COMPARISON OF TWO CLASSIFICATIONS OF A CLASS OF ODE'S IN THE FIRST CASE OF INTERMEDIATE DEGENERATION.

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Abstract. Two classifications of second order ODE's cubic with respect to the first order derivative are compared in the first case of intermediate degeneration. The correspondence of vectorial, pseudovectorial, scalar, and pseudoscalar invariants of the equations in this case is established.

1. Introduction.

Since the epoch of classical papers (see [1] and [2]) it is known that the class of second order differential equations cubic with respect to the first order derivative

$$y'' = P(x,y) + 3\,Q(x,y)\,y' + 3\,R(x,y)\,(y')^2 + S(x,y)\,(y')^3 \tag{1.1}$$

is closed with respect to transformations of the form

$$\begin{cases} \tilde{x} = \tilde{x}(x,y),\\ \tilde{y} = \tilde{y}(x,y). \end{cases} \tag{1.2}$$

About 19 years ago in [3] and [4] the equations (1.1) were classified using their scalar invariants. They were subdivided into **nine subclasses** closed with respect to transformations of the form (1.2). The richest class comprising almost all equations of the form (1.1) consists of the equations of **general position**. The smallest class is composed by the equations of **maximal degeneration**. The rest of the equations (1.1) are distributed among seven subclasses composed by the equations of **intermediate degeneration**.

Recently in 2013 Yu. Yu. Bagderina in [5] presented her own classification of the equations (1.1) again subdividing them into **nine subclasses** closed with respect to transformations of the form (1.2). She uses Sophus Lie's method of infinitesimal transformations adapted to equations of the form (1.1) by N. H. Ibragimov in [6].

In [5] Yu. Yu. Bagderina does not mention the previously existing classification from [3] and [4]. She cites the paper [4] only as a source of invariants and for criticism of its method. The present paper is the second one in a series of papers intended to examine the results of [5] and compare them with the prior results from [3, 4] and [7]. In the previous paper [8] it was shown that items 1 and 9 in Bagderina's classification Theorem 2 in [5] do coincide with the case of **general position**

2000 *Mathematics Subject Classification.* 34A34, 34A26, 34C20, 34C14.

and the case of **maximal degeneration** respectively from the previously existing classification in [3] and [4]. It was also revealed that in the case of **general position** most structures and most formulas from Bagderina's paper [5] do coincide or are very closely related to those in [7], though they are given in different notations.

In the present paper we consider item 2 of Bagderina's classification Theorem in [5] and compare it with the **first case of intermediate degeneration** in [3] and [4]. Then we study the structures and formulas from item 2 of Thheirem 2 in [5] and establish their correspondence to the structures and formulas of the previously existing classification in [3] and [4].

2. Some notations and definitions.

Transformations of the form (1.2) are called point transformations. They are assumed to be locally invertible. The inverse transformations for them are also point transformations. They are written as follows:

$$\begin{cases} x = \tilde{x}(\tilde{x}, \tilde{y}), \\ y = \tilde{y}(\tilde{x}, \tilde{y}). \end{cases} \tag{2.1}$$

According to [3, 4] and [7], we use dot index notations for partial derivatives, e. g. having two functions $f(x, y)$ and $g(\tilde{x}, \tilde{y})$ we write

$$f_{p.q} = \frac{\partial^{p+q} f}{\partial x^p\, \partial y^q}, \qquad\qquad g_{p.q} = \frac{\partial^{p+q} g}{\partial \tilde{x}^p\, \partial \tilde{y}^q}. \tag{2.2}$$

In terms of the notations (2.2) the Jacoby matrices of the direct and inverse point transformations (1.2) and (2.1) are written as follows:

$$S = \begin{Vmatrix} x_{1.0} & x_{0.1} \\ y_{1.0} & y_{0.1} \end{Vmatrix}, \qquad\qquad T = \begin{Vmatrix} \tilde{x}_{1.0} & \tilde{x}_{0.1} \\ \tilde{y}_{1.0} & \tilde{y}_{0.1} \end{Vmatrix}. \tag{2.3}$$

In geometry the transformations (1.2) and (2.1) are interpreted as changes of local curvilinear coordinates on the plane $\mathbb{R}^2$ or on some two-dimensional manifold. Their Jacoby matrices are called the direct and inverse transition matrices (see [9]).

Tensorial and pseudotensorial fields in local coordinates are presented as arrays of functions whose arguments are x, y or $\tilde{x}$, $\tilde{y}$ respectively. These arrays of functions are called their components. They obey some definite transformation rules.

Definition 2.1. A pseudotensorial field of the type (r, s) and weight m is an array of functions $F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ which under the change of coordinates (1.2) transforms as

$$F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = (\det T)^m \sum_{\substack{p_1\ldots p_r\\ q_1\ldots q_s}} S^{i_1}_{p_1} \ldots\, S^{i_r}_{p_r}\, T^{q_1}_{j_1} \,\ldots\, T^{q_s}_{j_s}\, \tilde{F}^{p_1\ldots\,p_r}_{q_1\ldots\,q_s}. \tag{2.4}$$

Tensorial fields are those pseudotensorial fields whose weight m in (2.4) is zero. The prefix "pseudo" always indicates the nonzero weight $m \neq 0$.

Tensorial and pseudotensorial fields of the type $(1, 0)$ are called vectorial and pseudovectorial fields. Tensorial and pseudotensorial fields of the type $(0, 1)$ are called covectorial and pseudocovectorial fields. And finally, scalar and pseudoscalar fields are those fields whose type is $(0, 0)$.

Definition 2.2. Tensorial and pseudotensorial fields whose components are expressed through y', through the coefficients P, Q, R, S of the equation (1.1), and through their partial derivatives are called tensorial and pseudotensorial invariants of this equation respectively.

Remark. Typically, in differential geometry components of tensorial and pseudotensorial fields depend on a point of the base manifold only, i. e. on x and y or on $\tilde{x}$ and $\tilde{y}$ in our particular case. If the dependence of other parameters is included, this makes an extension of the concept. So, in Definition 2.2 we have extended tensorial and pseudotensorial fields.

3. Some basic structures.

In [5] Yu. Yu. Bagderina introduces a long list of special notations. In order to distinguish her notations from those in [3, 4] and [7] we use the upper mark «Bgd» for her notations. The first order expressions introduced in [5] are

$$\begin{aligned} \alpha_0^{\text{Bgd}} &= Q_{1.0} - P_{0.1} + 2\,P\,R - 2\,Q^2,\\ \alpha_1^{\text{Bgd}} &= R_{1.0} - Q_{0.1} + P\,S - Q\,R,\\ \alpha_2^{\text{Bgd}} &= S_{1.0} - R_{0.1} + 2\,Q\,S - 2\,R^2. \end{aligned} \tag{3.1}$$

The order of an expression is determined by the highest order of partial derivatives of P, Q, R, S in it. As it was noted in [8], Bagderina's alpha quantities (3.1) coincide with the components of the symmetric two-dimensional array Ω from [7]:

$$\alpha_0^{\text{Bgd}} = \Omega_{11}, \qquad \alpha_1^{\text{Bgd}} = \Omega_{12} = \Omega_{21}, \qquad \alpha_2^{\text{Bgd}} = \Omega_{22}. \tag{3.2}$$

The quantities Ω_{ij} in (3.2) constitute nether a tensorial invariant nor a pseudotensor invariant. However, their derivatives are used in constructing both tensorial and pseudotensor invariants.

The second order expressions are given by the formulas (2.2) in [5]:

$$\begin{aligned} \beta_1^{\text{Bgd}} &= \partial_x\alpha_1^{\text{Bgd}} - \partial_y\alpha_0^{\text{Bgd}} + R\,\alpha_0^{\text{Bgd}} - 2\,Q\,\alpha_1^{\text{Bgd}} + P\,\alpha_2^{\text{Bgd}},\\ \beta_2^{\text{Bgd}} &= \partial_x\alpha_2^{\text{Bgd}} - \partial_y\alpha_1^{\text{Bgd}} + S\,\alpha_0^{\text{Bgd}} - 2\,R\,\alpha_1^{\text{Bgd}} + Q\,\alpha_2^{\text{Bgd}}. \end{aligned} \tag{3.3}$$

As it was noted in [8], Bagderina's beta quantities (3.3) coincide with the components of the pseudocovectorial field $\boldsymbol{\alpha}$ of the weight 1 constructed in [7]:

$$\beta_1^{\text{Bgd}} = \alpha_1 = A, \qquad \beta_2^{\text{Bgd}} = \alpha_2 = B. \tag{3.4}$$

It is convenient to express the components (3.4) of the field $\boldsymbol{\alpha}$ directly through P, Q, R, S, as it was done in [3, 4] and [7], rather than through (3.1):

$$\begin{aligned} A &= P_{0.2} - 2\,Q_{1.1} + R_{2.0} + 2\,P\,S_{1.0} + S\,P_{1.0} -\\ &\qquad - 3\,P\,R_{0.1} - 3\,R\,P_{0.1} - 3\,Q\,R_{1.0} + 6\,Q\,Q_{0.1},\\ B &= S_{2.0} - 2\,R_{1.1} + Q_{0.2} - 2\,S\,P_{0.1} - P\,S_{0.1} +\\ &\qquad + 3\,S\,Q_{1.0} + 3\,Q\,S_{1.0} + 3\,R\,Q_{0.1} - 6\,R\,R_{1.0}. \end{aligned} \tag{3.5}$$

Bagderina's third order expressions are given by the formulas (2.3) in [5]:

$$
\begin{aligned}
&\gamma_{10}^{\text{Bgd}} = \partial_x\beta_1^{\text{Bgd}} - Q\,\beta_1^{\text{Bgd}} + P\,\beta_2^{\text{Bgd}},\\
&\gamma_{11}^{\text{Bgd}} = \partial_x\beta_2^{\text{Bgd}} - R\,\beta_1^{\text{Bgd}} + Q\,\beta_2^{\text{Bgd}},\\
&\gamma_{20}^{\text{Bgd}} = \partial_y\beta_1^{\text{Bgd}} - R\,\beta_1^{\text{Bgd}} + Q\,\beta_2^{\text{Bgd}},\\
&\gamma_{21}^{\text{Bgd}} = \partial_y\beta_2^{\text{Bgd}} - S\,\beta_1^{\text{Bgd}} + R\,\beta_2^{\text{Bgd}}.
\end{aligned}
\tag{3.6}
$$

The expressions (3.6) are used in order to define other third order expressions. They are given by the formulas (2.17) in [5]:

$$
\begin{aligned}
&\Gamma_0^{\text{Bgd}} = 3\,\beta_2^{\text{Bgd}}\,\gamma_{10}^{\text{Bgd}} + \beta_1^{\text{Bgd}}\,(\gamma_{20}^{\text{Bgd}} - 4\,\gamma_{11}^{\text{Bgd}}),\\
&\Gamma_1^{\text{Bgd}} = \beta_2^{\text{Bgd}}\,(4\,\gamma_{20}^{\text{Bgd}} - \gamma_{11}^{\text{Bgd}}) - 3\,\beta_1^{\text{Bgd}}\,\gamma_{21}^{\text{Bgd}}.
\end{aligned}
\tag{3.7}
$$

As it was shown in [8], Bagderina's gamma quantities (3.7) coincide with the components of the pseudocovectorial field $\boldsymbol\beta$ of the weight 3 constructed in [7]:

$$
\Gamma_0^{\text{Bgd}} = \beta_1 = -H, \qquad\qquad \Gamma_1^{\text{Bgd}} = \beta_2 = G.
\tag{3.8}
$$

The quantities H and G in (3.8) can be expressed through A and B from (3.5) in a more explicit way. They are given by the following formulas taken from [7]:

$$
\begin{aligned}
&G = -B\,B_{1.0} - 3\,A\,B_{0.1} + 4\,B\,A_{0.1} + 3\,S\,A^2 - 6\,R\,B\,A + 3\,Q\,B^2,\\
&H = -A\,A_{0.1} - 3\,B\,A_{1.0} + 4\,A\,B_{1.0} - 3\,P\,B^2 + 6\,Q\,A\,B - 3\,R\,A^2,
\end{aligned}
\tag{3.9}
$$

Bagderina's fourth order expressions are given by the formulas (2.4) in [5]:

$$
\begin{aligned}
&\delta_{10}^{\text{Bgd}} = \partial_x\gamma_{10}^{\text{Bgd}} - 2\,Q\,\gamma_{10}^{\text{Bgd}} + P\,(\gamma_{20}^{\text{Bgd}} + \gamma_{11}^{\text{Bgd}}) - 5\,\alpha_0^{\text{Bgd}}\,\beta_1^{\text{Bgd}},\\
&\delta_{20}^{\text{Bgd}} = \partial_x\gamma_{20}^{\text{Bgd}} - R\,\gamma_{10}^{\text{Bgd}} + P\,\gamma_{21}^{\text{Bgd}} - 4\,\alpha_1^{\text{Bgd}}\,\beta_1^{\text{Bgd}} - \alpha_0^{\text{Bgd}}\,\beta_2^{\text{Bgd}},\\
&\delta_{30}^{\text{Bgd}} = \partial_y\gamma_{20}^{\text{Bgd}} - S\,\gamma_{10}^{\text{Bgd}} + Q\,\gamma_{21}^{\text{Bgd}} - 4\,\alpha_2^{\text{Bgd}}\,\beta_1^{\text{Bgd}} - \alpha_1^{\text{Bgd}}\,\beta_2^{\text{Bgd}},\\
&\delta_{11}^{\text{Bgd}} = \partial_x\gamma_{11}^{\text{Bgd}} - R\,\gamma_{10}^{\text{Bgd}} + P\,\gamma_{21}^{\text{Bgd}} - \alpha_1^{\text{Bgd}}\,\beta_1^{\text{Bgd}} - 4\,\alpha_0^{\text{Bgd}}\,\beta_2^{\text{Bgd}},\\
&\delta_{21}^{\text{Bgd}} = \partial_x\gamma_{21}^{\text{Bgd}} - R\,(\gamma_{20}^{\text{Bgd}} + \gamma_{11}^{\text{Bgd}}) + 2\,Q\,\gamma_{21}^{\text{Bgd}} - 5\,\alpha_1^{\text{Bgd}}\,\beta_2^{\text{Bgd}},\\
&\delta_{31}^{\text{Bgd}} = \partial_y\gamma_{21}^{\text{Bgd}} - S\,(\gamma_{20}^{\text{Bgd}} + \gamma_{11}^{\text{Bgd}}) + 2\,R\,\gamma_{21}^{\text{Bgd}} - 5\,\alpha_2^{\text{Bgd}}\,\beta_2^{\text{Bgd}}.
\end{aligned}
\tag{3.10}
$$

The fifth order expressions by Bagderina are given by the formulas (2.5) in [5]:

$$
\begin{aligned}
&\epsilon_{10}^{\text{Bgd}} = \partial_x\delta_{10}^{\text{Bgd}} - 3\,Q\,\delta_{10}^{\text{Bgd}} + P\,(2\,\delta_{20}^{\text{Bgd}} + \delta_{11}^{\text{Bgd}}) - 12\,\alpha_0^{\text{Bgd}}\,\gamma_{10}^{\text{Bgd}},\\
&\epsilon_{20}^{\text{Bgd}} = \partial_y\delta_{10}^{\text{Bgd}} - 3\,R\,\delta_{10}^{\text{Bgd}} + Q\,(2\,\delta_{20}^{\text{Bgd}} + \delta_{11}^{\text{Bgd}}) - 12\,\alpha_1^{\text{Bgd}}\,\gamma_{10}^{\text{Bgd}},\\
&\epsilon_{11}^{\text{Bgd}} = \partial_x\delta_{11}^{\text{Bgd}} - R\,\delta_{10}^{\text{Bgd}} - Q\,\delta_{11}^{\text{Bgd}} + 2\,P\,\delta_{21}^{\text{Bgd}} - 2\,\alpha_1^{\text{Bgd}}\,\gamma_{10}^{\text{Bgd}} -\\
&\qquad -\,10\,\alpha_0^{\text{Bgd}}\,\gamma_{11}^{\text{Bgd}} - 10\,(\beta_1^{\text{Bgd}})^2.
\end{aligned}
\tag{3.11}
$$

And finally, her sixth order expression is given by the formula (2.6) in [5]:

$$
\lambda_{10}^{\text{Bgd}} = \partial_x\epsilon_{10}^{\text{Bgd}} - 4\,Q\,\epsilon_{10}^{\text{Bgd}} + P\,(3\,\epsilon_{20}^{\text{Bgd}} + \epsilon_{11}^{\text{Bgd}}) - 21\,\alpha_0^{\text{Bgd}}\,\delta_{10}^{\text{Bgd}}.
\tag{3.12}
$$

In $\mathbb{R}^2$ and in any two-dimensional manifold there are two pseudotensorial fields with constant components. They are denoted by the same symbol $\mathbf{d}$ and are given by the same skew-symmetric matrix in any local coordinates:

$$d_{ij} = \left\| \begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array} \right\|, \qquad\qquad d^{ij} = \left\| \begin{array}{rr} 0 & 1 \\ -1 & 0 \end{array} \right\|. \tag{3.13}$$

The components d_{ij} in (3.13) correspond to the pseutotensorial field $\mathbf{d}$ of the type $(0, 2)$ and the weight -1. The components d^{ij} in (3.13) correspond to the pseutotensorial field $\mathbf{d}$ of the type $(2, 0)$ and the weight 1. These two fields are used for raising and lowering indices of other pseutotensorial fields. In particular, we have

$$\alpha^i = \sum_{k=1}^{2} d^{ik}\,\alpha_k, \qquad\qquad \beta^i = \sum_{k=1}^{2} d^{ik}\,\beta_k, \tag{3.14}$$

In explicit form the equalities (3.14) are written as follows:

$$\alpha^1 = B = \beta_2^{\,\mathrm{Bgd}}, \qquad\qquad \alpha^2 = -A = -\beta_1^{\,\mathrm{Bgd}}, \tag{3.15}$$

$$\beta^1 = G = \Gamma_1^{\,\mathrm{Bgd}}, \qquad\qquad \beta^2 = H = -\Gamma_0^{\,\mathrm{Bgd}}. \tag{3.16}$$

The quantities (3.15) are the components of the pseudovectorial field $\boldsymbol{\alpha}$ of the weight 2. The quantities (3.16) are the components of the pseudovectorial field $\boldsymbol{\beta}$ of the weight 4. In [4] these quantities are used in order to define a pseudoscalar field F of the weight 1. This field is defined by means of the formula

$$3\,F^5 = \sum_{i=1}^{2} \alpha_i\,\beta^i = -\sum_{i=1}^{2} \beta_i\,\alpha^i = A\,G + B\,H. \tag{3.17}$$

One can define F more explicitly by applying (3.5) and (3.9) to (3.17):

$$\begin{aligned} F^5 = A\,B\,A_{0.1} + B\,A\,B_{1.0} - A^2\,B_{0.1} - B^2\,A_{1.0} - {}\\ - P\,B^3 + 3\,Q\,A\,B^2 - 3\,R\,A^2\,B + S\,A^3. \end{aligned} \tag{3.18}$$

Yu. Yu. Bagderina introduces her own quantity J_0 (see (2.16) in [5]):

$$J_0^{\,\mathrm{Bgd}} = (\beta_2^{\,\mathrm{Bgd}})^2\,\gamma_{10}^{\,\mathrm{Bgd}} - \beta_1^{\,\mathrm{Bgd}}\,\beta_2^{\,\mathrm{Bgd}}\,(\gamma_{20}^{\,\mathrm{Bgd}} + \gamma_{11}^{\,\mathrm{Bgd}}) + (\beta_1^{\,\mathrm{Bgd}})^2\,\gamma_{21}^{\,\mathrm{Bgd}}. \tag{3.19}$$

As it was noted in [8], Bagderina's quantity $J_0^{\,\mathrm{Bgd}}$ in (3.19) is related to the quantity F in (3.17) and in (3.18) in the following way:

$$J_0^{\,\mathrm{Bgd}} = -F^5. \tag{3.20}$$

Using $J_0^{\,\mathrm{Bgd}}$, Yu. Yu. Bagderina introduces her mu quantity $\mu_1^{\,\mathrm{Bgd}}$:

$$\mu_1^{\,\mathrm{Bgd}} = (J_0^{\,\mathrm{Bgd}})^{1/5}. \tag{3.21}$$

Comparing the formula (3.20) with the formula (3.21), we see that Bagderina's mu quantity differs from F only in sign:

$$\mu_1^{\text{Bgd}} = -F.$$

The pseudocovectorial and pseudovectorial fields $\boldsymbol{\alpha}$ with the components (3.4) and (3.15), the pseudocovectorial and pseudovectorial fields $\boldsymbol{\beta}$ with the components (3.8) and (3.16), and the pseudoscalar field F in (3.17) and (3.18) constitute basic structures associated with any equation of the form (1.1). All of them are presented by Yu. Yu. Bagderina in [5] using her own notations. However, none of them is new in [5] as compared to [3, 4] and [7].

4. Cases of intermediate degeneration.

According to the classification from [3, 4] the class of equations of the form (1.1) is subdivided into **nine subclasses** which are called cases. The case of **general position** corresponds to the richest subclass of all nine. Any equation (1.1) taken by chance falls into the case of general position with the probability 1. The other eight classes are thin classes. Their total measure (probability) is zero.

The case of **general position** is defined by the condition

$$F \neq 0. \tag{4.1}$$

Looking at (3.17), we can write (4.1) as

$$3\,F^5 = \det \begin{Vmatrix} A & B \\ -H & G \end{Vmatrix} \neq 0. \tag{4.2}$$

A matrix with nonzero determinant cannot have zero rows. It cannot have proportional rows either. Note that A and B are components of the pseudocovectorial field $\boldsymbol{\alpha}$ in (3.4), while $-H$ and G are components of the pseudocovectorial field $\boldsymbol{\beta}$ in (3.8). Therefore, using (4.2), the condition (4.1) implies

$$\boldsymbol{\alpha} \neq 0, \qquad\qquad \boldsymbol{\beta} \neq 0, \qquad\qquad \boldsymbol{\alpha} \nparallel \boldsymbol{\beta}. \tag{4.3}$$

Conversely, using (3.4), (3.8), and (4.2), the conditions (4.3) imply (4.1).

The case of **maximal degeneration** is opposite to (4.3). It is given by the following condition for the pseudocovectorial field $\boldsymbol{\alpha}$:

$$\boldsymbol{\alpha} = 0. \tag{4.4}$$

Using using (3.4), (3.8), and (3.9), the condition (4.4) implies $\boldsymbol{\beta} = 0$ and $F = 0$.

Recently in [8] it was shown that the item 1 of Bagderina's classification Theorem 2 in [5] is equivalent to the case of **general position** from the prior papers [3, 4] and [7]. Also in [8] it was shown that the item 9 of this classification Theorem 2 in [5] is equivalent to the case of **maximal degeneration** from [3, 4]. Here we continue our comparison work and proceed to the cases of **intermediate degeneration**.

The cases of **intermediate degeneration** are splitted off from (4.1) and (4.4) by setting the following two conditions:

$$F = 0, \qquad\qquad \boldsymbol{\alpha} \neq 0. \tag{4.5}$$

There are seven cases of **intermediate degeneration**. In this paper we consider the first of them and compare it with item 2 in Bagderina's Theorem 2 in [5].

As we noted above, the conditions (4.3) taken altogether imply (4.1). Therefore, if the conditions (4.5) are fulfilled, then at least one of the last two conditions in (4.3) should be broken. A zero vector (or a zero pseudovector) is parallel to any other vector (or pseudovector). Therefore $\boldsymbol{\beta}=0$ is just a subcase of a more general case $\boldsymbol{\beta}\parallel\boldsymbol{\alpha}$. Hence the conditions (4.5) are equivalent to the following conditions:

$$\boldsymbol{\alpha}\neq 0, \qquad\qquad \boldsymbol{\beta}\parallel\boldsymbol{\alpha}. \tag{4.6}$$

The conditions (4.6) mean that there is a factor N such that

$$\boldsymbol{\beta}=3\,N\,\boldsymbol{\alpha}. \tag{4.7}$$

The factor N in (4.7) is a pseudoscalar field of the weight 2. The field N was discovered in [3] in the form $N=Q$ in some special coordinates (see (6.2) in [3]). The formula (4.7) was presented in [4] (see (4.2) in [4]). The formulas

$$N=\frac{G}{3\,B}, \qquad\qquad N=-\frac{H}{3\,A} \tag{4.8}$$

are immediate from (4.7), (4.8), (3.4), and (3.8), (see (4.3) in [4]). The first formula applies in the case $B\neq 0$, the second one in the case $A\neq 0$. If both A and B are nonzero, both formulas are applicable. Note that A and B cannot vanish simultaneously since A and B are components of the field $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}\neq 0$ (see (4.5) and (4.6) above).

Remark. The pseudoscalar field N, i.e. an object with proper geometric behavior, arises only if the conditions (4.5) are fulfilled. Otherwise the formulas (4.8) yield two quantities with no meaning at all.

Apart from N there are some other pseudotensorial fields and objects of different geometric nature associated with the equations (1.1) in all cases of intermediate degeneration. Here are the quantities φ_1 and φ_2:

$$\begin{aligned}&\varphi_1=-3\,A\,\frac{A\,S-B_{0.1}}{5\,B^2}-3\,\frac{A_{0.1}+B_{1.0}-3\,A\,R}{5\,B}-\frac{6}{5}\,Q,\\&\varphi_2=3\,\frac{A\,S-B_{0.1}}{5\,B}-\frac{3}{5}\,R.\end{aligned} \tag{4.9}$$

The formulas (4.9) apply in the case $B\neq 0$. If $A\neq 0$, we use the formulas

$$\begin{aligned}&\varphi_1=-3\,\frac{B\,P+A_{1.0}}{5\,A}+\frac{3}{5}\,Q,\\&\varphi_2=3\,B\,\frac{B\,P+A_{1.0}}{5\,A^2}-3\,\frac{B_{1.0}+A_{0.1}+3\,B\,Q}{5\,A}+\frac{6}{5}\,R.\end{aligned} \tag{4.10}$$

If both A and B are nonzero, then both formulas (4.9) and (4.10) are applicable.

The quantities φ_1 and φ_2 do not form a pseudotensorial field. They are transformed as follows under the point transformations (1.2):

$$\varphi_i=\sum_{j=1}^{2}T^j_i\,\tilde{\varphi}_j-\frac{\partial\ln\det T}{\partial x^i}. \tag{4.11}$$

Here $x^1 = x$, $x^2 = y$ and T is the transition matrix defined in (2.3).

The quantities φ_1 and φ_2 were introduced in [3] using some special coordinates. The formulas (4.9) and (4.10) were derived in [4]. These formulas are applicable in arbitrary coordinates x and y. Due to the transformation rule (4.11) the quantities φ_1 and φ_2 can be combined with the components of the array θ defined in [7] in order to form connection components (see (6.10) in [3] or (4.22) in [4]):

$$\Gamma^k_{ij} = \theta^k_{ij} - \frac{\varphi_i\,\delta^k_j + \varphi_j\,\delta^k_i}{3}. \tag{4.12}$$

Note that the connection (4.12) is different from the connection used in [7] and later in [8] for the case of general position. The quantities φ_1 and φ_2 here are also different from those used in the case of general position.

The second field introduced in [3] is M (see (6.7) in [3]). The first was N that was introduced as $N = Q$ in some special coordinates (see (6.2) in [3]). The field M was also first introduced in that special coordinates. The formulas for M in arbitrary coordinates were derived in [4] (see (4.28) and (4.29) in [4]):

$$\begin{aligned} M = &-\frac{12\,A\,N\,(A\,S - B_{0.1})}{5\,B} - A\,N_{0.1} + \frac{24}{5}\,A\,N\,R-\\ &-\frac{6}{5}\,N\,A_{0.1} - \frac{6}{5}\,N\,B_{1.0} + B\,N_{1.0} - \frac{12}{5}\,B\,N\,Q, \end{aligned} \tag{4.13}$$

$$\begin{aligned} M = &-\frac{12\,B\,N\,(B\,P + A_{1.0})}{5\,A} + B\,N_{1.0} + \frac{24}{5}\,B\,N\,Q+\\ &+\frac{6}{5}\,N\,B_{1.0} + \frac{6}{5}\,N\,A_{0.1} - A\,N_{0.1} - \frac{12}{5}\,A\,N\,R. \end{aligned} \tag{4.14}$$

The formula (4.13) applies in the case $B \neq 0$. If $A \neq 0$, we use the formula (4.14).

In the cases of **intermediate degeneration** we loose $\boldsymbol{\beta}$ as an independent field. It becomes parallel to $\boldsymbol{\alpha}$ (see (4.6)). However, exactly at that instant another pseudocovectorial field arises. It was discovered in [3] and was denoted through $\boldsymbol{\gamma}$. Initially $\boldsymbol{\gamma}$ was presented in some special coordinates. Then in [4] it was expressed by explicit formulas in arbitrary coordinates (see (4.30) and (4.31) in [4]):

$$\begin{aligned} \gamma_1 = &\frac{6\,A\,N\,(A\,S - B_{0.1})}{5\,B^2} - \frac{18\,N\,A\,R}{5\,B}+\\ &+\frac{6\,N\,(A_{0.1} + B_{1.0})}{5\,B} - N_{1.0} + \frac{12}{5}\,N\,Q - 2\,\Omega\,A. \end{aligned} \tag{4.15}$$

$$\gamma_2 = -\frac{6\,N\,(A\,S - B_{0.1})}{5\,B} - N_{0.1} + \frac{6}{5}\,N\,R - 2\,\Omega\,B, \tag{4.16}$$

The formulas (4.15) and (4.16) are used if $B \neq 0$. If $A \neq 0$, we write:

$$\gamma_1 = \frac{6\,N\,(B\,P + A_{1.0})}{5\,A} - N_{1.0} - \frac{6}{5}\,N\,Q - 2\,\Omega\,A, \tag{4.17}$$

$$\begin{aligned}\gamma_2 = &-\frac{6\,B\,N\,(B\,P + A_{1.0})}{5\,A^2} + \frac{18\,N\,B\,Q}{5\,A} + \\ &+\frac{6\,N\,(B_{1.0} + A_{0.1})}{5\,A} - N_{0.1} - \frac{12}{5}\,N\,R - 2\,\Omega\,B.\end{aligned}\tag{4.18}$$

The weight of the pseudocovectorial field $\boldsymbol{\gamma}$ given by the formulas (4.15) and (4.16) or by the formulas (4.17) and (4.18) is equal to 2. Note that the formulas (4.15), (4.16), (4.17), (4.18) in [4] are given in a pseudovectorial form, i. e. with upper indices (see (4.30), (4.31), (4.32), and (4.33) in [4]).

The pseudoscalar field Ω is the third field common for all cases of **intermediate degeneration**. This field was introduced in [3] by means of the formulas

$$\Omega = \frac{5}{6}\sum_{i=1}^{2}\sum_{j=1}^{2}\omega_{ij}\,d^{ij}, \text{ where } \omega_{ij} = \frac{\partial\varphi_i}{\partial x^j} - \frac{\partial\varphi_j}{\partial x^i},\tag{4.19}$$

in some special coordinates (see (6.17) and (6.18) in [3]). It turns out that the formulas (4.19) are applicable in arbitrary coordinates as well (see (4.15) and (4.16) in [4]). The matter is that Ω is related to the curvature tensor of the connection (4.12). The well-known formula for the curvature tensor (see [9]) is written as

$$R^k_{rij} = \frac{\partial\Gamma^k_{jr}}{\partial x^i} - \frac{\partial\Gamma^k_{ir}}{\partial x^j} + \sum_{q=1}^{2}\Gamma^k_{iq}\Gamma^q_{jr} - \sum_{q=1}^{2}\Gamma^k_{jq}\Gamma^q_{ir}.\tag{4.20}$$

Like in (4.11), here $x^1 = x$ and $x^2 = y$ are coordinates. Substituting (4.12) into the formula (4.19), we calculate R^k_{rij} and then find that

$$\omega_{ij} = \sum_{k=1}^{2} R^k_{kij}\tag{4.21}$$

Due to (4.21) the quantities ω_{ij} are components of a tensor, while Ω in (4.19) is a pseudoscalar of the weight 1. Here are explicit formulas for Ω:

$$\begin{aligned}\Omega = &\frac{2\,A\,B_{0.1}(A\,S - B_{0.1})}{B^3} + \frac{(2\,A_{0.1} - 3\,A\,R)\,B_{0.1}}{B^2} + \\ &+\frac{(B_{1.0} - 2\,A_{0.1})\,A\,S}{B^2} + \frac{A\,B_{0.2} - A^2\,S_{0.1}}{B^2} - \frac{A_{0.2}}{B} + \\ &+\frac{3\,A_{0.1}\,R + 3\,A\,R_{0.1} - A_{1.0}\,S - A\,S_{1.0}}{B} + R_{1.0} - 2\,Q_{0.1},\end{aligned}\tag{4.22}$$

$$\begin{aligned}\Omega = &\frac{2\,B\,A_{1.0}(B\,P + A_{1.0})}{A^3} - \frac{(2\,B_{1.0} + 3\,B\,Q)\,A_{1.0}}{A^2} + \\ &+\frac{(A_{0.1} - 2\,B_{1.0})\,B\,P}{A^2} - \frac{B\,A_{2.0} + B^2\,P_{1.0}}{A^2} + \frac{B_{2.0}}{A} + \\ &+\frac{3\,B_{1.0}\,Q + 3\,B\,Q_{1.0} - B_{0.1}\,P - B\,P_{0.1}}{A} + Q_{0.1} - 2\,R_{1.0}\end{aligned}\tag{4.23}$$

(see (4.17) and (4.21) in [4]). The formula (4.22) applies in the case $B \neq 0$. If $A \neq 0$, we apply the formula (4.23).

It is important to note that the fields $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and M obey the relationship:

$$M=\sum_{i=1}^{2}\alpha_i\,\gamma^i=\sum_{i=1}^{2}\sum_{j=1}^{2}\alpha_i\,d^{ij}\,\gamma_j. \tag{4.24}$$

The relationship (4.24) is easily derived from (4.27) in [4]. Since $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ in the right hand side of (4.24) are pseudocovectorial fields of the weights 1 and 2 respectively, and $\mathbf{d}$ is a pseudotensorial field of the weight 1, we see that M is a pseudoscalar field of the weight 4. This fact is known since [3].

5. Special coordinates.

Let's recall that the cases of **intermediate degeneration** were introduced and studied in [3] using some special coordinates where $A=0$ and $B=1$. However, Bagderina's classification Theorem 2 in [5] and her formulas are derived under the restriction $\beta_1^{\text{Bgd}}\neq 0$, which corresponds to $A\neq 0$ (see (3.4) above). In order to compare our formulas with those in Bagderina's paper [5] and in order to make this comparison comfortable for us we need some other special coordinates, which are similar to those in [3], but different from them.

Theorem 5.1. *For any equation (1.1) with $\boldsymbol{\alpha}\neq 0$ in (3.4) there are some variables x and y such that $A=1$ and $B=0$ in these variables.*

Proof. Note that $\boldsymbol{\alpha}$ in (3.4) is a peudocovectorial field of the weight 1 associated with the equation (1.1) through the formulas (3.5). Raising indices according to (3.14), we get the pseudovectorial field $\boldsymbol{\alpha}$ of the weight 2 with the components (3.15). Let $\mathbf{X}$ be some nonzero vector field such that $\mathbf{X}\parallel\boldsymbol{\alpha}$. We can choose such a field by fixing some coordinates x and y and setting $X^1=\alpha^1$ and $X^2=\alpha^2$ in these coordinates. Being fulfilled in some particular coordinates, due to (2.4) the parallelism $\mathbf{X}\parallel\boldsymbol{\alpha}$ holds in arbitrary coordinates.

It is well-known that any vector field $\mathbf{X}$ can be straighten (see [10]), i. e. there are some coordinates x and y such that

$$X^1=0, \qquad\qquad X^2=1 \tag{5.1}$$

in these coordinates. Combining (5.1) with $\mathbf{X}\parallel\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}\neq 0$, we get

$$\alpha^1=B=0, \qquad\qquad \alpha^2=-A\neq 0. \tag{5.2}$$

Now let's perform a special transformation of the form (1.2) given by the formulas

$$\tilde{x}=x \qquad\qquad \tilde{y}=\tilde{y}(x,y). \tag{5.3}$$

For the transformed components of $\boldsymbol{\alpha}$ in (5.2) from (2.4) and (5.3) we derive

$$\begin{Vmatrix}0\\ -\tilde{A}\end{Vmatrix}=(\det T)^{-2}\begin{Vmatrix}1 & 0\\ \tilde{y}_{1.0} & \tilde{y}_{0.1}\end{Vmatrix}\cdot\begin{Vmatrix}0\\ -A\end{Vmatrix}. \tag{5.4}$$

Applying (2.3), we find that (5.4) is equivalent to $\tilde{B}=0$ and $\tilde{A}=(\tilde{y}_{0.1})^{-1}A$. It is clear that, choosing a proper function $\tilde{y}(x,y)$ in (5.3), we can reach the required equality $\tilde{A}=1$ in the transformed coordinates $\tilde{x}$ and $\tilde{y}$. $\square$

Thus, due to Theorem 5.1 proved just above we have special coordinates such that the following equalities are fulfilled in them:

$$\alpha^1 = B = 0, \qquad\qquad \alpha^2 = -A = -1. \tag{5.5}$$

Assuming that such special coordinates are chosen for x and y, we shall apply (5.5) to various formulas from previous sections.

We cannot apply (5.5) to the formulas (4.8) since B is in the denominator of the first of them. However we can apply (5.5) to (4.7). This yields

$$\beta^1 = 3\,N\,\alpha^1 = 0, \qquad\qquad \beta^2 = 3\,N\,\alpha^2 = -3\,N. \tag{5.6}$$

Taking into account (3.16), from (5.6) we derive

$$G = \Gamma^{\mathrm{Bgd}}_1 = 0, \qquad\qquad H = -\Gamma^{\mathrm{Bgd}}_0 = -3\,N. \tag{5.7}$$

On the other hand, substituting (5.6) into the formulas (3.9), we obtain

$$G = 3\,S, \qquad\qquad H = -3\,R. \tag{5.8}$$

Comparing (5.8) with (5.7), we find that in our special coordinates

$$S = 0, \qquad\qquad N = R. \tag{5.9}$$

The next step is to apply (5.5) to (4.23). As a result for the pseudoscalar field Ω we derive the following very simple formula:

$$\Omega = Q_{0.1} - 2\,R_{1.0}. \tag{5.10}$$

The pseudoscalar field M is given by the formula (4.14). Applying (5.5), (5.9), and (5.10) to this formula, we obtain the following expression for M:

$$M = -R_{0.1} - \frac{12}{5}\,R^2. \tag{5.11}$$

The components of the pseudocovectorial field $\boldsymbol{\gamma}$ are given by the formulas (4.17) and (4.18). Applying (5.5), (5.9), and (5.10) to them and using (5.11), we get

$$\gamma_1 = 3\,R_{1.0} - 2\,Q_{0.1} - \frac{6}{5}\,R\,Q, \qquad\qquad \gamma_2 = M. \tag{5.12}$$

Unlike γ_1 and γ_2 in (5.12), the quantities φ_1 and φ_2 do not represent components of a pseudotensorial field. Nevertheless, applying (5.5) to (4.10), we derive

$$\varphi_1 = \frac{3}{5}\,Q, \qquad\qquad \varphi_2 = \frac{6}{5}\,R. \tag{5.13}$$

The non-tensorial quantities (5.13) are used in (4.12) to define a connection.

Now let's return to the section 3. The formulas (3.1) and (3.2) for Bagderina's alpha quantities from [5] remain unchanged. The formulas (3.4) express the following comparison lemma coinciding with Lemma 3.2 in [8].

Lemma 5.1. *Bagderina's beta quantities* (3.3) *coincide with the components of the pseudocovectorial field* $\boldsymbol\alpha$ *of the weight* 1 *constructed in* [7].

The formulas (3.4) are affected by (5.5) in our special coordinates. They reduce to

$$\beta_1^{\,\text{Bgd}}=1,\qquad\qquad \beta_2^{\,\text{Bgd}}=0. \tag{5.14}$$

The formulas (3.3) are equivalent to (3.5). Due to (5.14) or (5.5) and due to $S=0$ in (5.9) they lead to the following differential equations:

$$\begin{aligned}
&P_{0.2}-2\,Q_{1.1}+R_{2.0}-3\,P\,R_{0.1}-\\
&\qquad -\,3\,R\,P_{0.1}-3\,Q\,R_{1.0}+6\,Q\,Q_{0.1}=1,\\
&-2\,R_{1.1}+Q_{0.2}+3\,R\,Q_{0.1}-6\,R\,R_{1.0}=0.
\end{aligned}\tag{5.15}$$

The equations (5.15) can be used in order to express higher order derivatives through lower order ones.

Bagderina's gamma quantities (3.6) become very simple in our special coordinates. They are given by the following formulas:

$$\begin{aligned}
&\gamma_{10}^{\,\text{Bgd}}=-Q, &&\gamma_{11}^{\,\text{Bgd}}=-R,\\
&\gamma_{20}^{\,\text{Bgd}}=-R, &&\gamma_{21}^{\,\text{Bgd}}=0.
\end{aligned}\tag{5.16}$$

Bagderina's gamma quantities (3.7) are described by the following comparison lemma coinciding with Lemma 3.5 in [8].

Lemma 5.2. *Bagderina's gamma quantities* (3.7) *coincide with the components of the pseudocovectorial field* $\boldsymbol\beta$ *of the weight* 3 *constructed in* [7].

Combining (5.7) and (5.8), for Bagderina's gamma quantities (3.7) in our special coordinates we derive the following formulas:

$$\Gamma_0^{\,\text{Bgd}}=3\,R,\qquad\qquad \Gamma_1^{\,\text{Bgd}}=0. \tag{5.17}$$

The formulas (5.17) are equally simple as (5.16).

Apart from (3.1), (3.3), (3.6), (3.10), (3.11), and (3.12), Yu. Yu. Bagderina uses the quantity $J_0^{\,\text{Bgd}}$ in [5] (see (3.19)). The quantity $J_0^{\,\text{Bgd}}$ in (3.19) is described by the following comparison lemma coinciding with Lemma 3.3 in [8].

Lemma 5.3. *Bagderina's quantity* $J_0^{\,\text{Bgd}}$ *from* (3.19) *is related to the pseudoscalar field* F *of the weight* 1 *constructed in* [7] *by means of the formula*

$$J_0^{\,\text{Bgd}}=-F^5. \tag{5.18}$$

In addition to (3.19) Yu. Yu. Bagderina uses four other quantities in item 2 of her classification Theorem 2 in [5]. They are given by the formulas (2.17) in [5]:

$$j_0^{\,\text{Bgd}}=\frac{3}{\beta_1^{\,\text{Bgd}}}\biggl(\frac{\beta_2^{\,\text{Bgd}}}{\beta_1^{\,\text{Bgd}}}\,\delta_{10}^{\,\text{Bgd}}-\delta_{11}^{\,\text{Bgd}}\biggr)+\frac{6\,\gamma_{10}^{\,\text{Bgd}}}{(\beta_1^{\,\text{Bgd}})^2}\biggl(\gamma_{11}^{\,\text{Bgd}}-\frac{\beta_2^{\,\text{Bgd}}}{\beta_1^{\,\text{Bgd}}}\,\gamma_{11}^{\,\text{Bgd}}\biggr). \tag{5.19}$$

The quantity (5.19) is the first of these additional quantities from (2.17) in [5]. The second one is given by the following formula:

$$j_1^{\,\mathrm{Bgd}}=\frac{5}{6}\biggl(2\,\beta_2^{\,\mathrm{Bgd}}\,\delta_{20}^{\,\mathrm{Bgd}}-\beta_1^{\,\mathrm{Bgd}}\,\delta_{30}^{\,\mathrm{Bgd}}-\frac{(\beta_2^{\,\mathrm{Bgd}})^2}{\beta_1^{\,\mathrm{Bgd}}}\,\delta_{10}^{\,\mathrm{Bgd}}\biggr)+\biggl(\gamma_{20}^{\,\mathrm{Bgd}}-\\-\frac{2}{3}\,\gamma_{11}^{\,\mathrm{Bgd}}-\frac{\beta_2^{\,\mathrm{Bgd}}}{3\,\beta_1^{\,\mathrm{Bgd}}}\,\gamma_{10}^{\,\mathrm{Bgd}}\biggr)\biggl(\gamma_{20}^{\,\mathrm{Bgd}}+\gamma_{11}^{\,\mathrm{Bgd}}-2\,\frac{\beta_2^{\,\mathrm{Bgd}}}{\beta_1^{\,\mathrm{Bgd}}}\,\gamma_{10}^{\,\mathrm{Bgd}}\biggr).\tag{5.20}$$

The rest two quantities are given by the formulas

$$\begin{aligned}j_2^{\,\mathrm{Bgd}}&=\frac{1}{\beta_1^{\,\mathrm{Bgd}}}\biggl(\delta_{20}^{\,\mathrm{Bgd}}-\frac{\beta_2^{\,\mathrm{Bgd}}}{\beta_1^{\,\mathrm{Bgd}}}\,\delta_{10}^{\,\mathrm{Bgd}}\biggr)+\\&\quad+\frac{\gamma_{10}^{\,\mathrm{Bgd}}}{5\,(\beta_1^{\,\mathrm{Bgd}})^2}\biggl(7\,\frac{\beta_2^{\,\mathrm{Bgd}}}{\beta_1^{\,\mathrm{Bgd}}}\,\gamma_{10}^{\,\mathrm{Bgd}}-6\,\gamma_{20}^{\,\mathrm{Bgd}}-\gamma_{11}^{\,\mathrm{Bgd}}\biggr),\\ j_3^{\,\mathrm{Bgd}}&=\frac{3}{5}\biggl(\frac{\delta_{10}^{\,\mathrm{Bgd}}}{(\beta_1^{\,\mathrm{Bgd}})^3}-\frac{6\,(\gamma_{10}^{\,\mathrm{Bgd}})^2}{5\,(\beta_1^{\,\mathrm{Bgd}})^4}\biggr).\end{aligned}\tag{5.21}$$

The quantity $j_0^{\,\mathrm{Bgd}}$ in (5.19) is described by the following comparison lemma.

Lemma 5.4. *If the conditions* (4.5) *are fulfilled, i. e. in the cases of intermediate degeneration, Bagderina's jay quantity* $j_0^{\,\mathrm{Bgd}}$ *from* (5.19) *behaves as a pseudoscalar field of the weight* 1*. It is related to the pseudoscalar field* Ω *introduced in* [3] *as*

$$j_0^{\,\mathrm{Bgd}}=-3\,\Omega\,.\tag{5.22}$$

Lemma 5.4 is proved by verifying the formula (5.22). This could be done directly using some symbolic algebra package. In my case that was Maple[1].

Lemma 5.5. *If the conditions* (4.5) *are fulfilled, i. e. in the cases of intermediate degeneration, Bagderina's jay quantity* $j_1^{\,\mathrm{Bgd}}$ *from* (5.20) *behaves as a pseudoscalar field of the weight* 4*. It is related to the pseudoscalar field* M *introduced in* [3] *as*

$$j_1^{\,\mathrm{Bgd}}=\frac{5}{2}\,M.\tag{5.23}$$

Lemma 5.5 is similar to Lemma 5.4. It is proved by verifying the formula (5.23) by means of direct computations.

Bagderina's quantities $j_2^{\,\mathrm{Bgd}}$ and $j_3^{\,\mathrm{Bgd}}$ in (5.21) are different. They do not behave as pseudoscalar fields. However, some definite combination of them do. On page 27 of her paper [5] Yu. Yu. Bagderina introduces the following quantity:

$$j_5^{\,\mathrm{Bgd}}=5\,\bigl(2\,j_1^{\,\mathrm{Bgd}}\,j_3^{\,\mathrm{Bgd}}+(j_2^{\,\mathrm{Bgd}}-j_0^{\,\mathrm{Bgd}}/6)^2\bigr).\tag{5.24}$$

Lemma 5.6. *If the conditions* (4.5) *are fulfilled, i. e. in the cases of intermediate degeneration, Bagderina's jay quantity* $j_5^{\,\mathrm{Bgd}}$ *from* (5.24) *behaves as a pseudoscalar field of the weight* 2*.*

[1] Maple is a trademark of Waterloo Maple Inc.

The relation of Bagderina's field j_5^{Bgd} to the fields introduced in [3] and [4] is studied below. Now we write the explicit formulas for j_2^{Bgd} and j_3^{Bgd} from (5.21) in our special coordinates introduced according to Theorem 5.1:

$$\begin{aligned} &j_2^{\text{Bgd}} = 4\,Q_{0.1} - 5\,R_{1.0} + \frac{18}{5}\,Q\,R,\\ &j_3^{\text{Bgd}} = 3\,P_{0.1} - \frac{18}{5}\,Q_{1.0} - \frac{36}{5}\,P\,R + \frac{162}{25}\,Q^2. \end{aligned} \tag{5.25}$$

The formula for j_5^{Bgd} in these special coordinates is more complicated than (5.25):

$$\begin{aligned} j_5^{\text{Bgd}} = 180\,R\,P\,R_{0.1} - 216\,Q\,R\,R_{1.0} - \bigl(180\,R_{0.0}^2 + 75\,R_{0.1}\bigr)\,P_{0.1} +\\ + \bigl(216\,R^2 + 90\,R_{0.1}\bigr)\,Q_{1.0} - \bigl(270\,R_{1.0} - 162\,Q\,R\bigr)\,Q_{0.1} -\\ - 162\,R_{0.1}\,Q^2 + 180\,R_{1.0}^2 + 432\,R^3\,P - 324\,Q^2\,R^2 + \frac{405}{4}\,Q_{0.1}^2. \end{aligned} \tag{5.26}$$

Apart from j_5^{Bgd}, on page 27 of her paper [5] Yu. Yu. Bagderina introduces the quantity j_4^{Bgd} by means of the following formula:

$$j_4^{\text{Bgd}} = \frac{\Gamma_0^{\text{Bgd}}}{\beta_1^{\text{Bgd}}}. \tag{5.27}$$

Lemma 5.7. *If the conditions* (4.5) *are fulfilled, i. e. in the cases of intermediate degeneration, under the auxiliary condition* $\beta_1^{\text{Bgd}} \neq 0$ *Bagderina's jay quantity* j_4^{Bgd} *from* (5.27) *is related to the pseudoscalar field* N *of the weight* 2 *introduced in* [3] *and effectively calculated in* [4] *by means of the formula*

$$j_4^{\text{Bgd}} = 3\,N. \tag{5.28}$$

The comparison Lemma 5.7 is immediate from Lemma 5.1 and Lemma 5.2 due to the formulas (3.4), (3.8), and (4.8).

6. The first case of intermediate degeneration and Bagderina's type two equations.

The **first case of intermediate degeneration** is determined by the conditions

$$F = 0, \qquad\qquad \boldsymbol\alpha \neq 0, \qquad\qquad M \neq 0, \tag{6.1}$$

where $F = 0$ and $\boldsymbol\alpha \neq 0$ are common for all cases **of intermediate degeneration** (see (4.5)). From $M \neq 0$, using either (4.13) or (4.14), we derive

$$N \neq 0, \tag{6.2}$$

where N is given by the formulas (4.8). The equality (4.24) can be written as

$$M = \det\begin{Vmatrix} \alpha_1 & \alpha_2 \\ \gamma_1 & \gamma_2 \end{Vmatrix}, \tag{6.3}$$

which is similar to (4.2). From $M \neq 0$ and (6.3) we derive

$$\boldsymbol{\alpha} \neq 0, \qquad\qquad \boldsymbol{\gamma} \neq 0, \qquad\qquad \boldsymbol{\alpha} \nparallel \boldsymbol{\gamma}. \tag{6.4}$$

Conversely, due to (6.3) the conditions (6.4) imply $M \neq 0$, i.e. they are equivalent to the inequality $M \neq 0$.

Bagderina's type two equations are defined in item 2 of her classification Theorem 2 in [5]. They are given by the following conditions:

$$J_0^{\text{Bgd}} \neq 0, \qquad \beta_1^{\text{Bgd}} \neq 0, \qquad j_0^{\text{Bgd}} \neq 0, \qquad \Gamma_0^{\text{Bgd}} \neq 0. \tag{6.5}$$

Applying the comparison lemmas (see Lemma 5.3, Lemma 5.1, Lemma 5.4, Lemma 5.7 and the formulas (5.18), (3.15), (5.22), (5.27), (5.28)), we can write the conditions (6.5) in terms of the fields introduced in [3, 4] and [7]:

$$F \neq 0, \qquad \boldsymbol{\alpha} \neq 0, \qquad \Omega \neq 0, \qquad N \neq 0. \tag{6.6}$$

Comparing (6.6) with (6.1), we see that the conditions do not coincide. This means that Bagderina's classification in [5] is slightly different from that of [3, 4]. For the further comparison purposes we draw the following table.

<table>
<tr><td colspan="2">R. A. Sharipov's classification
1997-1998</td><td colspan="2">Yu. Yu. Bagderina's classification
2013</td></tr>
<tr><td colspan="2">ShrGP</td><td colspan="2">BgdET1</td></tr>
<tr><td>ShrID1</td><td rowspan="7"></td><td rowspan="7"></td><td>BgdET2</td></tr>
<tr><td>ShrID2</td><td>BgdET3</td></tr>
<tr><td>ShrID3</td><td>BgdET4</td></tr>
<tr><td>ShrID4</td><td>BgdET5</td></tr>
<tr><td>ShrID5</td><td>BgdET6</td></tr>
<tr><td>ShrID6</td><td>BgdET7</td></tr>
<tr><td>ShrID7</td><td>BgdET8</td></tr>
<tr><td colspan="2">ShrMD</td><td colspan="2">BgET9</td></tr>
</table>

The abbreviations in the above table read as follows:

– ShrGP stands for Sharipov's case of general position;
– ShrMD stands for Sharipov's case of maximal degeneration;
– ShrID1 stands for Sharipov's case of intermediate degeneration 1;
. . . .
– ShrID1 stands for Sharipov's case of intermediate degeneration 7;
– BgdET1 stands for Bagderina's equations of type 1;
. . . .
– BgdET9 stands for Bagderina's equations of type 9.

Again looking at (6.6) and (6.1), we see that generally speaking the equation classes ShrID1 and BgdET2 do not coincide, but they have a substantial overlap. Their overlap is described by the following conditions:

$$F \neq 0, \qquad \boldsymbol{\alpha} \neq 0, \qquad M \neq 0, \qquad \Omega \neq 0. \tag{6.7}$$

Indeed, $M \neq 0$ in (6.1) implies $N \neq 0$ in (6.6) (see (6.2)). However $M \neq 0$ in (6.1) does not imply $\Omega \neq 0$ in (6.6), unless some deeper mutual relations of these field will be discovered. Conversely, $N \neq 0$ and $\Omega \neq 0$ in (6.6) do not imply $M \neq 0$ in (6.1). Below we shall study the intersection class ShrID1 $\cap$ BgdET2.

Let's consider Bagderina's invariant differentiation operator $\mathcal{D}_1^{\text{Bgd}}$. It is given by the first formula (2.8) from Bagderina's classification Theorem 2 in [5]:

$$\mathcal{D}_1^{\text{Bgd}} = \frac{\beta_2^{\text{Bgd}}}{(\mu_1^{\text{Bgd}})^2}\,\frac{\partial}{\partial x} - \frac{\beta_1^{\text{Bgd}}}{(\mu_1^{\text{Bgd}})^2}\,\frac{\partial}{\partial y}. \tag{6.8}$$

Taking into account (3.15), (5.22) and Bagderina's formula $\mu_1^{\text{Bgd}} = j_0^{\text{Bgd}}$ from (2.10) in item 2 of Theorem 2 in [5] (which is different from (3.21)), we write (6.8) as

$$\mathcal{D}_1^{\text{Bgd}} = \frac{\alpha^1}{(3\,\Omega)^2}\,\frac{\partial}{\partial x} + \frac{\alpha^2}{(3\,\Omega)^2}\,\frac{\partial}{\partial y}. \tag{6.9}$$

Invariant differentiation operators were not considered in [3] and [4] for the first case of intermediate degeneration ShrID1. Instead of them covariant differentiation operators along pseudovectorial fields were considered. In particular we have

$$\nabla_{\boldsymbol{\alpha}} = \alpha^1\,\nabla_1 + \alpha^2\,\nabla_2, \qquad \nabla_{\boldsymbol{\gamma}} = \gamma^1\,\nabla_1 + \gamma^2\,\nabla_2 \tag{6.10}$$

(see (6.13) in [3] and (5.2) in [4]). The covariant derivatives in (6.10) extend partial derivatives from (6.9). They are defined by means of the formula

$$\begin{aligned}\nabla_k F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} &= \frac{\partial F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^k} + \sum_{n=1}^{r}\sum_{v_n=1}^{2} \Gamma^{i_n}_{k\,v_n}\,F^{i_1\ldots\,v_n\ldots\,i_r}_{j_1\ldots\,j_s} - \\ &- \sum_{n=1}^{s}\sum_{w_n=1}^{2} \Gamma^{w_n}_{k\,j_n}\,F^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s} + m\,\varphi_k\,F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\end{aligned} \tag{6.11}$$

(see (6.11) in [3] or (4.23) in [4]). The covariant derivative ∇_k in (6.11) is applied to a pseudotensorial field of the type (r,s) and the weight m. The connection components Γ^k_{ij} in (6.11) are defined by (4.12). They are canonically associated with a given equation equation (1.1).

Due to the covariant derivatives ∇_1 and ∇_2 in (6.10) the differential operators (6.10) are applicable not only to scalar invariants, but to any tensorial and pseudotensorial invariants as well. Yu. Yu. Bagderina's operator (6.9) can also be extended in this manner using covariant derivatives ∇_1 and ∇_2:

$$\mathcal{D}_1^{\text{Bgd}} = \frac{\alpha^1}{(3\,\Omega)^2}\,\nabla_1 + \frac{\alpha^2}{(3\,\Omega)^2}\,\nabla_2. \tag{6.12}$$

Lemma 6.1. *Within the intersection class* ShrID1 ∩ BgdET2*, i. e. if the conditions* (6.7) *are fulfilled, Bagderina's invariant differentiation operator* $\mathcal{D}_1^{\text{Bgd}}$ *from* (6.8) *extended in* (6.12) *is related to the differentiation operator* $\nabla_{\boldsymbol{\alpha}}$ *from* [3] *as*

$$\mathcal{D}_1^{\text{Bgd}} = \frac{1}{(3\,\Omega)^2}\,\nabla_{\boldsymbol{\alpha}}. \tag{6.13}$$

Lemma 6.1 and the formula (6.13) are immediate from (6.9) and (6.10). So we can proceed to the second invariant differentiation operator by Yu. Yu. Bagderina. It is given by the second formula (2.8) in Bagderina's Theorem 2 in [5]:

$$\mathcal{D}_2^{\text{Bgd}} = \left(\mu_2^{\text{Bgd}}\,\beta_2^{\text{Bgd}} - 3\,\frac{\mu_2^{\text{Bgd}}}{\beta_1^{\text{Bgd}}}\right)\frac{\partial}{\partial x} + \beta_1^{\text{Bgd}}\,\frac{\partial}{\partial y}. \tag{6.14}$$

The quantity μ_2^{Bgd} in (6.14) is given by one of the formulas (2.10) in item 2 of Bagderina's classification Theorem 2 in [5]:

$$\mu_2^{\text{Bgd}} = \frac{3\,\beta_1^{\text{Bgd}}\,e_1^{\text{Bgd}}}{\Gamma_0^{\text{Bgd}}}. \tag{6.15}$$

The quantity e_1^{Bgd} in (6.15) is expressed by one of the formulas (2.18) in [5]:

$$\begin{gathered} e_1^{\text{Bgd}} = \frac{5}{(\beta_1^{\text{Bgd}})^2}\left(\frac{\beta_2^{\text{Bgd}}}{\beta_1^{\text{Bgd}}}\,\epsilon_{10}^{\text{Bgd}} - \epsilon_{11}^{\text{Bgd}}\right) + \\ + \frac{15}{(\beta_1^{\text{Bgd}})^3}\left(\gamma_{11}^{\text{Bgd}} - \frac{\beta_2^{\text{Bgd}}}{\beta_1^{\text{Bgd}}}\,\gamma_{10}^{\text{Bgd}}\right) - \frac{6\,\gamma_{10}^{\text{Bgd}}}{(\beta_1^{\text{Bgd}})^2}\,j_0^{\text{Bgd}}. \end{gathered} \tag{6.16}$$

The notations used in (6.16) are given above in (3.3), (3.6), (3.10), (3.11) and in (5.19). In order to transform (6.14) we use our special coordinates introduced according to Theorem 5.1. Upon replacing partial derivatives in (6.14) by covariant derivatives ∇_1 and ∇_2 in these special coordinates we get

$$\begin{gathered} \mathcal{D}_2^{\text{Bgd}} = \bigl(9\,Q_{0.1} - 18\,R_{1.0}\bigr)\,\nabla_1 + \left(\frac{10\,P_{0.2}}{R} - 30\,P_{0.1} - \right. \\ \left. - \frac{15\,Q_{1.1}}{R} - \frac{36\,Q\,R_{1.0}}{R} + \frac{63\,Q\,Q_{0.1}}{R} - \frac{30\,P\,R_{0.1}}{R} - \frac{60}{R}\right)\nabla_2. \end{gathered} \tag{6.17}$$

In order to reveal the invariant nature of the operator (6.17) we calculate the covariant derivatives of the pseudoscalar field Ω in our special coordinates:

$$\begin{aligned} &\nabla_1\Omega = 2\,P_{0.2} - 3\,Q_{1.1} - 6\,R\,P_{0.1} - \\ &\qquad\qquad - \frac{36\,Q\,R_{1.0}}{5} + \frac{63\,Q\,Q_{0.1}}{5} - 6\,P\,R_{0.1} - 2, \\ &\nabla_2\Omega = \frac{18\,R\,R_{1.0}}{5} - \frac{9\,R\,Q_{0.1}}{5}. \end{aligned} \tag{6.18}$$

The quantities (6.18) are components of the pseudocovectorial field $\nabla\Omega$ of the weight 1. In order to apply them to (6.17) we need to raise their indices:

$$\nabla^i\Omega = \sum_{k=1}^{2} d^{ik}\,\nabla_k\Omega\,. \tag{6.19}$$

The quantities (6.19) are components of the pseudovectorial field $\nabla\Omega$ of the weight 2. Due to (3.13) the formula (6.19) simplifies to

$$\nabla^1\Omega = \nabla_2\Omega, \qquad\qquad \nabla^2\Omega = -\nabla_1\Omega\,. \tag{6.20}$$

Taking into account (5.5) and (6.20), then comparing (6.18) with (6.17), we get

$$\mathcal{D}_2^{\,\mathrm{Bgd}} = \left(\frac{50\,\alpha^1}{N} - \frac{5\,\nabla^1\Omega}{N}\right)\nabla_1 + \left(\frac{50\,\alpha^2}{N} - \frac{5\,\nabla^2\Omega}{N}\right)\nabla_2. \tag{6.21}$$

The denominator R in (6.17) is replaced by the denominator N in (6.21) since $N = R$ in our special coordinates (see (5.9)).

Note that (6.21) is a proper tensorial formula. Therefore, being derived in our special coordinates, it remains valid in arbitrary coordinates.

Any vectorial and/or pseudovectorial field on the plane $\mathbb{R}^2$ or in a two-dimensional manifold can be expressed as a linear combination of any other two non-parallel vectorial and/or pseudovectorial field. In our case ShrID1 $\cap$ BgdET2 this means that $\nabla\Omega$ is expressed through $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ since $\boldsymbol{\alpha} \nparallel \boldsymbol{\gamma}$ due to $M \neq 0$ in (6.7). This expression for $\nabla\Omega$ can be written explicitly:

$$\nabla\Omega = \frac{\nabla_{\boldsymbol{\gamma}}\Omega}{M}\,\boldsymbol{\alpha} - \frac{\nabla_{\boldsymbol{\alpha}}\Omega}{M}\,\boldsymbol{\gamma}. \tag{6.22}$$

Now, applying (6.22) to (6.21), we derive the following formula:

$$\mathcal{D}_2^{\,\mathrm{Bgd}} = \left(\frac{50}{N} - \frac{5\,\nabla_{\boldsymbol{\gamma}}\Omega}{M\,N}\right)\nabla_{\boldsymbol{\alpha}} + \left(\frac{5\,\nabla_{\boldsymbol{\alpha}}\Omega}{M\,N}\right)\nabla_{\boldsymbol{\gamma}}. \tag{6.23}$$

Lemma 6.2. *Within the intersection class* ShrID1 $\cap$ BgdET2*, i. e. if the conditions* (6.7) *are fulfilled, Bagderina's invariant differentiation operator* $\mathcal{D}_2^{\,\mathrm{Bgd}}$ *in* (6.14) *is related to the covariant differentiation operators* $\nabla_{\boldsymbol{\alpha}}$ *and* $\nabla_{\boldsymbol{\gamma}}$ *introduced in* [3] *according to the formula* (6.23)*.*

7. Curvature tensor and additional fields.

Let's return to the field Ω associated with the curvature tensor (4.20) of the connection (4.20). Following the receipt of [3] we write

$$R^k_{qij} = R^k_q\,d_{ij}, \ \text{ where } \ R^k_q = \frac{1}{2}\sum_{i=1}^{2}\sum_{j=1}^{2} R^k_{qij}\,d^{ij} \tag{7.1}$$

(see (7.1) and (7.2) in [3]). The quantities R^k_q in (7.1) are components of a pseudotensorial field of the type $(1,1)$ and the the weight 1. This field has two pseudoscalar invariants — its trace and its determinant. The trace of this field reduces to the pseudoscalar field Ω according to the formula:

$$\operatorname{tr}(R)=\frac{3}{5}\,\Omega. \tag{7.2}$$

Its determinant is a new field. In the framework of the **second case of intermediate degenerations** (see [4]), i. e. if $M=0$, this field can be expressed through the field Λ given by the formulas (6.10) in [4]:

$$\det(R)=-\frac{9}{25}\,\Lambda\,(\Omega+\Lambda). \tag{7.3}$$

In the present paper we deal with the case $M\neq 0$. Therefore we shall not use the formulas (7.2) and (7.3) and we shall treat $\det(R)$ as an separate pseudoscalar field of the weight 2. This field can be easily calculated in arbitrary coordinates using the formulas (4.12), (4.20), and (7.1). However, we choose our special coordinates introduced through Theorem 5.1. In these coordinates we have

$$\begin{aligned}\det(R)=-\frac{36}{35}\,R\,P\,R_{0.1}+\frac{216}{125}\,Q\,R\,R_{1.0}+\left(\frac{36}{25}\,R_{0.0}^{2}+\frac{3}{5}\,R_{0.1}\right)P_{0.1}\,-\\-\left(\frac{216}{125}\,R^{2}+\frac{18}{25}\,R_{0.1}\right)Q_{1.0}+\left(\frac{9}{5}\,R_{1.0}-\frac{162}{125}\,Q\,R\right)Q_{0.1}\,+\\+\frac{162}{125}\,R_{0.1}\,Q^{2}-\frac{27}{25}\,R_{1.0}^{2}-\frac{432}{125}\,R^{3}\,P+\frac{324}{125}\,Q^{2}\,R^{2}-\frac{18}{25}\,Q_{0.1}^{2}.\end{aligned} \tag{7.4}$$

Comparing (7.4) with (5.26), we derive the following formula:

$$j_5^{\,\mathrm{Bgd}}=-125\,\det(R)+\frac{45}{4}\,\Omega^{2}. \tag{7.5}$$

This formula (7.5) proves Lemma 5.6. It expresses Bagderina's quantity $j_5^{\,\mathrm{Bgd}}$ through the pseudotensorial field R in (7.1) previously known in [4].

Our further efforts are toward expressing $\det(R)$ through M, N, Ω, $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and their proper tensorial derivatives. For this purpose we need a little bit of theory.

For a while assume that $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are arbitrary two pseudovectorial fields with the weights m and n respectively. Let $\mathbf{X}$ be a third pseudovectorial field with the weight k. Then we have the following identities:

$$[\nabla_{\boldsymbol{\alpha}},\nabla_{\boldsymbol{\gamma}}]\mathbf{X}-\nabla_{[\boldsymbol{\alpha},\boldsymbol{\gamma}]}\mathbf{X}=\mathbf{R}(\boldsymbol{\alpha},\boldsymbol{\gamma})\mathbf{X}-k\;\omega(\boldsymbol{\alpha},\boldsymbol{\gamma})\;\mathbf{X}, \tag{7.6}$$

$$\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma}-\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha}=[\boldsymbol{\alpha},\boldsymbol{\gamma}]+\mathbf{T}(\boldsymbol{\alpha},\boldsymbol{\gamma})\mathbf{X}. \tag{7.7}$$

Here $\mathbf{R}(\boldsymbol{\alpha},\boldsymbol{\gamma})$ and $\mathbf{T}(\boldsymbol{\alpha},\boldsymbol{\gamma})$ are the curvature operator and the torsion operator respectively (see [11]). The term $k\,\omega(\boldsymbol{\alpha},\boldsymbol{\gamma})\,\mathbf{X}$ is determined by the skew symmetric form ω whose components are given in (4.19). The connection components (4.12) are symmetric. Therefore we have no torsion here:

$$\mathbf{T}(\boldsymbol{\alpha},\boldsymbol{\gamma})=0. \tag{7.8}$$

The formulas (7.6) and (7.7) are well known in differential geometry, though they are usually applied to vectorial fields rather then to pseudovectorial ones.

Their application to pseudovectorial fields have some features. In particular, the covariant derivatives (6.11) and the commutator of pseudovectorial fields requires some auxiliary quantities φ_i obeying the transformation rules (4.11):

$$[\boldsymbol{\alpha},\boldsymbol{\gamma}]=\sum_{i=1}^{2}\left(\sum_{s=1}^{2}\alpha^s\,\frac{\partial\gamma^i}{\partial x^s}-\gamma^s\,\frac{\partial\alpha^i}{\partial x^s}+n\,\alpha^s\,\varphi_s\,\gamma^i-m\,\gamma^s\,\varphi_s\,\alpha^i\right)\frac{\partial}{\partial x^i}. \tag{7.9}$$

The formula (7.9) can be treated as a definition of commutator in the case of pseudotensorial fields.

Returning to our previously defined pseudovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ we should remind that their weights are 2 and 3 respectively. Note that they were originally defined as pseudocovectorial fields of the weights 1 and 2. But having the skew-symmetric metric pseudotensors (3.13), we can always raise and lower indices of any pseudotensorial field. We should also note that

$$\nabla\mathbf{d}=0 \tag{7.10}$$

for both metric pseudotensors with the components (3.13). Due to (7.10) the operations of raising and lowering indices commute with covariant differentiations.

Now let's calculate the curvature operator $\mathbf{R}(\boldsymbol{\alpha},\boldsymbol{\gamma})$ applied to some pseudovectorial field $\mathbf{X}$ taking into account the special structure of R^k_{qij} in (7.1):

$$\mathbf{R}(\boldsymbol{\alpha},\boldsymbol{\gamma})\mathbf{X}=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{s=1}^{2}\sum_{q=1}^{2}R^s_q\,d_{ij}\,\alpha^i\,\gamma^j\,X^q\,\frac{\partial}{\partial x^s}. \tag{7.11}$$

Here R^s_q are the components of that very matrix whose determinant is applied in (7.5). Taking into account (4.24), we write (7.11) as

$$\mathbf{R}(\boldsymbol{\alpha},\boldsymbol{\gamma})\mathbf{X}=\sum_{s=1}^{2}\sum_{q=1}^{2}M\,R^s_q\,X^q\,\frac{\partial}{\partial x^s}=M\,R(\mathbf{X}). \tag{7.12}$$

Here R is the linear operator whose matrix is formed by R^s_q.

At this moment we can apply (7.6) to (7.12). As a result we get

$$R(\mathbf{X})=\frac{1}{M}\Bigl([\nabla_{\boldsymbol{\alpha}},\nabla_{\boldsymbol{\gamma}}]\mathbf{X}-\nabla_{[\boldsymbol{\alpha},\boldsymbol{\gamma}]}\mathbf{X}+k\;\omega(\boldsymbol{\alpha},\boldsymbol{\gamma})\;\mathbf{X}\Bigr). \tag{7.13}$$

Let's recall the formula (4.21). This formula combined with (7.1) and (4.24) yields

$$\omega(\boldsymbol{\alpha},\boldsymbol{\gamma})=\operatorname{tr}(R)\,\sum_{i=1}^{2}\sum_{j=1}^{2}d_{ij}\,\alpha^i\,\gamma^j=\operatorname{tr}(R)\,M. \tag{7.14}$$

Substituting (7.14) into (7.13), we derive

$$R(\mathbf{X})=\frac{1}{M}\Bigl([\nabla_{\boldsymbol{\alpha}},\nabla_{\boldsymbol{\gamma}}]\mathbf{X}-\nabla_{[\boldsymbol{\alpha},\boldsymbol{\gamma}]}\mathbf{X}\Bigr)+k\;\operatorname{tr}(R)\;\mathbf{X}. \tag{7.15}$$

The commutator $[\boldsymbol{\alpha},\boldsymbol{\gamma}]$ in (7.15) can be calculated with the use of (7.7) and (7.8):

$$[\boldsymbol{\alpha},\boldsymbol{\gamma}]=\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma}-\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha}. \tag{7.16}$$

The covariant derivatives $\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma}$ and $\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha}$ are that very derivatives used in [3, 4]:

$$\begin{aligned} &\nabla_{\boldsymbol{\alpha}}\boldsymbol{\alpha} = \Gamma^1_{11}\,\boldsymbol{\alpha} + \Gamma^2_{11}\,\boldsymbol{\gamma}, && \nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma} = \Gamma^1_{12}\,\boldsymbol{\alpha} + \Gamma^2_{12}\,\boldsymbol{\gamma},\\ &\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha} = \Gamma^1_{21}\,\boldsymbol{\alpha} + \Gamma^2_{21}\,\boldsymbol{\gamma}, && \nabla_{\boldsymbol{\gamma}}\boldsymbol{\gamma} = \Gamma^1_{22}\,\boldsymbol{\alpha} + \Gamma^2_{22}\,\boldsymbol{\gamma} \end{aligned} \tag{7.17}$$

(see (6.13) in [3] or (5.2) in [4]). The coefficients Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} are pseudoscalar fields uniquely determined by $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ since $\boldsymbol{\alpha} \nparallel \boldsymbol{\gamma}$ due to $M \neq 0$ in (6.7). Applying (7.17) to (7.16), we get

$$[\boldsymbol{\alpha},\boldsymbol{\gamma}] = (\Gamma^1_{12} - \Gamma^1_{21})\,\boldsymbol{\alpha} + (\Gamma^2_{12} - \Gamma^2_{21})\,\boldsymbol{\gamma} \tag{7.18}$$

Then we use (7.18) in order to calculate $\nabla_{[\boldsymbol{\alpha},\boldsymbol{\gamma}]}\mathbf{X}$ in (7.15):

$$\nabla_{[\boldsymbol{\alpha},\boldsymbol{\gamma}]}\mathbf{X} = (\Gamma^1_{12} - \Gamma^1_{21})\,\nabla_{\boldsymbol{\alpha}}\mathbf{X} + (\Gamma^2_{12} - \Gamma^2_{21})\,\nabla_{\boldsymbol{\gamma}}\mathbf{X}. \tag{7.19}$$

The first term $[\nabla_{\boldsymbol{\alpha}}\nabla_{\boldsymbol{\gamma}}]\mathbf{X}$ in (7.15) is expanded as follows:

$$[\nabla_{\boldsymbol{\alpha}},\nabla_{\boldsymbol{\gamma}}]\mathbf{X} = \nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\mathbf{X}\bigr) - \nabla_{\boldsymbol{\gamma}}\bigl(\nabla_{\boldsymbol{\alpha}}\mathbf{X}\bigr). \tag{7.20}$$

Taking into account (7.19) and (7.20), the formula (7.15) turns to

$$\begin{gathered} R(\mathbf{X}) = \frac{\nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\mathbf{X}\bigr) - \nabla_{\boldsymbol{\gamma}}\bigl(\nabla_{\boldsymbol{\alpha}}\mathbf{X}\bigr)}{M} - \\ - \frac{(\Gamma^1_{12} - \Gamma^1_{21})}{M}\,\nabla_{\boldsymbol{\alpha}}\mathbf{X} - \frac{(\Gamma^2_{12} - \Gamma^2_{21})}{M}\,\nabla_{\boldsymbol{\gamma}}\mathbf{X} + k\ \operatorname{tr}(R)\ \mathbf{X}. \end{gathered} \tag{7.21}$$

Now, using the formula (7.21), we can calculate two pseudovectoial fields $R(\boldsymbol{\alpha})$ and $R(\boldsymbol{\gamma})$, expressing them back through $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ as linear combinations:

$$R(\boldsymbol{\alpha}) = F^1_1\,\boldsymbol{\alpha} + F^2_1\,\boldsymbol{\gamma}, \qquad\qquad R(\boldsymbol{\gamma}) = F^1_2\,\boldsymbol{\alpha} + F^2_2\,\boldsymbol{\gamma}. \tag{7.22}$$

Indeed, for $R(\boldsymbol{\alpha})$ in (7.22) we have the following expression (with $k = 2$):

$$\begin{gathered} R(\boldsymbol{\alpha}) = \frac{\nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha}\bigr) - \nabla_{\boldsymbol{\gamma}}\bigl(\nabla_{\boldsymbol{\alpha}}\boldsymbol{\alpha}\bigr)}{M} - \\ - \frac{(\Gamma^1_{12} - \Gamma^1_{21})}{M}\,\nabla_{\boldsymbol{\alpha}}\boldsymbol{\alpha} - \frac{(\Gamma^2_{12} - \Gamma^2_{21})}{M}\,\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha} + k\ \operatorname{tr}(R)\ \boldsymbol{\alpha}. \end{gathered} \tag{7.23}$$

The derivatives $\nabla_{\boldsymbol{\alpha}}\boldsymbol{\alpha}$ and $\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha}$ are taken from (7.17). The second order derivatives $\nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha}\bigr)$ and $\nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha}\bigr)$ are transformed as follows:

$$\begin{gathered} \nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha}\bigr) = \nabla_{\boldsymbol{\alpha}}\bigl(\Gamma^1_{21}\,\boldsymbol{\alpha} + \Gamma^2_{21}\,\boldsymbol{\gamma}\bigr) = \nabla_{\boldsymbol{\alpha}}\bigl(\Gamma^1_{21}\bigr)\,\boldsymbol{\alpha} + \\ + \Gamma^1_{21}\,\bigl(\Gamma^1_{11}\,\boldsymbol{\alpha} + \Gamma^2_{11}\,\boldsymbol{\gamma}\bigr) + \nabla_{\boldsymbol{\alpha}}\bigl(\Gamma^2_{21}\bigr)\,\boldsymbol{\gamma} + \Gamma^2_{21}\,\bigl(\Gamma^1_{12}\,\boldsymbol{\alpha} + \Gamma^2_{12}\,\boldsymbol{\gamma}\bigr), \end{gathered} \tag{7.24}$$

$$\begin{gathered} \nabla_{\boldsymbol{\gamma}}\bigl(\nabla_{\boldsymbol{\alpha}}\boldsymbol{\alpha}\bigr) = \nabla_{\boldsymbol{\gamma}}\bigl(\Gamma^1_{11}\,\boldsymbol{\alpha} + \Gamma^2_{11}\,\boldsymbol{\gamma}\bigr) = \nabla_{\boldsymbol{\gamma}}\bigl(\Gamma^1_{11}\bigr)\ \boldsymbol{\alpha} + \\ + \Gamma^1_{11}\,\bigl(\Gamma^1_{21}\,\boldsymbol{\alpha} + \Gamma^2_{21}\,\boldsymbol{\gamma}\bigr) + \nabla_{\boldsymbol{\gamma}}\bigl(\Gamma^2_{11}\bigr)\,\boldsymbol{\gamma} + \Gamma^2_{11}\,\bigl(\Gamma^1_{22}\,\boldsymbol{\alpha} + \Gamma^2_{22}\,\boldsymbol{\gamma}\bigr). \end{gathered} \tag{7.25}$$

The pseudovectorial field $R(\boldsymbol{\gamma})$ in (7.22) is treated similarly. From (7.21) we derive the following expression for this field (with $k=3$):

$$\begin{gathered} R(\boldsymbol{\gamma})=\frac{\nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\boldsymbol{\gamma}\bigr)-\nabla_{\boldsymbol{\gamma}}\bigl(\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma}\bigr)}{M}-\\ -\frac{(\Gamma^1_{12}-\Gamma^1_{21})}{M}\,\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma}-\frac{(\Gamma^2_{12}-\Gamma^2_{21})}{M}\,\nabla_{\boldsymbol{\gamma}}\boldsymbol{\gamma}+k\ \operatorname{tr}(R)\ \boldsymbol{\gamma}. \end{gathered}\tag{7.26}$$

The derivatives $\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma}$ and $\nabla_{\boldsymbol{\gamma}}\boldsymbol{\gamma}$ are taken from (7.17). The second order derivatives $\nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\boldsymbol{\gamma}\bigr)$ and $\nabla_{\boldsymbol{\gamma}}\bigl(\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma}\bigr)$ are transformed as follows:

$$\begin{gathered} \nabla_{\boldsymbol{\alpha}}\bigl(\nabla_{\boldsymbol{\gamma}}\boldsymbol{\gamma}\bigr)=\nabla_{\boldsymbol{\alpha}}\bigl(\Gamma^1_{22}\,\boldsymbol{\alpha}+\Gamma^2_{22}\,\boldsymbol{\gamma}\bigr)=\nabla_{\boldsymbol{\alpha}}\bigl(\Gamma^1_{22}\bigr)\,\boldsymbol{\alpha}+\\ +\Gamma^1_{22}\,\bigl(\Gamma^1_{11}\,\boldsymbol{\alpha}+\Gamma^2_{11}\,\boldsymbol{\gamma}\bigr)+\nabla_{\boldsymbol{\alpha}}\bigl(\Gamma^2_{22}\bigr)\,\boldsymbol{\gamma}+\Gamma^2_{22}\,\bigl(\Gamma^1_{12}\,\boldsymbol{\alpha}+\Gamma^2_{12}\,\boldsymbol{\gamma}\bigr), \end{gathered}\tag{7.27}$$

$$\begin{gathered} \nabla_{\boldsymbol{\gamma}}\bigl(\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma}\bigr)=\nabla_{\boldsymbol{\gamma}}\bigl(\Gamma^1_{12}\,\boldsymbol{\alpha}+\Gamma^2_{12}\,\boldsymbol{\gamma}\bigr)=\nabla_{\boldsymbol{\gamma}}\bigl(\Gamma^1_{12}\bigr)\ \boldsymbol{\alpha}+\\ +\Gamma^1_{12}\,\bigl(\Gamma^1_{21}\,\boldsymbol{\alpha}+\Gamma^2_{21}\,\boldsymbol{\gamma}\bigr)+\nabla_{\boldsymbol{\gamma}}\bigl(\Gamma^2_{12}\bigr)\,\boldsymbol{\gamma}+\Gamma^2_{12}\,\bigl(\Gamma^1_{22}\,\boldsymbol{\alpha}+\Gamma^2_{22}\,\boldsymbol{\gamma}\bigr). \end{gathered}\tag{7.28}$$

Summarizing the formulas (7.23), (7.24), (7.25), (7.26), (7.27), and (7.28), we can formulate the following lemma.

Lemma 7.1. *The coefficients F^1_1, F^2_1, F^1_2, F^2_2 of the linear combinations (7.22) are expressed through M, through the pseudoscalar fields Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} from (7.17), and through covariant derivatives of them.*

Note that $\boldsymbol{\alpha}\nparallel\boldsymbol{\gamma}$ due to $M\neq 0$ in (6.7). Hence the values of the fields $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ constitute a basis at each point of $\mathbb{R}^2$ or a two-dimensional manifold. In this case the coefficients F^1_1, F^2_1, F^1_2, F^2_2 of the linear combinations (7.22) constitute the matrix of the linear operator R in (7.22). It is well-known (see [12]) that the determinant of a linear operator does not depend on a basis where its matrix is calculated. Therefore we have the following equality:

$$\det(R)=\det\begin{Vmatrix} R^1_1 & R^1_2\\ R^2_1 & R^2_2\end{Vmatrix}=\det\begin{Vmatrix} F^1_1 & F^1_2\\ F^2_1 & F^2_2\end{Vmatrix}.\tag{7.29}$$

Applying (7.29) along with Lemma 7.1 to the formula (7.5), we derive a theorem.

Theorem 7.1. *Within the intersection class* ShrID1 ∩ BgdET2*, i. e. if the conditions (6.7) are fulfilled, Bagderina's pseudoscalar field j^{Bgd}_5 from (5.24) is expressed through M, through Ω, through the pseudoscalar fields Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} from (7.17), and through covariant derivatives of them along the pseudovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$.*

8. Comparison of invariants.

Two scalar invariants in the first case of intermediate degeneration are very simple. They are given as two ratios of the pseudoscalar fields M, N and Ω:

$$I_1=\frac{M}{N^2},\qquad\qquad I_2=\frac{\Omega^2}{N}\tag{8.1}$$

(see (6.8) and (6.19) in [3] or (5.1) in [4]). Other scalar invariants are more complicated. The formulas (7.17) and the coefficients Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} therein are used for introducing them (see (6.13) in [3] or (5.2) in [4]). Some of the coefficients in (7.17) are identically zero:

$$\Gamma^2_{11}=0, \qquad\qquad \Gamma^1_{21}=0. \tag{8.2}$$

Some others are expressed through the pseudoscalar field N:

$$\Gamma^1_{11}=-\frac{3}{5}\,N, \qquad\qquad \Gamma^2_{21}=-\frac{3}{5}\,N. \tag{8.3}$$

And some of them are bound to the invariants I_1 and I_2 in (8.1) and to the pseudoscalar field N by more complicated relationships

$$I_1\,\Gamma^2_{12}=I_4\,N+\frac{3}{5}\,I_1\,N+2\,I_1^2\,N, \tag{8.4}$$

$$\begin{gathered}\bigl(I_1\,\Gamma^2_{22}\bigr)^4+\bigl(I_7\,N^3\bigr)^2+\bigl(16\,I_2\,N^3\,{I_1}^4\bigr)^2=\\ =32\,I_7\,N^6\,I_2\,{I_1}^4+2\,\bigl(I_7\,N^3+16\,I_2\,N^3\,{I_1}^4\bigr)\,\bigl(I_1\,\Gamma^2_{22}\bigr)^2\end{gathered} \tag{8.5}$$

(see (6.22) and (6.23) in [3] or (5.6) and (5.7) in [4]). Unfortunately the formula (6.22) in [3] is mistyped and then copied to (5.6) in [4]. The minus signs in the right hand side of this formula should be altered for pluses. Here we present the correct formula (8.4). As for the quantities I_4 and I_7 in (8.5), they are higher order invariants defined among others by the formulas

$$\begin{aligned}&I_4=\frac{\nabla_{\boldsymbol\alpha}I_1}{N}, &&I_5=\frac{\nabla_{\boldsymbol\alpha}I_2}{N}, &&I_6=\frac{\nabla_{\boldsymbol\alpha}I_3}{N},\\ &I_7=\frac{(\nabla_{\boldsymbol\gamma}I_1)^2}{N^3}, &&I_8=\frac{(\nabla_{\boldsymbol\gamma}I_2)^2}{N^3}, &&I_9=\frac{(\nabla_{\boldsymbol\gamma}I_3)^2}{N^3}.\end{aligned} \tag{8.6}$$

Apart from (8.2), (8.3), (8.4), (8.5), we have the relationship

$$\Gamma^1_{12}=-\Gamma^2_{22}. \tag{8.7}$$

(see (6.15) in [3]). Due to (8.2), (8.3), (8.4), (8.5), and (8.7) the only nontrivial coefficient in (7.17) is Γ^1_{22}. It is used in order to produce the invariant I_3 in (8.6):

$$I_3=\frac{\Gamma^1_{22}\,N^2}{M^2} \tag{8.8}$$

(see (6.20) in [3]). The formula (5.5) for I_3 is mistyped. The exponent of M in the denominator is dropped. Here we present the correct formula (8.8).

In item 2 of her classification Theorem 2 Yu. Yu. Bagderina presents two basic invariants. They are given by means of the formulas

$$I_1^{\,\mathrm{Bgd}}=\frac{\Gamma_0^{\,\mathrm{Bgd}}}{\beta_1^{\,\mathrm{Bgd}}\,(j_0^{\,\mathrm{Bgd}})^2}, \tag{8.9}$$

$$I_2^{\,\mathrm{Bgd}}=\frac{5}{(j_0^{\,\mathrm{Bgd}})^2}\bigl(2\,j_1^{\,\mathrm{Bgd}}\,j_3^{\,\mathrm{Bgd}}+(j_2^{\,\mathrm{Bgd}}-j_0^{\,\mathrm{Bgd}}/6)^2\bigr). \tag{8.10}$$

The first Bagderina's invariant is simple. Applying (5.27), (5.28), (5.22), we find

$$I_1^{\text{Bgd}}=\frac{N}{3\,\Omega^2}. \tag{8.11}$$

Lemma 8.1. *Within the intersection class* ShrID1 ∩ BgdET2, *i. e. if the conditions* (6.7) *are fulfilled, the first Bagderina's invariant* I_1^{Bgd} *in* (8.9) *is related to the invariant* I_2 *introduced in* [3] *according to the formula*

$$I_1^{\text{Bgd}}=\frac{1}{3\,I_2}. \tag{8.12}$$

Lemma 8.1 and the formula (8.12) in it are immediate from (8.1) and (8.11).

The second Bagderina's invariant I_2^{Bgd} in (8.10) is more complicated. Applying (5.24) and (5.22) to it, we can write the formula (8.10) as follows:

$$I_2^{\text{Bgd}}=\frac{j_5^{\text{Bgd}}}{(j_0^{\text{Bgd}})^2}=\frac{j_5^{\text{Bgd}}}{9\,\Omega^2}. \tag{8.13}$$

The numerator in (8.13) is described by Theorem 7.1. At this moment we know that the pseudoscalar fields Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} from (7.17) are expressed through scalar invariants I_1, I_2, I_3, I_4, I_7 and the field N. Their covariant derivatives along $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are expressed through the covariant derivatives of I_1, I_2, I_3, I_4, I_7 and through the covariant derivatives of N.

Higher order covariant derivatives of I_1, I_2, I_3 along $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ form higher order invariants in (8.6) and in the recurrent formulas

$$I_{k+3}=\frac{\nabla_{\boldsymbol{\alpha}}I_k}{N}, \qquad\qquad I_{k+6}=\frac{\bigl(\nabla_{\boldsymbol{\gamma}}I_k\bigr)^2}{N^3}. \tag{8.14}$$

that should be applied in triples in some commonly negotiated order (see (6.21) in [3]). Therefore covariant derivatives of I_1, I_2, I_3, I_4, I_7 of any order can be expressed back through the sequence of higher order scalar invariants, through N and through covariant derivatives of N.

Let's consider the covariant derivatives along $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ for N and for the other two fields M and Ω. In the case of N we have

$$\nabla_{\boldsymbol{\alpha}}N=M, \qquad\qquad \nabla_{\boldsymbol{\gamma}}N=-2\,M\,\Omega\,. \tag{8.15}$$

These formulas are derived by direct calculations in our special coordinates introduced through Theorem 5.1. In order to find covariant derivatives of M we express it through N and the scalar invariant I_1 by means of the first formula (8.1):

$$M=I_1\,N^2. \tag{8.16}$$

Differentiating (8.16), we derive

$$\begin{aligned}&\nabla_{\boldsymbol{\alpha}}M=(\nabla_{\boldsymbol{\alpha}}I_1)\,N^2+2\,I_1\,N\,(\nabla_{\boldsymbol{\alpha}}N)=I_4\,N^3+2\,I_1\,N\,M,\\&\nabla_{\boldsymbol{\gamma}}M=(\nabla_{\boldsymbol{\gamma}}I_1)\,N^2+2\,I_1\,N\,(\nabla_{\boldsymbol{\gamma}}N)=\sqrt{N^3\,I_7}\,N^2-4\,I_1\,N\,M\,\Omega\,.\end{aligned} \tag{8.17}$$

In the case of Ω, we use the second formula (8.1). It yields

$$\Omega^2 = I_2\,N. \tag{8.18}$$

Differentiating (8.18), we derive the following formulas:

$$\begin{aligned}
&\nabla_{\boldsymbol\alpha}\Omega = \frac{(\nabla_{\boldsymbol\alpha} I_2)\,N + I_2\,(\nabla_{\boldsymbol\alpha} N)}{2\,\Omega} = \frac{I_5\,N^2 + I_2\,M}{2\,\Omega},\\
&\nabla_{\boldsymbol\gamma}\Omega = \frac{(\nabla_{\boldsymbol\gamma} I_2)\,N + I_2\,(\nabla_{\boldsymbol\gamma} N)}{2\,\Omega} = \frac{\sqrt{N^3\,I_8}\;N - 2\,I_2\,M\,\Omega}{2\,\Omega}.
\end{aligned} \tag{8.19}$$

Looking at (8.15), (8.17), and (8.19), we can formulate the following lemma.

Lemma 8.2. *Covariant derivatives of the pseudoscalar fields M, N, and Ω along pseudovectorial fields $\boldsymbol\alpha$ and $\boldsymbol\gamma$ are expressed through the scalar invariants I_1, I_2, I_3, I_4 etc in the recurrent sequence* (8.14) *and through these fields themselves.*

Note that Lemma 8.2 applies not only to first order covariant derivatives, but to higher order derivatives as well, since the formulas (8.15), (8.17), and (8.19) can be applied recursively. Combining this lemma with the above considerations just after the formula (8.13), we derive a theorem.

Theorem 8.1. *Within the intersection class* ShrID1 ∩ BgdET2*, i. e. if the conditions* (6.7) *are fulfilled, the second Bagderina's invariant I_2^{Bgd} in* (8.10) *can be expressed through I_1, I_2, I_3 and through higher order invariants I_4, I_5, I_6 etc in the recurrent sequence* (8.14)*.*

The above proof of Theorem 8.1 is half constructive. One can make it constructive by continuing the calculations ended with (7.29), though the resulting expression could be enormously large.

9. Conclusions.

Comparing the classification of the equations (1.1) suggested by Yu. Yu. Bagderina in [5] with the previously known classification suggested in [3] we find that the case of intermediate degeneration from [3] does not completely coincide with the corresponding item 2 of Bagderina's classification theorem in [5]. So, formally, Bagderina's classification is new. However, the case of intermediate degeneration from [3] has a substantial intersection with item 2 in Bagderina's classification. Denoting the intersection class through ShrID1 ∩ BgdET2, we compared the two classifications within this intersection class. As a result we have found that most basic structures and basic formulas from Bagderina's paper [5] do coincide or are very closely related to those in [3] and [4], though they are given in different notations (see Lemma 5.1, Lemma 5.2, Lemma 5.3, Lemma 5.4, Lemma 5.5, Lemma 5.7, Lemma 6.1, Lemma 6.2, and Lemma 8.1).

In item 2 of her Teorem 2 in [5] Yu. Yu. Bagderina presents two basic invariants I_1^{Bgd} and I_2^{Bgd}, while in [3] three basic invariants I_1, I_2, and I_3 were presented. Yu. Yu. Bagderina claims that her two invariants are sufficient for expressing all of the invariants, including I_1, I_2, I_3, through them and through their invariant derivatives. However, in [5] there are no explicit formulas expressing I_1, I_2, I_3 through I_1^{Bgd} and I_2^{Bgd}. Some formulas of this sort are given in [13], but again for the case $\Omega = 0$, which is outside our present intersection class ShrID1 ∩ BgdET2.

In the present paper we solve the basic invariants problem from our side by showing that both Bagderina's invariants $I_1^{\rm Bgd}$ and $I_2^{\rm Bgd}$ can be expressed through the invariants I_1, I_2, I_3 and through proper invariant derivatives of them (see Lemma 8.1 and Theorem 8.1 above). It would be best if Yu. Yu. Bagderina presents some explicit formulas or an algorithm for expressing I_1, I_2, I_3 through her invariants $I_1^{\rm Bgd}$ and $I_2^{\rm Bgd}$ in the intersection class $\rm ShrID1 \cap BgdET2$. Otherwise her claim that her invariants are basic is open to question.

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