mathrm{Bgd}}) + \\
&\quad + \frac{7}{5}\,(\gamma_{20}^{\mathrm{Bgd}} - \gamma_{11}^{\mathrm{Bgd}})^2 + \frac{3}{5}\,(\gamma_{20}^{\mathrm{Bgd}}\,\gamma_{11}^{\mathrm{Bgd}} - \gamma_{10}^{\mathrm{Bgd}}\,\gamma_{21}^{\mathrm{Bgd}}), \\
J_2^{\mathrm{Bgd}} &= \Gamma_1^{\mathrm{Bgd}}\,(\delta_{20}^{\mathrm{Bgd}} - \delta_{11}^{\mathrm{Bgd}}) + \Gamma_0^{\mathrm{Bgd}}\,(\delta_{21}^{\mathrm{Bgd}} - \delta_{30}^{\mathrm{Bgd}}) + 3\,(\gamma_{20}^{\mathrm{Bgd}} - \\
&\quad - \gamma_{11}^{\mathrm{Bgd}})\,(\gamma_{10}^{\mathrm{Bgd}}\,\gamma_{21}^{\mathrm{Bgd}} - \gamma_{20}^{\mathrm{Bgd}}\,\gamma_{11}^{\mathrm{Bgd}}) + \frac{4}{3}\,(\gamma_{20}^{\mathrm{Bgd}} - \gamma_{11}^{\mathrm{Bgd}})^3, \\
J_3^{\mathrm{Bgd}} &= (\beta_2^{\mathrm{Bgd}})^3\,\delta_{10}^{\mathrm{Bgd}} - \beta_1^{\mathrm{Bgd}}\,(\beta_2^{\mathrm{Bgd}})^2\,(2\,\delta_{20}^{\mathrm{Bgd}} + \delta_{11}^{\mathrm{Bgd}}) + \\
&\quad + (\beta_1^{\mathrm{Bgd}})^2\,\beta_2^{\mathrm{Bgd}}\,(\delta_{30}^{\mathrm{Bgd}} + 2\,\delta_{21}^{\mathrm{Bgd}}) - (\beta_1^{\mathrm{Bgd}})^3\,\delta_{31}^{\mathrm{Bgd}} + \\
&\quad + 4\,(\gamma_{20}^{\mathrm{Bgd}} - \gamma_{11}^{\mathrm{Bgd}})\,J_0^{\mathrm{Bgd}}, \\
J_4^{\mathrm{Bgd}} &= -\beta_2^{\mathrm{Bgd}}\,(\beta_2^{\mathrm{Bgd}}\,\Gamma_0^{\mathrm{Bgd}} + 2\,\beta_1^{\mathrm{Bgd}}\,\Gamma_1^{\mathrm{Bgd}})\,(2\,\delta_{20}^{\mathrm{Bgd}} + \delta_{11}^{\mathrm{Bgd}}) + \\
&\quad + \beta_1^{\mathrm{Bgd}}\,(2\,\beta_2^{\mathrm{Bgd}}\,\Gamma_0^{\mathrm{Bgd}} + \beta_1^{\mathrm{Bgd}}\,\Gamma_1^{\mathrm{Bgd}})\,(\delta_{30}^{\mathrm{Bgd}} + 2\,\delta_{21}^{\mathrm{Bgd}}) + \\
&\quad + 3\,(\beta_2^{\mathrm{Bgd}})^2\,\Gamma_1^{\mathrm{Bgd}}\,\delta_{10}^{\mathrm{Bgd}} - 3\,(\beta_1^{\mathrm{Bgd}})^2\,\Gamma_0^{\mathrm{Bgd}}\,\delta_{31}^{\mathrm{Bgd}} + \frac{66}{5}\,\times \\
&\quad \times\,(\gamma_{20}^{\mathrm{Bgd}} - \gamma_{11}^{\mathrm{Bgd}})^2\,J_0^{\mathrm{Bgd}} + \frac{36}{5}\,(\gamma_{10}^{\mathrm{Bgd}}\,\gamma_{21}^{\mathrm{Bgd}} - \gamma_{20}^{\mathrm{Bgd}}\,\gamma_{11}^{\mathrm{Bgd}})\,J_0^{\mathrm{Bgd}}.
\end{aligned}
\tag{3.13}
$$

The quantities Γ_0^{Bgd} and Γ_1^{Bgd} from (3.13) are given by the formulas (2.17) in [5]:

$$
\begin{aligned}
\Gamma_0^{\mathrm{Bgd}} &= 3\,\beta_2^{\mathrm{Bgd}}\,\gamma_{10}^{\mathrm{Bgd}} + \beta_1^{\mathrm{Bgd}}\,(\gamma_{20}^{\mathrm{Bgd}} - 4\,\gamma_{11}^{\mathrm{Bgd}}), \\
\Gamma_1^{\mathrm{Bgd}} &= \beta_2^{\mathrm{Bgd}}\,(4\,\gamma_{20}^{\mathrm{Bgd}} - \gamma_{11}^{\mathrm{Bgd}}) - 3\,\beta_1^{\mathrm{Bgd}}\,\gamma_{21}^{\mathrm{Bgd}}.
\end{aligned}
\tag{3.14}
$$

Lemma 3.3. *Bagderina's quantity* J_0^{Bgd} *from* (3.13) *is related to the pseudoscalar field F of the weight 1 constructed in* [7] *by means of the formula*

$$
J_0^{\mathrm{Bgd}} = -F^5.
\tag{3.15}
$$

Lemma 3.3 is proved by direct calculations with the use of the first formula (3.13) and the formula (2.25) from [7].

Theorem 3.2. *The case of the equations of the first type of Bagderina's classification in* [5] *coincide with the case of* **general position** *in* [3, 4].

According to [3] (see section 3 on page 6), the case of **general position** is defined by the non-vanishing condition for the pseudoscalar invariant F:

$$
F \neq 0.
\tag{3.16}
$$

Due to (3.15) in order to prove Theorem 3.2 now it is sufficient to compare the condition (3.16) with the condition $J_0^{\mathrm{Bgd}} \neq 0$ in item 1 of Theorem 2 in [5].

The equality (3.15) means that Bagderina's quantity J_0^{Bgd} is a pseudoscalar field of the weight 5. For the case of general position Yu. Yu. Bagderina defines the following quantity in her paper [5] (see item 1 of theortem 2 in [5]):

$$\mu_1^{\mathrm{Bgd}} = (J_0^{\mathrm{Bgd}})^{1/5}. \tag{3.17}$$

Lemma 3.4. *Bagderina's quantity μ_1^{Bgd} from (3.17) is related to the pseudoscalar field F of the weight 1 constructed in [7] by means of the formula*

$$\mu_1^{\mathrm{Bgd}} = -F. \tag{3.18}$$

The equality (3.18) and Lemma 3.4 are immediate from (3.15) and (3.17).

Lemma 3.5. *Bagderina's gamma quantities (3.14) coincide with the components of the pseudocovectorial field $\boldsymbol{\beta}$ of the weight 3 constructed in [7]:*

$$\Gamma_0^{\mathrm{Bgd}} = \beta_1 = -H, \qquad\qquad \Gamma_1^{\mathrm{Bgd}} = \beta_2 = G. \tag{3.19}$$

The pseudovectorial field $\boldsymbol{\beta}$ is produced from the pseudocovectorial field $\boldsymbol{\beta}$ in [7] by raising the index of its components with the use of the formula

$$\beta^i = \sum_{k=1}^{2} d^{ik}\, \beta_k. \tag{3.20}$$

Then due to (3.20) the formulas (3.19) are written as follows:

$$\Gamma_1^{\mathrm{Bgd}} = \beta^1 = G, \qquad\qquad \Gamma_0^{\mathrm{Bgd}} = -\beta^2 = -H \tag{3.21}$$

The weight of the pseudovectorial field $\boldsymbol{\beta}$ produced by means of the above formula (3.20) in [3] is equal to 4.

In her paper [5] Yu. Yu. Bagderina defines two differential operators (see Theorem 2). They are given by the formulas (2.8) in [5]:

$$\begin{aligned}
\mathcal{D}_1^{\mathrm{Bgd}} &= (\mu_1^{\mathrm{Bgd}})^{-2}(\beta_2^{\mathrm{Bgd}}\, D_x - \beta_1^{\mathrm{Bgd}}\, D_y), \\
\mathcal{D}_2^{\mathrm{Bgd}} &= \mu_2^{\mathrm{Bgd}}\,(\beta_2^{\mathrm{Bgd}}\, D_x - \beta_1^{\mathrm{Bgd}}\, D_y) - 3\,(\beta_1^{\mathrm{Bgd}})^{-1}\,\mu_1^{\mathrm{Bgd}}\, D_x,
\end{aligned} \tag{3.22}$$

where D_x and D_y can be understood as partial derivatives $\partial/\partial x$ and $\partial/\partial y$ acting upon functions of the form $f(x, y)$. In the case of **general position** the quantity μ_2^{Bgd} is defined by one of the formulas (2.9) in [5]:

$$\mu_2^{\mathrm{Bgd}} = \Gamma_0^{\mathrm{Bgd}}\,(\beta_1^{\mathrm{Bgd}})^{-1}\,(J_0^{\mathrm{Bgd}})^{-4/5}. \tag{3.23}$$

Applying (3.11) and (3.18) to the first formula (3.22), we get

$$\mathcal{D}_1^{\mathrm{Bgd}} = \frac{\alpha^1}{F^2}\,\frac{\partial}{\partial x} + \frac{\alpha^2}{F^2}\,\frac{\partial}{\partial y}. \tag{3.24}$$

The operator $\mathcal{D}_2^{\text{Bgd}}$ is a little bit more complicated. In order to transform it one should use the following formula from [5] (see Remark 1):

$$3\, J_0^{\text{Bgd}} = \beta_2^{\text{Bgd}}\, \Gamma_0^{\text{Bgd}} - \beta_1^{\text{Bgd}}\, \Gamma_1^{\text{Bgd}}. \tag{3.25}$$

The formula (3.25) coincides with the formula (2.24) in [7]. Applying this formula in calculating the operator $\mathcal{D}_2^{\text{Bgd}}$, we get the formula

$$\mathcal{D}_2^{\text{Bgd}} = (J_0^{\text{Bgd}})^{-4/5}\, (\Gamma_1^{\text{Bgd}}\, D_x - \Gamma_0^{\text{Bgd}}\, D_y). \tag{3.26}$$

The formula (3.26) is comprised in [5] (see Remark 1). Applying the formulas (3.21) and (3.15) to the above formula (3.26), we get

$$\mathcal{D}_2^{\text{Bgd}} = \frac{\beta^1}{F^4}\frac{\partial}{\partial x} + \frac{\beta^2}{F^4}\frac{\partial}{\partial y}. \tag{3.27}$$

Theorem 3.3. *Bagderina's differential operators* (3.22) *coincide with the vectorial fields* $\mathbf{X}$ *and* $\mathbf{Y}$ *constructed in* [7]:

$$\mathcal{D}_1^{\text{Bgd}} = \mathbf{X}, \qquad\qquad\qquad \mathcal{D}_2^{\text{Bgd}} = \mathbf{Y}. \tag{3.28}$$

The components of the vector fields $\mathbf{X}$ and $\mathbf{Y}$ are determined by the formulas (3.1) in [7]. Therefore in order to prove the relationships (3.28) in Theorem 3.3 it is sufficient to compare the formulas (3.1) from [7] with (3.24) and (3.27).

4. Scalar invariants.

According to Definition 2.2, scalar and pseudoscalar invariants are those whose type is $(0,0)$. Scalar invariants differ from pseudoscalar ones by their weight. The weight of scalar invariants is zero, while pseudoscalar invariants have nonzero weights. In [7] ten scalar invariants are considered. They are denoted through I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, L and K. Not all of them are independent. We have

$$I_1 = -4\, I_6, \qquad I_2 = \frac{1}{3}, \qquad I_4 = 4\, I_6, \qquad I_5 = -I_8. \tag{4.1}$$

The formulas (4.1) coincide with the formulas (3.10) in [4]. Apart from the formulas (4.1) we have the following relationships:

$$I_5 = I_3 - L, \qquad\qquad\qquad I_6 = -I_1 + K. \tag{4.2}$$

The formulas (4.2) are taken from (3.13) and (3.14) in [7]. From (4.1) and (4.2) we derive the following expressions for L and K:

$$L = I_3 + I_8, \qquad\qquad\qquad K = -3\, I_6. \tag{4.3}$$

Due to (4.1) and (4.3) all of the ten invariants I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8, L, K are expressed through four of them: I_3, I_6, I_7, I_8.

The invariants I_3, I_6, I_7, I_8 are given by explicit formulas taken from [7] and [4]. The invariant I_3 is given by the following formula:

$$
\begin{aligned}
I_3 = {} & \frac{B\,(H\,G_{1.0} - G\,H_{1.0})}{3\,F^9} - \frac{A\,(H\,G_{0.1} - G\,H_{0.1})}{3\,F^9} + \frac{H\,F_{0.1} + G\,F_{1.0}}{3\,F^5} + \\
& + \frac{B\,G^2\,P}{3\,F^9} - \frac{(A\,G^2 - 2\,H\,B\,G)\,Q}{3\,F^9} + \frac{(B\,H^2 - 2\,H\,A\,G)\,R}{3\,F^9} - \frac{A\,H^2\,S}{3\,F^9}.
\end{aligned}
\tag{4.4}
$$

The formula (4.4) coincides with the formula (3.8) in [7] and the formula (3.6) in [4]. The quantities A and B in (4.4) are given by the formulas (3.8) and (3.2). The quantities G and H are given by the formulas (3.21) and (3.14). The quantity F is given by the formulas (3.15) and (3.18). Here are more explicit formulas for all of these five quantities A, B, G, H, and F:

$$
\begin{aligned}
A = {} & P_{0.2} - 2\,Q_{1.1} + R_{2.0} + 2\,P\,S_{1.0} + S\,P_{1.0} - \\
& - 3\,P\,R_{0.1} - 3\,R\,P_{0.1} - 3\,Q\,R_{1.0} + 6\,Q\,Q_{0.1}, \\
B = {} & S_{2.0} - 2\,R_{1.1} + Q_{0.2} - 2\,S\,P_{0.1} - P\,S_{0.1} + \\
& + 3\,S\,Q_{1.0} + 3\,Q\,S_{1.0} + 3\,R\,Q_{0.1} - 6\,R\,R_{1.0},
\end{aligned}
\tag{4.5}
$$

$$
\begin{aligned}
G = {} & -B\,B_{1.0} - 3\,A\,B_{0.1} + 4\,B\,A_{0.1} + 3\,S\,A^2 - 6\,R\,B\,A + 3\,Q\,B^2, \\
H = {} & -A\,A_{0.1} - 3\,B\,A_{1.0} + 4\,A\,B_{1.0} - 3\,P\,B^2 + 6\,Q\,A\,B - 3\,R\,A^2,
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
F^5 = {} & A\,B\,A_{0.1} + B\,A\,B_{1.0} - A^2\,B_{0.1} - B^2\,A_{1.0} - \\
& - P\,B^3 + 3\,Q\,A\,B^2 - 3\,R\,A^2\,B + S\,A^3.
\end{aligned}
\tag{4.7}
$$

The formulas (4.5) coincide with (2.19) in [7] and with (1.6) in [4]. Similarly, the formulas (4.6) coincide with (2.23) in [7] and with (2.4) in [4]. The formula (4.7) coincides with (2.25) in [7] and (2.2) in [4].

Like in (2.2), (2.3), (2.4), and (3.1), double indices in (4.4), (4.5), (4.6), and (4.7) are used to denote partial derivatives. For a given function $f(x, y)$ we write

$$
f_{p.q} = \frac{\partial^{p+q} f}{\partial x^p\, \partial y^q}.
\tag{4.8}
$$

The invariant I_6 is given by the formula (3.7) in [4]. Unfortunately this formula is mistyped. The sign of the last term should be altered. The correct formula is

$$
\begin{aligned}
I_6 = {} & \frac{A\,(G\,A_{0.1} + H\,B_{0.1})}{12\,F^7} - \frac{B\,(G\,A_{1.0} + H\,B_{1.0})}{12\,F^7} - \frac{4\,(A\,F_{0.1} - B\,F_{1.0})}{12\,F^3} - \\
& - \frac{G\,B^2\,P}{12\,F^7} - \frac{(H\,B^2 - 2\,G\,B\,A)\,Q}{12\,F^7} - \frac{(G\,A^2 - 2\,H\,B\,A)\,R}{12\,F^7} - \frac{H\,A^2\,S}{12\,F^7}.
\end{aligned}
\tag{4.9}
$$

Partial derivatives in (4.9) are denoted according to the convention (4.8). There is another formula for the invariant I_6 in [4], which is more simple:

$$
I_6 = \frac{A_{0.1} - B_{1.0}}{3\,F^2} - \frac{A\,F_{0.1} - B\,F_{1.0}}{3\,F^3}.
\tag{4.10}
$$

The formula (4.10) coincides with (3.11) in [4]. Both formulas (4.9) and (4.10) give the same result, though they look different.

The invariants I_7 and I_8 are given by the following two formulas:

$$I_7 = \frac{G\,H\,G_{1.0} - G^2\,H_{1.0} + H^2\,G_{0.1} - H\,G\,H_{0.1}}{3\,F^{11}} +$$
$$+ \frac{G^3\,P + 3\,G^2\,H\,Q + 3\,G\,H^2\,R + H^3\,S}{3\,F^{11}}, \tag{4.11}$$

$$I_8 = \frac{G\,(A\,G_{1.0} + B\,H_{1.0})}{3\,F^9} + \frac{H\,(A\,G_{0.1} + B\,H_{0.1})}{3\,F^9} - \frac{10\,(H\,F_{0.1} + G\,F_{1.0})}{3\,F^5} -$$
$$- \frac{B\,G^2\,P}{3\,F^9} + \frac{(A\,G^2 - 2\,H\,B\,G)\,Q}{3\,F^9} - \frac{(B\,H^2 - 2\,H\,A\,G)\,R}{3\,F^9} + \frac{A\,H^2\,S}{3\,F^9}. \tag{4.12}$$

The formula (4.11) coincides with the formula (3.9) in [7] and with the formula (3.8) in [4]. The formula (4.12) is taken from (3.9) in [4].

Yu. Yu. Bagderina presented her own invariants in [5] . In the case of **general position** they are given by the formulas

$$I_1^{\mathrm{Bgd}} = \frac{J_1^{\mathrm{Bgd}}}{(J_0^{\mathrm{Bgd}})^{4/5}}, \qquad\qquad I_2^{\mathrm{Bgd}} = \frac{J_2^{\mathrm{Bgd}}}{(J_0^{\mathrm{Bgd}})^{6/5}},$$
$$\tag{4.13}$$
$$I_3^{\mathrm{Bgd}} = \frac{J_3^{\mathrm{Bgd}}}{(J_0^{\mathrm{Bgd}})^{7/5}}, \qquad\qquad I_4^{\mathrm{Bgd}} = \frac{J_4^{\mathrm{Bgd}}}{(J_0^{\mathrm{Bgd}})^{9/5}},$$

where J_0^{Bgd}, J_1^{Bgd}, J_2^{Bgd}, J_3^{Bgd}, J_0^{Bgd} are taken from (3.13) (see Theorem 2 in [5]). On page 3 of her paper [5] Yu. Yu. Bagderina give the comparison formulas relating her invariants with I_3, I_6, I_7, and I_8. In order to verify these comparison formulas we need to introduce some special coordinates.

5. Special coordinates.

For the case of **general position** in [7] two vectorial fields $\mathbf{X}$ and $\mathbf{Y}$ were constructed. Their components are given by the formulas

$$X^1 = \frac{\alpha^1}{F^2} = \frac{B}{F^2}, \qquad\qquad X^2 = \frac{\alpha^2}{F^2} = -\frac{A}{F^2}, \tag{5.1}$$

$$Y^1 = \frac{\beta^1}{F^4} = \frac{G}{F^4}, \qquad\qquad Y^2 = \frac{\beta^2}{F^4} = \frac{H}{F^4}. \tag{5.2}$$

According to Theorem 3.3 Bagderina's differential operators $\mathcal{D}_1^{\mathrm{Bgd}}$ and $\mathcal{D}_2^{\mathrm{Bgd}}$ coincide with the vector fields $\mathbf{X}$ and $\mathbf{Y}$ whose components are given in (5.1) and (5.2). The commutator of the vector fields $\mathbf{X}$ and $\mathbf{Y}$ is given by the formula

$$[\mathbf{X},\,\mathbf{Y}] = L\,\mathbf{X} - K\,\mathbf{Y}, \tag{5.3}$$

where L and K are scalar invariants from (4.3). The formula (5.3) coincides with (3.15) in [7]. There it is used in order to define the invariants L and K.

Theorem 5.1. *For any two vector field obeying the relationship* (5.3) *there are two scalar functions u and v such that the vector fields*

$$\tilde{\mathbf{X}} = \frac{\mathbf{X}}{u}, \qquad\qquad \tilde{\mathbf{Y}} = \frac{\mathbf{Y}}{v}, \qquad\qquad (5.4)$$

do commute with each other, i. e. their commutator is zero: $[\tilde{\mathbf{X}}, \tilde{\mathbf{Y}}] = 0$.

Theorem 5.1 is a rather well-known result. It can be considered as a two-dimensional reduction of the well-known Frobenius theorem on integrability of involutive distributions (see Proposition 1.2 in Chapter I of [10]). The proof of Theorem 5.1 is sketched on page 8 of [7].

Two commuting vector fields (5.4) on the plane define a local curvilinear coordinate system $(\tilde{x}, \tilde{y})$. The coordinates $\tilde{x}$ and $\tilde{y}$ are related to the initial coordinates x and y by means of some transformation (1.2). They are special coordinates for the equation (1.1) since they are related to the vectorial invariants of this equation. Transforming to these special coordinates $\tilde{x}$ and $\tilde{y}$, we would straighten the vector fields (5.4), i. e. we would have the following formulas for their components:

$$\tilde{X}^1 = 1, \qquad\qquad \tilde{X}^2 = 0, \qquad\qquad (5.5)$$

$$\tilde{Y}^1 = 0, \qquad\qquad \tilde{Y}^2 = 1. \qquad\qquad (5.6)$$

Applying (5.4) to (5.5) and (5.6), we derive

$$X^1 = u, \qquad\qquad X^2 = 0, \qquad\qquad (5.7)$$

$$Y^1 = 0, \qquad\qquad Y^2 = v. \qquad\qquad (5.8)$$

Now we apply (5.7) and (5.8) to (5.3). As a result we derive

$$\mathbf{X}v = -K\,v, \qquad\qquad \mathbf{Y}u = -L\,u. \qquad\qquad (5.9)$$

The formulas (5.9) coincide with (3.17) in [7]. Being combined with (5.7) and (5.8), the formulas (5.9) lead to the differential equations

$$u\,v_{1.0} = -K\,v, \qquad\qquad v\,u_{0.1} = -L\,u. \qquad\qquad (5.10)$$

The partial derivatives in (5.10) refer to the special coordinates $\tilde{x}$ and $\tilde{y}$. However below we shall omit the tilde sign, e. g. assuming that these special coordinates were chosen from the very beginning.

The next step is to apply (5.1) and (5.2) to (5.7) and (5.8). This yields

$$u = \frac{B}{F^2}, \qquad A = 0, \qquad G = 0, \qquad v = \frac{H}{F^4}. \qquad (5.11)$$

Taking $A = 0$ from (5.11) and applying it to (4.6) and (4.7), we get

$$P = -\frac{F^5}{B^3}, \qquad\qquad H = -3\,P\,B^2, \qquad\qquad Q = \frac{B_{1.0}}{3\,B}. \qquad (5.12)$$

Combining the formulas (5.11) and (5.12), we can write

$$u = \frac{B}{F^2}, \qquad v = \frac{3\,F}{B}, \qquad P = -\frac{F^5}{B^3}, \qquad Q = \frac{B_{1.0}}{3\,B}. \qquad (5.13)$$

The formulas (5.13) do coincide with (3.20) in [7]. We use the first two of them in order to substitute them into (5.10). As a result we get

$$L = \frac{6\,B\,F_{0.1} - 3\,B_{0.1}\,F}{B^2}, \qquad\qquad K = \frac{F\,B_{1.0} - B\,F_{1.0}}{F^3}. \qquad (5.14)$$

The formulas (5.14) coincide with the formulas (3.22) in [7].

Note that L and K in (5.14) are scalar invariants considered above in section 4. The other invariants considered in section 4 are I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8. In [7] they are introduced through the following formulas:

$$\begin{aligned} \nabla_{\mathbf{X}}\mathbf{X} &= I_1\,\mathbf{X} + I_2\,\mathbf{Y}, & \nabla_{\mathbf{X}}\mathbf{Y} &= I_3\,\mathbf{X} + I_4\,\mathbf{Y}, \\ \nabla_{\mathbf{Y}}\mathbf{X} &= I_5\,\mathbf{X} + I_6\,\mathbf{Y}, & \nabla_{\mathbf{Y}}\mathbf{Y} &= I_7\,\mathbf{X} + I_8\,\mathbf{Y} \end{aligned} \qquad (5.15)$$

(see the formulas (3.5) in [7]). The covariant derivatives in the left hand sides of the formulas (5.15) are calculated with the use of the connection components Γ^k_{ij} introduced by the formulas (3.4) in [7]. Irrespective to the choice of a coordinate system these connection components are given by the formulas

$$\begin{aligned} \Gamma^1_{11} &= Q + \frac{2\,F_{1.0}}{3\,F}, & \Gamma^2_{11} &= -P, \\[2mm] \Gamma^1_{12} &= \Gamma^1_{21} = R + \frac{F_{0.1}}{3\,F}, & \Gamma^2_{12} &= \Gamma^2_{21} = -Q + \frac{F_{1.0}}{3\,F}, \\[2mm] \Gamma^1_{22} &= S, & \Gamma^2_{22} &= -R + \frac{2\,F_{0.1}}{3\,F}. \end{aligned} \qquad (5.16)$$

Applying (5.7), (5.8) and (5.16) to (5.15), one can easily calculate the invariants I_1, I_2, I_3, I_4, I_5, I_6, I_7, I_8 in special coordinates explicitly:

$$\begin{aligned} I_2 &= \frac{1}{3}, & I_1 &= \frac{4\,F\,B_{1.0} - 4\,B\,F_{1.0}}{3\,F^3}, \\[2mm] I_3 &= \frac{F_{0.1} + 3\,F\,B}{B}, & I_4 &= \frac{4\,B\,F_{1.0} - 4\,F\,B_{1.0}}{3\,F^3}, \\[2mm] I_6 &= \frac{B\,F_{1.0} - F\,B_{1.0}}{3\,F^3}, & I_5 &= \frac{3\,F\,B_{0.1} + 3\,F\,B\,R - 5\,B\,F_{0.1}}{B^2}, \\[2mm] I_7 &= \frac{9\,F^4\,S}{B^3}, & I_8 &= \frac{5\,B\,F_{0.1} - 3\,F\,B_{0.1} - 3\,F\,B\,R}{B^2}. \end{aligned} \qquad (5.17)$$

The formulas (4.1), (4.2), (4.3) are easily derived from (5.14) and (5.17). It is that very way they were derived in [7].

Note that the formulas (5.13) express P and Q through F and B. Differentiating these expressions, we can calculate partial derivatives of P and Q, expressing them

through partial derivatives of F and B. Here are the formulas for $P_{1.0}$ and $P_{0.1}$:

$$P_{1.0} = \frac{3\,F^5\,B_{1.0} - 5\,B\,F^4\,F_{1.0}}{B^4}, \qquad P_{0.1} = \frac{3\,F^5\,B_{0.1} - 5\,B\,F^4\,F_{0.1}}{B^4}. \qquad (5.18)$$

For the partial derivatives $Q_{1.0}$ and $Q_{0.1}$ we have

$$Q_{1.0} = \frac{B\,B_{2.0} - B_{1.0}^2}{3\,B^2}, \qquad Q_{0.1} = \frac{B\,B_{1.1} - B_{1.0}\,B_{0.1}}{3\,B^2}. \qquad (5.19)$$

Differentiating (5.18) and (5.19), we can derive formulas for all partial derivatives of P and Q in terms of F and B and in terms of partial derivatives of them:

$$P_{m.n} = P_{m.n}(F, B, \dots, F_{p.q}, B_{p.q}, \dots), \quad \text{where} \quad p + q \leqslant m + n, \qquad (5.20)$$

$$Q_{m.n} = Q_{m.n}(B, \dots, B_{p.q}, \dots), \quad \text{where} \quad p + q \leqslant m + n. \qquad (5.21)$$

In particular we have the following formulas:

$$\begin{aligned} P_{0.2} = {} & \frac{3\,F^5\,B_{0.2}}{B^4} - \frac{5\,F^4\,F_{0.2}}{B^3} - \frac{12\,F^5\,B_{0.1}^2}{B^5} + \\ & + \frac{30\,F^4\,F_{0.1}\,B_{0.1}}{B^4} - \frac{20\,F^3\,F_{0.1}^2}{B^3}, \end{aligned} \qquad (5.22)$$

$$Q_{1.1} = \frac{B_{2.1}}{3\,B} - \frac{B_{0.1}\,B_{2.0}}{3\,B} - \frac{2\,B_{1.0}\,B_{1.1}}{3\,B} + \frac{2\,B_{0.1}\,B_{1.0}^2}{3\,B}, \qquad (5.23)$$

$$Q_{0.2} = \frac{B_{1.2}}{3\,B} - \frac{B_{1.0}\,B_{0.2}}{3\,B} - \frac{2\,B_{0.1}\,B_{1.1}}{3\,B} + \frac{2\,B_{1.0}\,B_{0.1}^2}{3\,B}, \qquad (5.24)$$

Substituting (5.22), (5.23), (5.24) with (5.18) and (5.19) into (4.5) and taking into account $A = 0$ from (5.11), we derive the following expressions for $R_{2.0}$ and $R_{1.1}$:

$$\begin{aligned} R_{2.0} = {} & \frac{B_{1.0}}{B}\,R_{1.0} - \frac{3\,F^5}{B^3}\,R_{0.1} + \left(\frac{9\,F^5\,B_{0.1}}{B^4} - \frac{15\,F^4\,F_{0.1}}{B^3} \right) R + \\ & + \frac{2\,F^5}{B^3}\,S_{1.0} + \left(\frac{5\,F^4\,F_{1.0}}{B^3} - \frac{3\,F^5\,B_{1.0}}{B^4} \right) S + \frac{2\,B_{2.1}}{3\,B} - \frac{2\,B_{0.1}\,B_{2.0}}{3\,B^2} - \\ & - \frac{2\,B_{1.0}\,B_{1.1}}{B^2} - \frac{3\,F^5\,B_{0.2}}{B^4} + \frac{5\,F^4\,F_{0.2}}{B^3} + \frac{2\,B_{0.1}\,B_{1.0}^2}{B^3} + \frac{20\,F^3\,F_{0.1}^2}{B^3} - \\ & - \frac{30\,F^4\,F_{0.1}\,B_{0.1}}{B^4} + \frac{12\,F^5\,B_{0.1}^2}{B^5}, \end{aligned} \qquad (5.25)$$

$$\begin{aligned} R_{1.1} = {} & \frac{1}{2}\,S_{2.0} - 3\,R\,R_{1.0} + \frac{B_{1.0}}{2\,B}\,S_{1.0} + \frac{F^5}{2\,B^3}\,S_{0.1} + \left(\frac{B_{1.0}}{2\,B} - \right. \\ & \left. - \frac{B_{0.1}\,B_{1.0}}{2\,B^2} \right) R + \left(\frac{B_{2.0}}{2\,B} - \frac{B_{1.0}^2}{2\,B^2} - \frac{3\,F^5\,B_{0.1}}{B^4} + \frac{5\,F^4\,F_{0.1}}{B^3} \right) S + \\ & + \frac{B_{1.2}}{6\,B} - \frac{B_{0.1}\,B_{1.1}}{3\,B^2} - \frac{B_{1.0}\,B_{0.2}}{6\,B^2} + \frac{B_{1.0}\,B_{0.1}^2}{3\,B^2} - \frac{B}{2}. \end{aligned} \qquad (5.26)$$

Differentiating the relationships (5.25) and (5.26) we produce the expressions for higher order derivatives of R. Some of such expressions can be produced in two different ways, e. g. for the derivative $R_{2.1}$ we have

$$R_{2.1} = \frac{\partial R_{2.0}}{\partial y}, \qquad\qquad R_{2.1} = \frac{\partial R_{1.1}}{\partial x}. \qquad (5.27)$$

The right hand sides of (5.27) derived from (5.25) and (5.26) are not identically equal to each other. As a result we get a non-trivial equation:

$$\frac{\partial R_{2.0}}{\partial y} - \frac{\partial R_{1.1}}{\partial x} = 0. \qquad (5.28)$$

Equations like (5.28) are called compatibility conditions in the theory of differential equations. In our particular case it turns out that the equation (5.28) can be resolved with respect to the derivative $S_{3.0}$.

Higher order partial derivatives of P and Q are calculated by differentiating (5.18) and (5.19) (see (5.20) and (5.21)). They produce no compatibility conditions like (5.28), i. e. their compatibility conditions are identically fulfilled.

The fourth order partial derivatives $S_{4.0}$ and $S_{3.1}$ are calculated by differentiating the expression for $S_{3.0}$ which is derived from (5.28).

The third order partial derivatives $R_{3.0}$, $R_{2.1}$, $R_{1.2}$ and the fourth order partial derivatives $R_{4.0}$, $R_{3.1}$, $R_{2.2}$, $R_{1.3}$ are calculated by differentiating (5.25) and (5.26). They produce no compatibility conditions of the order 3 and 4 other than (5.28).

We do not present the results of calculations mentioned just above since they are very huge formulas. Upon completing these calculations one can proceed to Bagderina's formulas (3.1), (3.2), (3.3), (3.4). In the special coordinates introduced in [7] and used in the present section the formulas (3.1) yield

$$\alpha_0^{\mathrm{Bgd}} = \frac{B_{2.0}}{3\,B} - \frac{5\,B_{1.0}^2}{9\,B^2} - \frac{3\,F^5\,B_{0.1}}{B^4} + \frac{5\,F^4\,F_{0.1}}{B^3} - \frac{2\,F^5}{B^3}\,R,$$

$$\alpha_1^{\mathrm{Bgd}} = -\frac{B_{1.1}}{3\,B} + \frac{B_{0.1}\,B_{1.0}}{3\,B^2} + R_{1.0} - \frac{B_{1.0}}{3\,B}\,R - \frac{F^5}{B^3}\,S, \qquad (5.29)$$

$$\alpha_2^{\mathrm{Bgd}} = S_{1.0} - R_{0.1} - 2\,R^2 + \frac{2\,B_{1.0}}{3\,B}\,S.$$

The formulas (3.2) in our special coordinates simplify to

$$\beta_1^{\mathrm{Bgd}} = 0, \qquad\qquad \beta_2^{\mathrm{Bgd}} = B, \qquad (5.30)$$

which is not surprising due to Lemma 3.2 (see (3.8) and (5.11)). The formulas (3.3) in our special coordinates also yield very simple results:

$$\gamma_{10}^{\mathrm{Bgd}} = -\frac{F^5}{B^2}, \qquad\qquad \gamma_{11}^{\mathrm{Bgd}} = \frac{4}{3}\,B_{1.0},$$

$$\gamma_{20}^{\mathrm{Bgd}} = \frac{1}{3}\,B_{1.0}, \qquad\qquad \gamma_{21}^{\mathrm{Bgd}} = B_{0.1} + B\,R. \qquad (5.31)$$

The results of the formulas (3.4) in our special coordinates are not very simple.

However, they are not very complicated too. Here are these results:

$$\delta_{10}^{\mathrm{Bgd}} = \frac{F^5\,B_{1.0}}{B^3} - \frac{5\,F^4\,F_{1.0}}{B^2},$$

$$\delta_{20}^{\mathrm{Bgd}} = \frac{5\,B_{1.0}^2}{9\,B} + \frac{2\,F^5\,B_{0.1}}{B^3} - \frac{5\,F^4\,F_{0.1}}{B^2} + \frac{2\,F^5}{B^2}\,R,$$

$$\delta_{30}^{\mathrm{Bgd}} = \frac{2}{3}\,B_{1.1} - B\,R_{1.0} + \frac{2\,B_{1.0}}{3}\,R + \frac{2\,F^5}{B^2}\,S, \qquad (5.32)$$

$$\delta_{11}^{\mathrm{Bgd}} = \frac{20\,B_{1.0}^2}{9\,B} + \frac{11\,F^5\,B_{0.1}}{B^3} - \frac{20\,F^4\,F_{0.1}}{B^2} + \frac{8\,F^5}{B^2}\,R,$$

$$\delta_{21}^{\mathrm{Bgd}} = \frac{8}{3}\,B_{1.1} - \frac{B_{0.1}\,B_{1.0}}{B} - 4\,B\,R_{1.0} + \frac{5\,B_{1.0}}{3}\,R + \frac{5\,F^5}{B^2}\,S,$$

$$\delta_{31}^{\mathrm{Bgd}} = B_{0.2} + 6\,B\,R_{0.1} - 5\,B\,S_{1.0} + 12\,B\,R^2 + 3\,B_{0.1}\,R - 5\,B_{1.0}\,S.$$

The formulas (3.5) and (3.6) are not used in item 1 of Bagderina's Theorem 2 in [5], i.e. they do not refer to the case of **general position**. For this reason we shall not calculate their values here and, having completed (5.29), (5.30), (5.31), (5.32), we proceed to the quantities (3.14). The values of these quantities in our special coordinates are given by the following very simple formulas:

$$\Gamma_0^{\mathrm{Bgd}} = -\frac{3\,F^5}{B}, \qquad\qquad\qquad \Gamma_1^{\mathrm{Bgd}} = 0, \qquad\qquad (5.33)$$

The formulas (5.33) agree with Lemma 3.5 (see (3.19), (5.11), (5.12)).

Now let's proceed to the quantities (3.13). The first of them is J_0. According to Lemma 3.3 this Bagderina's quantity is related to the quantity F from (4.7) introduced in [7] by means of the formula (3.15). In our special coordinates the formula (3.15) is immediate from (3.14) due to (5.30) and (5.31). Then, using Bagderina's formula (3.17), we reproduce (3.18).

The formula (3.23) is inapplicable in our special coordinates since $\beta_1^{\mathrm{Bgd}} = 0$ (see (5.30)). There is a reservation for this case in Bagderina's paper (see Remark 1 on page 7 of [5]). According to this reservation, if $\beta_1^{\mathrm{Bgd}} = 0$, then the second formula in (3.22) is replaced by the following formula:

$$\mathcal{D}_2^{\mathrm{Bgd}} = (J_0^{\mathrm{Bgd}})^{-4/5}(\Gamma_1^{\mathrm{Bgd}}\,D_x - \Gamma_0^{\mathrm{Bgd}}\,D_y). \qquad\qquad (5.34)$$

Applying the first formula (3.22) and the formula (5.34), from (3.15), (3.18), (5.30), and (5.33) we get the following formulas for $\mathcal{D}_1^{\mathrm{Bgd}}$ and $\mathcal{D}_2^{\mathrm{Bgd}}$:

$$\mathcal{D}_1^{\mathrm{Bgd}} = \frac{B}{F^2}\,D_x, \qquad\qquad\qquad \mathcal{D}_2^{\mathrm{Bgd}} = \frac{3\,F}{B}\,D_y. \qquad\qquad (5.35)$$

The formulas (5.35) are in agreement with (3.24) and (3.27) due to (3.8), (3.21), (5.11), (5.12) and since D_x and D_y are just partial derivatives in x and y. The formulas (5.35) confirm Theorem 3.3.

Let's return to the formulas (3.13). As we noted above the first of them reduces to (3.18). The others are used in order to produce Bagderina's scalar invariants (4.13).

Applying (3.13) to (4.13), one can derive some huge formulas for the invariants I_1^{Bgd}, I_2^{Bgd}, I_3^{Bgd}, and I_4^{Bgd}. In our special coordinates these huge formulas reduce to the following formulas, which are rather simple:

$$I_1^{\mathrm{Bgd}} = \frac{15\,F_{0.1}}{B} - \frac{42\,F\,B_{0.1}}{5\,B^2} - \frac{27\,F\,R}{5\,B}, \tag{5.36}$$

$$I_2^{\mathrm{Bgd}} = -\frac{6\,B_{1.1}}{F\,B} + \frac{6\,B_{1.0}\,B_{0.1}}{F\,B^2} + \frac{9\,R_{1.0}}{F} - \frac{9\,F^4\,S}{B^3}, \tag{5.37}$$

$$I_3^{\mathrm{Bgd}} = -\frac{5\,B_{1.0}}{F^2} + \frac{5\,B\,F_{1.0}}{F^3}, \tag{5.38}$$

$$I_4^{\mathrm{Bgd}} = \frac{90\,F_{0.1}}{B} - \frac{261\,F\,B_{0.1}}{5\,B^2} - \frac{216\,F\,R}{5\,B}. \tag{5.39}$$

Theorem 5.2. *In the case of general position Bagderina's scalar invariants I_1^{Bgd}, I_2^{Bgd}, I_3^{Bgd}, I_4^{Bgd} from* [5] *given by the formulas* (4.13) *are expressed through the invariants I_3, I_6, I_7, I_8 from* [7] *by means of the formulas*

$$I_3^{\mathrm{Bgd}} = 15\,I_6, \qquad I_1^{\mathrm{Bgd}} = I_3 + \frac{14}{5}\,I_8, \qquad I_4^{\mathrm{Bgd}} = 3\,I_3 + \frac{87}{5}\,I_8, \tag{5.40}$$

$$I_2^{\mathrm{Bgd}} = -I_7 - 3\,\mathbf{X}I_8 + 15\,\mathbf{Y}I_6 + (24\,I_8 + 15\,I_3)\,I_6, \tag{5.41}$$

where $\mathbf{X} = \mathcal{D}_1^{\mathrm{Bgd}}$ and $\mathbf{Y} = \mathcal{D}_2^{\mathrm{Bgd}}$ are the operators of invariant differentiations given by the formulas (3.24) *and* (3.27).

The formulas (5.40) and (5.41) are proved by direct calculations using the explicit formulas for invariants (5.36), (5.37), (5.38), (5.39), and (5.17). In our special coordinates the operators $\mathcal{D}_1^{\mathrm{Bgd}}$ and $\mathcal{D}_2^{\mathrm{Bgd}}$ are given by the formulas (5.35). In the form of vector fields these operators were introduced in [7] (see Theorem 3.3).

On page 3 of her paper [5] Yu. Yu. Bagderina presents her own comparison formulas expressing I_3, I_6, I_7, I_8 through her invariants. We write them as

$$I_6 = \frac{I_3^{\mathrm{Bgd}}}{15}, \qquad I_3 = \frac{29\,I_1^{\mathrm{Bgd}}}{15} - \frac{14\,I_4^{\mathrm{Bgd}}}{45}, \qquad I_8 = \frac{I_4^{\mathrm{Bgd}}}{9} - \frac{I_1^{\mathrm{Bgd}}}{3}, \tag{5.42}$$

$$I_7 = 3\,I_2^{\mathrm{Bgd}} - \frac{I_5^{\mathrm{Bgd}}}{3}, \quad \text{where} \quad I_5^{\mathrm{Bgd}} = \frac{J_5^{\mathrm{Bgd}}}{(J_0^{\mathrm{Bgd}})^{11/5}}. \tag{5.43}$$

It is easy to see that the formulas (5.42) are converse to the formulas (5.40). They are easily derived from (5.40). The formula (5.43) is different. It refers the reader to some quantity J_5^{Bgd} outside the main Theorem 2 and involves a separate bunch of huge notations and calculations on page 25 of [5]. Using the relationships (5.41) and (5.42), we can replace (5.43) with the formula

$$\begin{aligned} I_7 = -I_2^{\mathrm{Bgd}} &- \frac{1}{3}\,\mathcal{D}_1^{\mathrm{Bgd}}(I_4^{\mathrm{Bgd}}) + \mathcal{D}_1^{\mathrm{Bgd}}(I_1^{\mathrm{Bgd}}) + \\ &+ \mathcal{D}_2^{\mathrm{Bgd}}(I_3^{\mathrm{Bgd}}) + \frac{1}{15}\,(21\,I_1^{\mathrm{Bgd}} - 2\,I_4^{\mathrm{Bgd}})\,I_3^{\mathrm{Bgd}}. \end{aligned} \tag{5.44}$$

Theorem 5.3. *In the case of general position the scalar invariants I_3, I_6, I_7, I_8 from* [7] *given by the formulas* (4.4)*,* (4.9)*,* (4.11)*,* (4.12) *are expressed through Bagderina's scalar invariants I_1^{Bgd}, I_2^{Bgd}, I_3^{Bgd}, I_4^{Bgd} from* [5] *by means of the formulas* (5.42) *and* (5.44)*, where $\mathcal{D}_1^{\mathrm{Bgd}} = \mathbf{X}$ and $\mathcal{D}_2^{\mathrm{Bgd}} = \mathbf{Y}$ are the operators of invariant differentiations given by the formulas* (3.24) *and* (3.27)*.*

Due to Theorems 5.2 and 5.3 the quadruples of invariants I_3, I_6, I_7, I_8 from [7] and I_1^{Bgd}, I_2^{Bgd}, I_3^{Bgd}, I_4^{Bgd} from [5] are equivalent to each other.

Apart from I_1^{Bgd}, I_2^{Bgd}, I_3^{Bgd}, I_4^{Bgd}, in [5] Yu. Yu. Bagderina considers two omega invariants Ω_1^{Bgd} and Ω_2^{Bgd} and writes the relationship

$$[\mathcal{D}_1^{\mathrm{Bgd}}, \mathcal{D}_2^{\mathrm{Bgd}}] = \Omega_1^{\mathrm{Bgd}} \mathcal{D}_1^{\mathrm{Bgd}} + \Omega_2^{\mathrm{Bgd}} \mathcal{D}_2^{\mathrm{Bgd}}. \tag{5.45}$$

Comparing (5.45) with (5.3) and applying Theorem 3.3, we find that

$$\Omega_1^{\mathrm{Bgd}} = L, \qquad\qquad\qquad \Omega_2^{\mathrm{Bgd}} = -K. \tag{5.46}$$

Theorem 5.4. *In the case of general position Bagderina's omega invariants Ω_1^{Bgd} and Ω_2^{Bgd} in* (5.46) *are expressed through the invariants L and K introduced in* [7] *by means of the formulas* (5.46)*.*

For Ω_1^{Bgd} and Ω_2^{Bgd} from (5.46) in item 1 of her Theorem 2 in [5] Yu. Yu. Bagderina writes the following formulas:

$$\Omega_1^{\mathrm{Bgd}} = \frac{1}{5}(8\,I_1^{\mathrm{Bgd}} - I_4^{\mathrm{Bgd}}), \qquad\qquad \Omega_2^{\mathrm{Bgd}} = \frac{1}{5}\,I_3^{\mathrm{Bgd}}. \tag{5.47}$$

Using (5.46) and (5.40), one easily finds that the formulas (5.47) are equivalent to the formulas (4.3).

Apart from the formulas discussed above, which are not new in [5] as compared to [7], [3], and [4], in item 1 of her Theorem 2 Yu. Yu. Bagderina provides a series of algebraic equations relating higher order invariants with each other (see (2.9) in [5]). These equations are new. But they do not affect the classification itself. We shall not verify these equations, just saying that this could be done with the use of the explicit formulas (5.36), (5.37), (5.38), (5.39), and (5.35), which are rather simple. In addition to them the equations (5.25) and (5.26) along with their compatibility condition (5.28) should be taken into account.

6. Conclusions.

The classification of the equations (1.1) suggested by Yu. Yu. Bagderina in [5] is not absolutely new. It coincides with the prior classification suggested in [3] at least in two items. The item 9 of Bagderina's classification Theorem 2 in [5] coincides with the case of **maximal degeneration** from [3]. The item 1 in Bagderina's classification Theorem 2 coincides with the case of **general position** from [3]. The comparison of the two classifications in other 7 cases (items), which are present in both classifications, will be continued in separate papers.

The detailed analysis carried out in the above sections shows that in the case of **general position** most structures and most formulas from Bagderina's paper [5] do coincide or are very closely related to those in [7], though they are given in different

notations (see Lemma 3.1, Lemma 3.2, Lemma 3.3, Theorem 3.2, Lemma 3.4, Lemma 3.5, Theorem 3.3, Theorem 5.4). Four basic scalar invariants from item 1 of Bagderina's Theorem 2 in [5] are equivalent to four invariants I_3, I_6, I_7, I_8 defined in [7] (see Theorem 5.2 and Theorem 5.3).

In addition to the coincidence facts revealed in lemmas and theorems listed just above, in section 5 of the present paper we show that most of the complicated formulas from [5] do simplify substantially in the special coordinates suggested in [7] more than 15 years before Bagderina's publication.

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