UMBILICAL AND ZERO CURVATURE EQUATIONS IN A CLASS OF SECOND ORDER ODE'S

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Abstract. The class of second order ODE's cubic with respect to the first order derivative is considered. Using geometric structures associated with these equations, the subclasses of umbilical equations, zero mean curvature equations, and zero Gaussian curvature equations are defined. Zero mean curvature equations are studied within the framework of the first case of intermediate degeneration with the stress on their pseudoscalar and scalar invariants.

1. Introduction.

Since the epoch of classical papers (see [1] and [2]) the class of second order differential equations cubic with respect to the first order derivative

$$y'' = P(x,y) + 3\,Q(x,y)\,y' + 3\,R(x,y)\,(y')^2 + S(x,y)\,(y')^3 \tag{1.1}$$

attracted the attention due to rich geometric structures associated with equations of this class. This class is closed with respect to transformations of the form

$$\begin{cases} \tilde{x} = \tilde{x}(x,y), \\ \tilde{y} = \tilde{y}(x,y), \end{cases} \tag{1.2}$$

which can be interpreted as changes of local curvilinear coordinates in $\mathbb{R}^2$ or in some two-dimensional manifolds. About 19 years ago in [3] and [4] the equations (1.1) were classified using their scalar invariants derived from geometric structures associated with them. They were subdivided into **nine subclasses** closed with respect to transformations of the form (1.2). Here is the list of these classes, which are called cases in [3] and [4]:

— the case of general position (the richest class);
— the first case of intermediate degeneration;
— the second case of intermediate degeneration;
— the third case of intermediate degeneration;
— the fourth case of intermediate degeneration;
— the fifth case of intermediate degeneration;
— the sixth case of intermediate degeneration;
— the seventh case of intermediate degeneration;
— the case of maximal degeneration (the smallest class);

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The case of general position was previously studied in [5] using the same geometric methods as in [3] and [4]. Some time later than [3] and [4] other approaches to classifications of the equations (1.1) were considered (see [6–10]) and Yu. Yu. Bagderina's classification in [11]).

In this paper we define three geometric subclasses of the equations (1.1) that can potentially intersect each of the last eight classes in the above classification and carefully study one of them in the first case of intermediate degeneration.

2. Some notations and definitions.

Transformations of the form (1.2) are interpreted as changes of local coordinates. They are assumed to be locally invertible. The inverse transformations for them are written similarly in the following form:

$$\begin{cases} x = \tilde{x}(\tilde{x}, \tilde{y}), \\ y = \tilde{y}(\tilde{x}, \tilde{y}). \end{cases} \tag{2.1}$$

Like in [3–5] and [12], here we use dot index notations for partial derivatives, e. g. having two functions $f(x, y)$ and $g(\tilde{x}, \tilde{y})$ we write

$$f_{p.q} = \frac{\partial^{p+q} f}{\partial x^p\, \partial y^q}, \qquad\qquad g_{p.q} = \frac{\partial^{p+q} g}{\partial \tilde{x}^p\, \partial \tilde{y}^q}. \tag{2.2}$$

Then we write the Jacoby matrices of the direct and inverse transformations (1.2) and (2.1) in terms of the above notations (2.2):

$$S = \begin{Vmatrix} x_{1.0} & x_{0.1} \\ y_{1.0} & y_{0.1} \end{Vmatrix}, \qquad\qquad T = \begin{Vmatrix} \tilde{x}_{1.0} & \tilde{x}_{0.1} \\ \tilde{y}_{1.0} & \tilde{y}_{0.1} \end{Vmatrix}. \tag{2.3}$$

In differential geometry the Jacoby matrices (2.3) are called the direct and inverse transition matrices respectively (see [13]).

Tensorial and pseudotensorial fields in local coordinates are presented as arrays of functions whose arguments are local coordinates x, y or $\tilde{x}$, $\tilde{y}$ respectively. These arrays of functions are called their components. They obey some definite transformation rules under a change of local coordinates.

Definition 2.1. A pseudotensorial field of the type (r, s) and weight m is an array of functions $F^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}$ which under the change of coordinates (1.2) transforms as

$$F^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = (\det T)^m \sum_{\substack{p_1 \ldots p_r \\ q_1 \ldots q_s}} S^{i_1}_{p_1} \ldots\, S^{i_r}_{p_r}\, T^{q_1}_{j_1} \ldots\, T^{q_s}_{j_s}\, \tilde{F}^{p_1 \ldots\, p_r}_{q_1 \ldots\, q_s}. \tag{2.4}$$

Tensorial fields are those pseudotensorial fields whose weight m in (2.4) is zero. The prefix "pseudo" always indicates the nonzero weight $m \neq 0$.

Tensorial and pseudotensorial fields of the type $(1, 0)$ are called vectorial and pseudovectorial fields. Tensorial and pseudotensorial fields of the type $(0, 1)$ are called covectorial and pseudocovectorial fields. And finally, scalar and pseudoscalar fields are those fields whose type is $(0, 0)$.

Definition 2.2. Tensorial and pseudotensorial fields whose components are expressed through y', through the coefficients P, Q, R, S of the equation (1.1), and through their partial derivatives are called tensorial and pseudotensorial invariants of this equation respectively.

3. Some basic structures.

Each equation of the form (1.1) is associated with two pseudocovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and with one pseudoscalar field F. The components of $\boldsymbol{\alpha}$ are

$$\begin{aligned}\alpha_1 = A = P_{0.2} - 2\,Q_{1.1} + R_{2.0} + 2\,P\,S_{1.0} + S\,P_{1.0} -\\ - 3\,P\,R_{0.1} - 3\,R\,P_{0.1} - 3\,Q\,R_{1.0} + 6\,Q\,Q_{0.1},\\ \alpha_2 = B = S_{2.0} - 2\,R_{1.1} + Q_{0.2} - 2\,S\,P_{0.1} - P\,S_{0.1} +\\ + 3\,S\,Q_{1.0} + 3\,Q\,S_{1.0} + 3\,R\,Q_{0.1} - 6\,R\,R_{1.0}.\end{aligned} \tag{3.1}$$

The weight of the field $\boldsymbol{\alpha}$ with the components (3.1) is equal to 1 (see [3–5]). The components of $\boldsymbol{\beta}$ are expressed through A and B taken from (3.1):

$$\beta_1 = -H, \qquad\qquad \beta_2 = G, \tag{3.2}$$

where G and H are given by the formulas

$$\begin{aligned}G = -B\,B_{1.0} - 3\,A\,B_{0.1} + 4\,B\,A_{0.1} + 3\,S\,A^2 - 6\,R\,B\,A + 3\,Q\,B^2,\\ H = -A\,A_{0.1} - 3\,B\,A_{1.0} + 4\,A\,B_{1.0} - 3\,P\,B^2 + 6\,Q\,A\,B - 3\,R\,A^2.\end{aligned} \tag{3.3}$$

The weight of the field $\boldsymbol{\beta}$ with the components (3.2) is equal to 3 (see [3–5]).

In $\mathbb{R}^2$ and in any two-dimensional manifold there are two pseudotensorial fields with constant components. They are denoted by the same symbol $\mathbf{d}$ and are given by the same skew-symmetric matrix in any local coordinates:

$$d_{ij} = \begin{Vmatrix} 0 & 1\\ -1 & 0\end{Vmatrix}, \qquad\qquad d^{ij} = \begin{Vmatrix} 0 & 1\\ -1 & 0\end{Vmatrix}. \tag{3.4}$$

The weight of the first field in (3.4) is -1, the weight of the second field is 1. The skew-symmetric fields (3.4) here play the same role as metric tensors in metric geometry. They are used for raising and lowering indices of other fields. Raising the indices of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, we get two pseudovectorial fields

$$\alpha^i = \sum_{k=1}^{2} d^{ik}\,\alpha_k, \qquad\qquad \beta^i = \sum_{k=1}^{2} d^{ik}\,\beta_k. \tag{3.5}$$

The weights of the fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with the components (3.5) are 2 and 4 respectively. These pseudovectorial fields are denoted with the same symbols $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as the pseudocovectorial fields in (3.1) and (3.2). The formulas (3.5) yield

$$\alpha^1 = B, \qquad\qquad \alpha^2 = -A, \tag{3.6}$$

$$\beta^1 = G, \qquad\qquad \beta^2 = H. \tag{3.7}$$

The pseudoscalar field F mentioned above is the third field associated with any equation of the form (1.1). It is expressed through the quantities A, B, G, H in (3.1), (3.2), (3.3), (3.6), (3.7) by means of the formulas

$$3\,F^5=\sum_{i=1}^{2}\alpha_i\,\beta^i=-\sum_{i=1}^{2}\beta_i\,\alpha^i=A\,G+B\,H. \tag{3.8}$$

The weight of the field F introduced by (3.8) is equal to 1 (see [3–5]). As for the formula (3.8) itself, it can be written in a more explicit form:

$$\begin{aligned} F=\Bigl(&A\,B\,A_{0.1}+B\,A\,B_{1.0}-A^2\,B_{0.1}-B^2\,A_{1.0}-\\ &-P\,B^3+3\,Q\,A\,B^2-3\,R\,A^2\,B+S\,A^3\Bigr)^{1/5}. \end{aligned} \tag{3.9}$$

The case of **general position** is introduced by the condition $F\neq 0$ in terms of the field (3.9) (see [3–5]). This condition is equivalent to

$$\boldsymbol\alpha\neq 0, \qquad\qquad \boldsymbol\beta\neq 0, \qquad\qquad \boldsymbol\alpha\nparallel\boldsymbol\beta. \tag{3.10}$$

The case of **maximal degeneration** is opposite to (3.10). It is given by the condition $\boldsymbol\alpha=0$, which implies $\boldsymbol\beta=0$ and $F=0$. As for the cases of **intermediate degeneration**, they are introduced by the conditions

$$\boldsymbol\alpha\neq 0, \qquad\qquad \boldsymbol\alpha\parallel\boldsymbol\beta, \tag{3.11}$$

which imply $F=0$. Each particular case of of intermediate degeneration is specified by some auxiliary conditions added to (3.11) (see [3, 4]).

4. Auxiliary structures common for all cases of intermediate degeneration.

As soon as the conditions (3.11) are fulfilled, some new geometric structures associated with the equations (1.1) arise. They consists of two pseudoscalar field M and N, a pseudocovectorial field $\boldsymbol\gamma$, and connection components Γ^k_{ij}. The pseudoscalar field N is introduced as a factor relating two parallel pseudocovectorial fields one of which is nonzero. It is given by the following formula:

$$\boldsymbol\beta=3\,N\,\boldsymbol\alpha. \tag{4.1}$$

In terms of the components of $\boldsymbol\alpha$ and $\boldsymbol\beta$ the formula (4.1) yields

$$N=\frac{G}{3\,B}, \qquad\qquad N=-\frac{H}{3\,A}. \tag{4.2}$$

The first formula applies in the case $B\neq 0$, the second one in the case $A\neq 0$. If both A and B are nonzero, both formulas are applicable. The quantities A and B cannot vanish simultaneously since they are components of the field $\boldsymbol\alpha$ and $\boldsymbol\alpha\neq 0$ in all cases of intermediate degeneration (see (3.11)). The weight of the field N in (4.1) and (4.2) is equal to 2.

The pseudoscalar field M is also introduced by means of two formulas one of which is for the case $B \neq 0$ and the second one is for $A \neq 0$ (see [4]):

$$\begin{aligned} M = -\frac{12\,A\,N\,(A\,S - B_{0.1})}{5\,B} - A\,N_{0.1} + \frac{24}{5}\,A\,N\,R- \\ -\frac{6}{5}\,N\,A_{0.1} - \frac{6}{5}\,N\,B_{1.0} + B\,N_{1.0} - \frac{12}{5}\,B\,N\,Q, \end{aligned} \tag{4.3}$$

$$\begin{aligned} M = -\frac{12\,B\,N\,(B\,P + A_{1.0})}{5\,A} + B\,N_{1.0} + \frac{24}{5}\,B\,N\,Q+ \\ +\frac{6}{5}\,N\,B_{1.0} + \frac{6}{5}\,N\,A_{0.1} - A\,N_{0.1} - \frac{12}{5}\,A\,N\,R. \end{aligned} \tag{4.4}$$

The weight of the field M in (4.3) and (4.4) is equal to 4.

The pseudocovectorial field $\boldsymbol{\gamma}$ is introduced by two pairs of formulas for its components, one pair is for $B \neq 0$ and the other is for $A \neq 0$ (see [4]):

$$\begin{aligned} \gamma_1 = & \frac{6\,A\,N\,(A\,S - B_{0.1})}{5\,B^2} - \frac{18\,N\,A\,R}{5\,B} + \\ & + \frac{6\,N\,(A_{0.1} + B_{1.0})}{5\,B} - N_{1.0} + \frac{12}{5}\,N\,Q - 2\,\Omega\,A, \end{aligned} \tag{4.5}$$

$$\gamma_2 = -\frac{6\,N\,(A\,S - B_{0.1})}{5\,B} - N_{0.1} + \frac{6}{5}\,N\,R - 2\,\Omega\,B. \tag{4.6}$$

The formulas (4.5) and (4.6) are used if $B \neq 0$. If $A \neq 0$, we write:

$$\gamma_1 = \frac{6\,N\,(B\,P + A_{1.0})}{5\,A} - N_{1.0} - \frac{6}{5}\,N\,Q - 2\,\Omega\,A, \tag{4.7}$$

$$\begin{aligned} \gamma_2 = & -\frac{6\,B\,N\,(B\,P + A_{1.0})}{5\,A^2} + \frac{18\,N\,B\,Q}{5\,A} + \\ & + \frac{6\,N\,(B_{1.0} + A_{0.1})}{5\,A} - N_{0.1} - \frac{12}{5}\,N\,R - 2\,\Omega\,B. \end{aligned} \tag{4.8}$$

The weight of the field $\boldsymbol{\gamma}$ with the components (4.5), (4.6), (4.7), (4.8) is 2. Raising indices in these formulas, we get a pseudovectorial field denoted by the same symbol:

$$\gamma^i = \sum_{k=1}^{2} d^{ik}\,\gamma_k. \tag{4.9}$$

The formula (4.9) can be written in a more explicit form:

$$\gamma^1 = C = \gamma_2, \qquad\qquad \gamma^2 = D = -\gamma_1. \tag{4.10}$$

Here C and D are notations for the components of the pseudovectorial field $\boldsymbol{\gamma}$. Its weight is 3. The formula (4.9) is an analog of the formulas (3.5), while the formulas (4.10) are analogous to the formulas (3.6) and (3.7).

The field M given by the formulas (4.3) and (4.4) is related to the fields $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ by means of the formulas similar to (3.8):

$$M=\sum_{i=1}^{2}\alpha_i\,\gamma^i=-\sum_{i=1}^{2}\gamma_i\,\alpha^i=A\,C+B\,D. \tag{4.11}$$

The connection components Γ^k_{ij} constitute the fourth auxiliary structure common for all cases of **intermediate degeneration**. They are given by the formula

$$\Gamma^k_{ij}=\theta^k_{ij}-\frac{\varphi_i\,\delta^k_j+\varphi_j\,\delta^k_i}{3}, \tag{4.12}$$

where δ^k_j and δ^k_i are Kronecker deltas. The quantities θ^k_{ij} in (4.12) are given by the components of a fully symmetric array θ_{ijk} upon raising one of its indices:

$$\theta^k_{ij}=\sum_{r=1}^{2}d^{kr}\,\theta_{rij}. \tag{4.13}$$

The components θ_{rij} of the array θ in (4.13) are given explicitly:

$$\begin{aligned}&\theta_{111}=P, &&\theta_{112}=\theta_{121}=\theta_{211}=Q,\\ &\theta_{122}=\theta_{212}=\theta_{221}=R, &&\theta_{222}=S.\end{aligned} \tag{4.14}$$

In addition to θ^k_{ij} defined through (4.13) and (4.14), the quantities φ_i and φ_j are used in (4.12). They are given by the formulas

$$\begin{aligned}&\varphi_1=-3\,A\,\frac{A\,S-B_{0.1}}{5\,B^2}-3\,\frac{A_{0.1}+B_{1.0}-3\,A\,R}{5\,B}-\frac{6}{5}\,Q,\\ &\varphi_2=3\,\frac{A\,S-B_{0.1}}{5\,B}-\frac{3}{5}\,R.\end{aligned} \tag{4.15}$$

The formulas (4.15) apply in the case $B\neq 0$. If $A\neq 0$, we use the formulas

$$\begin{aligned}&\varphi_1=-3\,\frac{B\,P+A_{1.0}}{5\,A}+\frac{3}{5}\,Q,\\ &\varphi_2=3\,B\,\frac{B\,P+A_{1.0}}{5\,A^2}-3\,\frac{B_{1.0}+A_{0.1}+3\,B\,Q}{5\,A}+\frac{6}{5}\,R.\end{aligned} \tag{4.16}$$

If both A and B are nonzero, then both formulas (4.15) and (4.16) are applicable. The quantities φ_1 and φ_2 do not form a pseudotensorial field. They are used for producing the connection components (4.12).

5. Derived structures common for all cases of intermediate degeneration.

The auxiliary structures given by the fields M, N, $\boldsymbol{\gamma}$ and the connection components Γ^k_{ij} are complemented by some more structures common for all cases of **intermediate degeneration**. It is known that each affine connection produces its curvature tensor (see [13] and [14]). The well-known formula expressing the curvature tensor through the connection components is written as

$$R^k_{rij} = \frac{\partial \Gamma^k_{jr}}{\partial x^i} - \frac{\partial \Gamma^k_{ir}}{\partial x^j} + \sum_{q=1}^{2} \Gamma^k_{iq} \Gamma^q_{jr} - \sum_{q=1}^{2} \Gamma^k_{jq} \Gamma^q_{ir}. \tag{5.1}$$

The type of the field with the components (5.1) is $(1,3)$, its weight is zero, i. e. it is a tensorial field.

The curvature tensor R in (5.1) has one upper index and three lower indices. Contracting it with respect to the upper index and the fist lower index, we get

$$\omega_{ij} = \sum_{k=1}^{2} R^k_{kij}. \tag{5.2}$$

The quantities (5.2) are components of a skew-symmetric tensorial field. The quantities (5.2) were first introduced through the quantities φ_i in [3]:

$$\omega_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}. \tag{5.3}$$

One can verify that (5.3) yields the same result as (5.2). The quantities (5.3) then were used in order to define a pseudoscalar field Ω (see [3, 4]):

$$\Omega = \frac{5}{6} \sum_{i=1}^{2} \sum_{j=1}^{2} \omega_{ij}\, d^{ij}. \tag{5.4}$$

The weight of the field Ω in (5.4) is equal to 1. Here are explicit formulas for Ω:

$$\begin{aligned} \Omega = {} & \frac{2\,A\,B_{0.1}(A\,S - B_{0.1})}{B^3} + \frac{(2\,A_{0.1} - 3\,A\,R)\,B_{0.1}}{B^2} + \\ & + \frac{(B_{1.0} - 2\,A_{0.1})\,A\,S}{B^2} + \frac{A\,B_{0.2} - A^2\,S_{0.1}}{B^2} - \frac{A_{0.2}}{B} + \\ & + \frac{3\,A_{0.1}\,R + 3\,A\,R_{0.1} - A_{1.0}\,S - A\,S_{1.0}}{B} + R_{1.0} - 2\,Q_{0.1}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} \Omega = {} & \frac{2\,B\,A_{1.0}(B\,P + A_{1.0})}{A^3} - \frac{(2\,B_{1.0} + 3\,B\,Q)\,A_{1.0}}{A^2} + \\ & + \frac{(A_{0.1} - 2\,B_{1.0})\,B\,P}{A^2} - \frac{B\,A_{2.0} + B^2\,P_{1.0}}{A^2} + \frac{B_{2.0}}{A} + \\ & + \frac{3\,B_{1.0}\,Q + 3\,B\,Q_{1.0} - B_{0.1}\,P - B\,P_{0.1}}{A} + Q_{0.1} - 2\,R_{1.0}. \end{aligned} \tag{5.6}$$

The formula (5.5) applies in the case $B \neq 0$. If $A \neq 0$, we apply the formula (5.6).

Note that we use the formula (5.1) in two-dimensional case. In two-dimensional case the curvature tensor of any connection is presented as

$$R^k_{qij} = R^k_q\, d_{ij}, \quad \text{where} \quad R^k_q = \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} R^k_{qij}\, d^{ij}. \tag{5.7}$$

The type of the pseudotensorial field R with the components R^k_q in (5.7) is $(1,1)$, its weight is 1. The field R is a pseudooperator field. There are two pseudoscalar fields associated with the field R — its trace $\operatorname{tr}(R)$ and its determinant $\det(R)$. The trace $\operatorname{tr}(R)$ is given by the following formula:

$$\operatorname{tr}(R) = \frac{3}{5}\,\Omega. \tag{5.8}$$

Unlike $\operatorname{tr}(R)$ in (5.8), the determinant $\det(R)$ is a new field. It was not studied in [3, 4], though the curvature tensor (5.1) and the pseudooperator field R from (5.7) were considered in [4]. The eigenvalues of the pseudooperator field R were also considered in [4]. They are given by the following characteristic equation:

$$\lambda^2 - \operatorname{tr}(R)\,\lambda + \det(R) = 0. \tag{5.9}$$

Let's denote through $\mathfrak{D}$ the discriminant of the quadratic equation (5.9). This discriminant is calculated by means of the following formula:

$$\mathfrak{D} = \operatorname{tr}(R)^2 - 4\,\det(R) = \frac{9}{25}\,\Omega^2 - 4\,\det(R). \tag{5.10}$$

The determinant $\det(R)$ and the discriminant $\mathfrak{D}$ in (5.10) both are pseudoscalar fields of the weight 2. Both of these two pseudoscalar fields are defined in all cases of **intermediate degeneration**.

6. Umbilical and zero curvature equations.

The term **umbilical points** in metric geometry of two-dimensional surfaces in a three-dimensional Euclidean space denotes those points of a surface where two principal curvatures λ_1 and λ_2 are equal to each other (see [15]). The principal curvatures are eigenvalues of a symmetric operator defined by the first fundamental form and the second fundamental form of a surface (see [16]). Using this analogy, we introduce the following definition of umbilical equations of the form (1.1).

Definition 6.1. An umbilical equation of the form (1.1) is an equation of that form whose discriminant field $\mathfrak{D}$ in (5.10) is identically equal to zero.

The equality $\mathfrak{D} = 0$ for the discriminant field (5.10) means that two eigenvalues λ_1 and λ_2 of the curvature pseudooperator field R solving the characteristic equation (5.9) are equal to each other thus supporting the above analogy with two-dimensional surfaces.

In [12] Yu. Yu. Bagderina's classification from [11] was compared with the previously existing classification from [3, 4]. Among other results in [12] the following equality for Bagderina's pseudoscalar field $j_5^{\,\mathrm{Bgd}}$ was derived:

$$j_5^{\,\mathrm{Bgd}} = -125\,\det(R) + \frac{45}{4}\,\Omega^2. \tag{6.1}$$

Comparing (6.1) with (5.10), we see that

$$j_5^{\,\mathrm{Bgd}} = \frac{125}{4}\,\mathfrak{D}. \tag{6.2}$$

The field $j_5^{\,\mathrm{Bgd}}$ is defined by Yu. Yu. Bagderina on page 27 of her paper [11]. It is used in item 2 of her Theorem 2 in order to define one of the two basic invariants:

$$I_2^{\,\mathrm{Bgd}} = \frac{j_5^{\,\mathrm{Bgd}}}{(j_0^{\,\mathrm{Bgd}})^2}. \tag{6.3}$$

The same field $j_5^{\,\mathrm{Bgd}}$ is used in item 4 of Yu. Yu. Bagderina's Theorem 2 in [11] again in order to define one of the two her basic invariants:

$$I_2^{\,\mathrm{Bgd}} = \frac{j_5^{\,\mathrm{Bgd}}}{(j_1^{\,\mathrm{Bgd}})^{1/2}}. \tag{6.4}$$

Along with its very restrictive use in (6.3) to (6.4), the field $j_5^{\,\mathrm{Bgd}}$ from (6.2) is used in the following lemma.

Lemma 6.1. *Umbilical equations constitute that very class where Yu. Yu. Bagderina's pseudoscalar field $j_5^{\,\mathrm{Bgd}}$ is identically zero: $j_5^{\,\mathrm{Bgd}} = 0$.*

Apart from $\lambda_1 = \lambda_2$, one can write two other relationships for principal curvatures λ_1 and λ_2, which are symmetric with respect to them:

$$\frac{\lambda_1 + \lambda_2}{2} = 0, \qquad\qquad \lambda_1\,\lambda_2 = 0. \tag{6.5}$$

In metric geometry the expressions in the left hand sides of the formulas (6.5) are called mean curvature and Gaussian curvature respectively (see [17] and [18]). Since

$$\frac{\lambda_1 + \lambda_2}{2} = \frac{\operatorname{tr}(R)}{2}, \qquad\qquad \lambda_1\,\lambda_2 = \det(R), \tag{6.6}$$

using the analogy to metric geometry, we can formulate two definitions.

Definition 6.2. A zero mean curvature equation of the form (1.1) is an equation of that form whose trace field $\operatorname{tr}(R)$ is identically equal to zero.

Definition 6.3. A zero Gaussian curvature equation of the form (1.1) is an equation of that form whose determinant field $\det(R)$ is identically equal to zero.

7. Zero mean curvature equations in the first case of intermediate degeneration.

Comparing (5.8) with (6.6) and taking into account Definition 6.2, we see that zero mean curvature equations are given by the condition

$$\Omega = 0. \tag{7.1}$$

The first case of intermediate degeneration is defined by the conditions

$$\boldsymbol{\alpha} \neq 0, \qquad\qquad F = 0, \qquad\qquad M \neq 0 \tag{7.2}$$

(see [3, 4]). The first two of the conditions (7.2) follow from (3.11). They are common for all cases of intermediate degeneration. The third condition is specific to the first case of intermediate degeneration. Due to (4.11) it implies $\boldsymbol{\alpha} \nparallel \boldsymbol{\gamma}$.

Combining (7.1) and (7.2), we find that zero mean curvature equations in the first case of intermediate degeneration are given by the following conditions:

$$\boldsymbol{\alpha}\neq 0,\qquad F=0,\qquad M\neq 0,\qquad \Omega=0.\tag{7.3}$$

Now let's refer to [12] where two classifications from [3, 4]) and [11] were compared. As a result of this comparison we have the following relationships

$$\beta_1^{\text{Bgd}}=\alpha_1\qquad J_0^{\text{Bgd}}=-F^5,\qquad j_1^{\text{Bgd}}=\frac{5}{2}\,M,\qquad j_0^{\text{Bgd}}=-3\,\Omega\,.\tag{7.4}$$

Comparing (7.3) and (7.4) with item 4 of Theorem 2 in [11], we can formulate the following comparison theorem.

Theorem 7.1. *Bagderina's type four equations from* [11] *coincide with the subclass of zero mean curvature equations within the first case of intermediate degeneration in the classification from* [3, 4]*.*

In terms of the notations introduced in the table on page 15 of [12] we have

$$\text{BgdET4}=\text{ShrID1}\cap\text{BgdET4}\subset\text{ShrID1}.\tag{7.5}$$

The class ShrID1 ∩ BgdET4 in (7.5) and in Theorem 7.1 is complementary to the intersection class ShrID1 ∩ BgdET2 considered in [12]:

$$\text{ShrID1}=\bigl(\text{ShrID1}\cap\text{BgdET4}\bigr)\cup\bigl(\text{ShrID1}\cap\text{BgdET2}\bigr).\tag{7.6}$$

Due to (7.6) the class of equations of the first case of intermediate degeneration ShrID1 does not intersect with the classes other than BgdET2 and BgdET4 in Bagderina's classification from [11].

Relying on Theorem 7.1, below we continue studying the intersection class ShrID1 ∩ BgdET4. In [11] Yu. Yu. Bagderina defines two invariant differentiation operators. The first of them is given by the formula

$$\mathcal{D}_1^{\text{Bgd}}=\frac{\beta_2^{\text{Bgd}}}{(\mu_1^{\text{Bgd}})^2}\,\frac{\partial}{\partial x}-\frac{\beta_1^{\text{Bgd}}}{(\mu_1^{\text{Bgd}})^2}\,\frac{\partial}{\partial y}.\tag{7.7}$$

The quantity μ_1^{Bgd} in (7.7) is given by one of the formulas (2.12) from [11]:

$$\mu_1^{\text{Bgd}}=\bigl(j_1^{\text{Bgd}}\bigr)^{1/4}.\tag{7.8}$$

Comparing the formula (7.8) with (7.4), we can write

$$\mu_1^{\text{Bgd}}=\sqrt[4]{\frac{5\,M}{2}}.\tag{7.9}$$

The quantities β_1^{Bgd} and β_2^{Bgd} in (7.7) coincide with the components (3.1) of the pseudoscalar field $\boldsymbol{\alpha}$ (see Lemma 3.2 in [19] or Lemma 5.1 in [12]):

$$\beta_1^{\text{Bgd}}=\alpha_1=A,\qquad \beta_2^{\text{Bgd}}=\alpha_2=B.\tag{7.10}$$

Raising indices in (7.10) according to (3.5) and then applying (7.10) and (7.9) to (7.7), we derive the following formula for $\mathcal{D}_1^{\,\text{Bgd}}$:

$$\mathcal{D}_1^{\,\text{Bgd}} = \sqrt{\frac{2}{5\,M}}\biggl(\alpha^1\,\frac{\partial}{\partial x} + \alpha^2\,\frac{\partial}{\partial y}\biggr). \tag{7.11}$$

Lemma 7.1. *If the conditions* (7.3) *are fulfilled, i. e. within the intersection class* ShrID1 ∩ BgdET4 *coinciding with the class of Bagderina's type four equations, Bagderina's invariant differentiation operator* $\mathcal{D}_1^{\,\text{Bgd}}$ *is expressed through the pseudovectorial field* $\boldsymbol\alpha$ *and pseudoscalar field* M *from* [3] *by means of the formula* (7.11).

The use of non-integer power exponents with even denominators by Bagderina in (6.4) and (7.8) is a bad practice. In order to avoid such a practice in [3, 4] covariant differentiation operators were considered instead of invariant differentiations. They are produced by pseudovectorial fields according to the following patterns:

$$\nabla_{\boldsymbol\alpha} = \alpha^1\,\nabla_1 + \alpha^2\,\nabla_2, \qquad\qquad \nabla_{\boldsymbol\gamma} = \gamma^1\,\nabla_1 + \gamma^2\,\nabla_2 \tag{7.12}$$

(see (6.13) in [3] and (5.2) in [4]). The covariant derivatives in (7.12) extend partial derivatives from (7.7). They are defined by means of the formula

$$\begin{aligned}\nabla_k F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \frac{\partial F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^k} + \sum^r_{n=1}\sum^2_{v_n=1} \varGamma^{i_n}_{k\,v_n}\,F^{i_1\ldots\,v_n\ldots\,i_r}_{j_1\ldots\,j_s} - \\ - \sum^s_{n=1}\sum^2_{w_n=1} \varGamma^{w_n}_{k\,j_n}\,F^{i_1\ldots\,i_r}_{j_1\ldots\,w_n\ldots\,j_s} + m\,\varphi_k\,F^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\end{aligned} \tag{7.13}$$

(see (6.11) in [3] or (4.23) in [4]). The covariant derivative ∇_k in (7.13) is applied to a pseudotensorial field of the type (r, s) and the weight m. The connection components $\varGamma^k_{ij}$ in (7.13) are defined by (4.12). They are canonically associated with a given equation (1.1) in all cases of intermediate degeneration.

Covariant differentiation operators like (7.12) are preferable with respect to invariant differentiation operators like (7.7). They can be applied to any pseudotensorial invariants, not only to scalar ones. Fortunately invariant differentiation operators can be extended to covariant differentiations. In the case of (7.11) we have

$$\mathcal{D}_1^{\,\text{Bgd}} = \sqrt{\frac{2}{5\,M}}\,\bigl(\alpha^1\,\nabla_1 + \alpha^2\,\nabla_2\bigr) = \sqrt{\frac{2}{5\,M}}\,\nabla_{\boldsymbol\alpha}. \tag{7.14}$$

Due to (7.14) we can reformulate Lemma 7.1 as follows.

Lemma 7.2. *If the conditions* (7.3) *are fulfilled, i. e. within the intersection class* ShrID1 ∩ BgdET4 *coinciding with the class of Bagderina's type four equations, Bagderina's invariant differentiation operator* $\mathcal{D}_1^{\,\text{Bgd}}$ *is expressed through the pseudovectorial field* $\boldsymbol\alpha$ *and pseudoscalar field* M *from* [3] *by means of the formula* (7.14).

The second invariant differentiation operator $\mathcal{D}_1^{\,\text{Bgd}}$ introduced by Yu. Yu. Bagderina in [11] upon replacing partial derivatives by covariant derivatives is

$$\mathcal{D}_2^{\,\text{Bgd}} = \biggl(\mu_2^{\,\text{Bgd}}\,\beta_2^{\,\text{Bgd}} - 3\,\frac{\mu_1^{\,\text{Bgd}}}{\beta_1^{\,\text{Bgd}}}\biggr)\,\nabla_1 - \mu_2^{\,\text{Bgd}}\,\beta_1^{\,\text{Bgd}}\,\nabla_2. \tag{7.15}$$

Note that the formula (6.14) for $\mathcal{D}_2^{\,\text{Bgd}}$ in [12] is mistyped. The author apologizes for this typo and presents the correct formula (7.15).

The quantity μ_2^{Bgd} in (7.15) is given by one of the formulas (2.12) in [11]:

$$\mu_2^{\text{Bgd}}=\frac{5\,j_2^{\text{Bgd}}}{2\left(j_1^{\text{Bgd}}\right)^{3/4}} \tag{7.16}$$

The quantity j_2^{Bgd} in (7.16) is expressed through the coefficients of the equation (1.1) by means of a series of auxiliary notations (see [11]). We reproduce them here without exceptions and reductions for the sake of completeness:

$$\begin{aligned} j_2^{\text{Bgd}}&=\frac{1}{\beta_1^{\text{Bgd}}}\left(\delta_{20}^{\text{Bgd}}-\frac{\beta_2^{\text{Bgd}}}{\beta_1^{\text{Bgd}}}\,\delta_{10}^{\text{Bgd}}\right)+\\ &+\ \frac{\gamma_{10}^{\text{Bgd}}}{5\,(\beta_1^{\text{Bgd}})^2}\left(7\,\frac{\beta_2^{\text{Bgd}}}{\beta_1^{\text{Bgd}}}\,\gamma_{10}^{\text{Bgd}}-6\,\gamma_{20}^{\text{Bgd}}-\gamma_{11}^{\text{Bgd}}\right),\\ j_3^{\text{Bgd}}&=\frac{3}{5}\left(\frac{\delta_{10}^{\text{Bgd}}}{(\beta_1^{\text{Bgd}})^3}-\frac{6\,(\gamma_{10}^{\text{Bgd}})^2}{5\,(\beta_1^{\text{Bgd}})^4}\right), \end{aligned} \tag{7.17}$$

$$\begin{aligned} \delta_{10}^{\text{Bgd}}&=\partial_x\gamma_{10}^{\text{Bgd}}-2\,Q\,\gamma_{10}^{\text{Bgd}}+P\,(\gamma_{20}^{\text{Bgd}}+\gamma_{11}^{\text{Bgd}})-5\,\alpha_0^{\text{Bgd}}\,\beta_1^{\text{Bgd}},\\ \delta_{20}^{\text{Bgd}}&=\partial_x\gamma_{20}^{\text{Bgd}}-R\,\gamma_{10}^{\text{Bgd}}+P\,\gamma_{21}^{\text{Bgd}}-4\,\alpha_1^{\text{Bgd}}\,\beta_1^{\text{Bgd}}-\alpha_0^{\text{Bgd}}\,\beta_2^{\text{Bgd}}, \end{aligned} \tag{7.18}$$

$$\begin{aligned} \gamma_{10}^{\text{Bgd}}&=\partial_x\beta_1^{\text{Bgd}}-Q\,\beta_1^{\text{Bgd}}+P\,\beta_2^{\text{Bgd}},\\ \gamma_{11}^{\text{Bgd}}&=\partial_x\beta_2^{\text{Bgd}}-R\,\beta_1^{\text{Bgd}}+Q\,\beta_2^{\text{Bgd}},\\ \gamma_{20}^{\text{Bgd}}&=\partial_y\beta_1^{\text{Bgd}}-R\,\beta_1^{\text{Bgd}}+Q\,\beta_2^{\text{Bgd}},\\ \gamma_{21}^{\text{Bgd}}&=\partial_y\beta_2^{\text{Bgd}}-S\,\beta_1^{\text{Bgd}}+R\,\beta_2^{\text{Bgd}}, \end{aligned} \tag{7.19}$$

$$\begin{aligned} \alpha_0^{\text{Bgd}}&=Q_{1.0}-P_{0.1}+2\,P\,R-2\,Q^2,\\ \alpha_1^{\text{Bgd}}&=R_{1.0}-Q_{0.1}+P\,S-Q\,R,\\ \alpha_2^{\text{Bgd}}&=S_{1.0}-R_{0.1}+2\,Q\,S-2\,R^2. \end{aligned} \tag{7.20}$$

We do no provide Bagderina's expressions for j_1^{Bgd}, β_1^{Bgd}, β_2^{Bgd} since they can be calculated using (7.4) and (7.10). Using the formulas (7.16), (7.17), (7.18), (7.19), (7.20) and applying them to (7.15), one can derive the following formula for $\mathcal{D}_2^{\,\text{Bgd}}$:

$$\mathcal{D}_2^{\,\text{Bgd}}=-3\,\sqrt[4]{\frac{5}{2\,M^3}}\,\bigl(\gamma^1\,\nabla_1+\gamma^2\,\nabla_2\bigr)=-3\,\sqrt[4]{\frac{5}{2\,M^3}}\,\nabla_{\boldsymbol\gamma}. \tag{7.21}$$

Lemma 7.3. *If the conditions* (7.3) *are fulfilled, i. e. within the intersection class* ShrID1 ∩ BgdET4 *coinciding with the class of Bagderina's type four equations, Bagderina's invariant differentiation operator* $\mathcal{D}_2^{\,\text{Bgd}}$ *is expressed through the pseudovectorial field* $\boldsymbol\gamma$ *and pseudoscalar field* M *from* [3] *by means of the formula* (7.21).

Lemmas 7.1, 7.2, and 7.3 are proved by direct calculations using some symbolic algebra package. In my case that was Maple[1].

[1] Maple is a trademark of Waterloo Maple Inc.

8. Scalar invariants.

The condition $\Omega = 0$ in (7.3) does not specify a special subcase within the first case of intermediate degeneration. The subcase $\Omega = 0$ is treated regularly, though one of the three invariants I_1, I_2, I_3 from [3, 4]) does vanish:

$$I_2 = \frac{\Omega^2}{N} = 0. \tag{8.1}$$

The invariant I_1 does not vanish due to (7.3). It is given by the formula

$$I_1 = \frac{M}{N^2}. \tag{8.2}$$

The third invariant I_3 is introduced through the coefficients in the expansions

$$\begin{aligned} &\nabla_{\boldsymbol{\alpha}}\boldsymbol{\alpha} = \Gamma^1_{11}\,\boldsymbol{\alpha} + \Gamma^2_{11}\,\boldsymbol{\gamma}, &&\nabla_{\boldsymbol{\alpha}}\boldsymbol{\gamma} = \Gamma^1_{12}\,\boldsymbol{\alpha} + \Gamma^2_{12}\,\boldsymbol{\gamma},\\ &\nabla_{\boldsymbol{\gamma}}\boldsymbol{\alpha} = \Gamma^1_{21}\,\boldsymbol{\alpha} + \Gamma^2_{21}\,\boldsymbol{\gamma}, &&\nabla_{\boldsymbol{\gamma}}\boldsymbol{\gamma} = \Gamma^1_{22}\,\boldsymbol{\alpha} + \Gamma^2_{22}\,\boldsymbol{\gamma}\end{aligned} \tag{8.3}$$

(see (6.13) in [3] or (5.2) in [4]). As it was shown in [3], only one of the coefficients Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} does matter. This is the coefficient Γ^1_{22}. It defines the invariant I_3 by means of the formula

$$I_3 = \frac{\Gamma^1_{22}\,N^2}{M^2} \tag{8.4}$$

(see (6.20) in [3]). The coefficients Γ^2_{11} and Γ^1_{21} in (8.3) are equal to zero

$$\Gamma^2_{11} = 0, \qquad\qquad \Gamma^1_{21} = 0. \tag{8.5}$$

The coefficients Γ^1_{11} and Γ^2_{21} are expressed through the pseudoscalar field N:

$$\Gamma^1_{11} = -\frac{3}{5}\,N, \qquad\qquad \Gamma^2_{21} = -\frac{3}{5}\,N. \tag{8.6}$$

The coefficients Γ^2_{12}, Γ^2_{22}, and Γ^1_{12} obey more complicated relationships

$$I_1\,\Gamma^2_{12} = I_4\,N + \frac{3}{5}\,I_1\,N + 2\,I_1^2\,N, \tag{8.7}$$

$$\begin{aligned} &\bigl(I_1\,\Gamma^2_{22}\bigr)^4 + \bigl(I_7\,N^3\bigr)^2 + \bigl(16\,I_2\,N^3\,I_1{}^4\bigr)^2 =\\ &\qquad = 32\,I_7\,N^6\,I_2\,I_1{}^4 + 2\,\bigl(I_7\,N^3 + 16\,I_2\,N^3\,I_1{}^4\bigr)\,\bigl(I_1\,\Gamma^2_{22}\bigr)^2,\end{aligned} \tag{8.8}$$

$$\Gamma^1_{12} = -\Gamma^2_{22}. \tag{8.9}$$

(see (6.22), (6.23), (6.15) and in [3] or (5.6) and (5.7) in [4]). Unfortunately the formula (6.22) in [3] is mistyped and then copied to (5.6) in [4]. The minus signs in the right hand side of this formula should be altered for pluses. The formula (8.7) here is a corrected version of this formula.

The quantities I_4 and I_7 in (8.7) and (8.8) are higher order invariants. They are taken from the following formulas:

$$\begin{aligned}
&I_4=\frac{\nabla_{\boldsymbol\alpha}I_1}{N}, && I_5=\frac{\nabla_{\boldsymbol\alpha}I_2}{N}, && I_6=\frac{\nabla_{\boldsymbol\alpha}I_3}{N},\\
&I_7=\frac{(\nabla_{\boldsymbol\gamma}I_1)^2}{N^3}, && I_8=\frac{(\nabla_{\boldsymbol\gamma}I_2)^2}{N^3}, && I_9=\frac{(\nabla_{\boldsymbol\gamma}I_3)^2}{N^3}.
\end{aligned}\tag{8.10}$$

More higher order invariants can be produced recursively

$$I_{k+3}=\frac{\nabla_{\boldsymbol\alpha}I_k}{N}, \qquad\qquad I_{k+6}=\frac{\bigl(\nabla_{\boldsymbol\gamma}I_k\bigr)^2}{N^3}.\tag{8.11}$$

The recurrent formulas (8.11) should be applied in triples in some commonly negotiated order (see (6.21) in [3]). As a result we shall have an infinite series of scalar invariants associated with the equation (1.1).

In [11] Yu. Yu. Bagderina introduces her own basic invariants for the class BgdET4 coinciding with the intersection class ShrID1 ∩ BgdET4 and given by the conditions (7.3). Her first invariant $I_1^{\,\mathrm{Bgd}}$ is given by the formula

$$I_1^{\,\mathrm{Bgd}}=\frac{\Gamma_0^{\,\mathrm{Bgd}}}{\beta_1^{\,\mathrm{Bgd}}\,(j_1^{\,\mathrm{Bgd}})^{1/2}}.\tag{8.12}$$

The fields $\beta_1^{\,\mathrm{Bgd}}$ and $j_1^{\,\mathrm{Bgd}}$ in (8.12) are taken from (7.10) and (7.4). As for the field $\Gamma_0^{\,\mathrm{Bgd}}$, in [19] it was shown that the $\Gamma_0^{\,\mathrm{Bgd}}$ coincides with the first component of the pseudocovectorial field $\boldsymbol\beta$ in (3.2) (see Lemma 3.5 in [19] or Lemma 5.2 in [12]). Therefore from (4.1) or from (4.2) in our present case we derive

$$\frac{\Gamma_0^{\,\mathrm{Bgd}}}{\beta_1^{\,\mathrm{Bgd}}}=3\,N.\tag{8.13}$$

Applying (8.13), (7.4) and (8.2) to (8.12), we derive the following formula:

$$I_1^{\,\mathrm{Bgd}}=\sqrt{\frac{18\,N^2}{5\,M}}=\sqrt{\frac{18}{5\,I_1}}.\tag{8.14}$$

The following lemma is formulated for further references.

Lemma 8.1. *If the conditions (7.3) are fulfilled, i. e. within the intersection class* ShrID1 ∩ BgdET4 *coinciding with the class of Bagderina's type four equations, Bagderina's basic invariant $I_1^{\,\mathrm{Bgd}}$ is expressed through the invariant I_1 from [3] by means of the formula (8.14).*

The formula in the right hand side of (8.14) is in agreement with the first formula (7.2) presented by Yu. Yu. Bagderina in [20], where she compares her results with the previously known results from [3–5]. The formula (8.1) is evidently in agreement with the second formula (7.2) from [20]. The third formula (7.2) in [20] is more complicated. We shall consider it below.

Apart from (7.2), there are the following comparison formulas in [20]:

$$A=\beta_1^{\,\mathrm{Bgd}},\qquad B=\beta_2^{\,\mathrm{Bgd}},\qquad F^5=-J_0^{\,\mathrm{Bgd}},\qquad G=\Gamma_1^{\,\mathrm{Bgd}},\qquad H=-\Gamma_0^{\,\mathrm{Bgd}}.\tag{8.15}$$

The formulas (8.15) are in agreement with the results from [19] and [12] (see Lemma 3.2, Lemma 3.3, and Lemma 3.5 in [19]). The formulas

$$N=\frac{\Gamma_0^{\text{Bgd}}}{3\,\beta_1^{\text{Bgd}}},\qquad\qquad \Omega=-\frac{j_0^{\text{Bgd}}}{3},\qquad\qquad M=\frac{2}{5}\,j_1^{\text{Bgd}}\tag{8.16}$$

are also presented in [20]. The formulas (8.16) are in agreement with the formulas (8.13) and (7.4) in the present paper. The formulas (7.1) from [20] look like

$$F=0\qquad\qquad A\neq 0,\qquad\qquad \Omega=0,\qquad\qquad M\neq 0.\tag{8.17}$$

The formulas (8.17) are equivalent to the condition (7.3). They define the class BgdET4 coinciding with the intersection class ShrID1 ∩ BgdET4.

Immediately after the formulas (7.1) in [20] we see the formulas

$$\gamma^1=\frac{2\,j_1^{\text{Bgd}}}{5\,\beta_1^{\text{Bgd}}}-\frac{\beta_2^{\text{Bgd}}\,j_2^{\text{Bgd}}}{3},\qquad\qquad \gamma^2=\frac{\beta_1^{\text{Bgd}}\,j_2^{\text{Bgd}}}{3}.\tag{8.18}$$

The formulas (8.18) are valid. They are verified by direct calculations.

Now we can proceed to the third formula (7.2) in [20]. It is written as follows:

$$I_3=\frac{I_{12}^{\text{Bgd}}}{18}+\frac{I_2^{\text{Bgd}}}{90\,I_1^{\text{Bgd}}}\Bigl(5\,I_{11}^{\text{Bgd}}-3\,\bigl(I_1^{\text{Bgd}}\bigr)^2-6\Bigr)+\frac{5}{3}.\tag{8.19}$$

The invariants I_{12}^{Bgd} and I_{11}^{Bgd} in (8.19) are calculated by applying the invariant differentiation operator $\mathcal{D}_1^{\text{Bgd}}$ from [11] to I_2^{Bgd} and I_1^{Bgd} respectively:

$$I_{12}^{\text{Bgd}}=\mathcal{D}_1^{\text{Bgd}}(I_2^{\text{Bgd}}),\qquad\qquad I_{11}^{\text{Bgd}}=\mathcal{D}_1^{\text{Bgd}}(I_1^{\text{Bgd}}).\tag{8.20}$$

Using Lemma 7.2, the formulas (8.20) can be rewritten as

$$I_{12}^{\text{Bgd}}=\sqrt{\frac{2}{5\,M}}\,\nabla_{\boldsymbol{\alpha}}I_2^{\text{Bgd}},\qquad\qquad I_{11}^{\text{Bgd}}=\sqrt{\frac{2}{5\,M}}\,\nabla_{\boldsymbol{\alpha}}I_1^{\text{Bgd}}.\tag{8.21}$$

The invariants I_2^{Bgd} and I_1^{Bgd} in (8.20) and (8.21) are two basic invariants defined by Yu. Yu. Bagderina in [11] in item 4 of her Theorem 2. The invariant I_2^{Bgd} is given by the formula (6.4), the invariant I_1^{Bgd} is given by the formula (8.12). We can apply the formula (6.1) in order to derive an explicit formula for j_5^{Bgd} in (6.4). The formulas (6.1), (6.4), (8.14), and (8.21) are sufficient in order to verify the formula (8.19) by means of direct computations. It turns out that the formula (8.19) is written for the case where I_3 is redefined as

$$I_3\to\frac{I_3}{I_1}=\frac{\Gamma^1_{22}}{M}.\tag{8.22}$$

Probably (8.22) would be a better choice for I_3. But, historically in [3, 4] it was introduced in its present form (8.4). We need to rewrite the formula (8.19) as

$$\frac{I_3}{I_1}=\frac{I_{12}^{\text{Bgd}}}{18}+\frac{I_2^{\text{Bgd}}}{90\,I_1^{\text{Bgd}}}\Bigl(5\,I_{11}^{\text{Bgd}}-3\,\bigl(I_1^{\text{Bgd}}\bigr)^2-6\Bigr)+\frac{5}{3}.\tag{8.23}$$

The formula (8.23) is valid. It is verified by means of direct computations. Looking at the formula (8.23) and at few other formulas in [20] expressing I_4, I_6, I_7, and I_{10} through her invariants, Yu. Yu. Bagderina detected that they do not comprise her basic invariant I_2^{Bgd}, but only some derivatives of I_2^{Bgd}. As a result she issued a criticism saying that the invariants from [3, 4] are impractical for the equivalence problem. However, she omitted the invariant I_9 in the sequence I_4, I_6, I_7, and I_{10}. The invariants I_5 and I_8, which are also omitted, are zero due to $I_2 = 0$ (see (8.10) and (8.1)). But the invariant I_9 is nonzero. If Yu. Yu. Bagderina would not omit this invariant, she would have the following formula:

$$\begin{aligned}\sqrt{I_9}=&-\frac{\sqrt{3}}{45}\,\frac{I_{212}^{\text{Bgd}}}{\bigl(I_1^{\text{Bgd}}\bigr)^{3/2}}-\frac{\sqrt{3}}{225}\,\frac{\bigl(5\,I_{11}^{\text{Bgd}}-3\,\bigl(I_1^{\text{Bgd}}\bigr)^2-6\bigr)\,I_{22}^{\text{Bgd}}}{\bigl(I_1^{\text{Bgd}}\bigr)^{5/2}}\,+\\&+\biggl(\frac{\sqrt{3}}{450}\,\frac{\bigl(15\,I_{11}^{\text{Bgd}}+10\,\bigl(I_1^{\text{Bgd}}\bigr)^2-6\bigr)\,I_{21}^{\text{Bgd}}}{\bigl(I_1^{\text{Bgd}}\bigr)^{7/2}}-\frac{\sqrt{3}}{45}\,\frac{I_{121}^{\text{Bgd}}}{\bigl(I_1^{\text{Bgd}}\bigr)^{5/2}}\biggr)\,I_2^{\text{Bgd}}.\end{aligned}\tag{8.24}$$

The invariant I_2^{Bgd} is explicitly present in the formula (8.24). Opposing Yu. Yu. Bagderina, below we prove that, in spite of $\Omega = 0$, in spite of (8.23), and in spite of other her formulas in [20], her basic invariants I_1^{Bgd} and I_2^{Bgd} can be expressed through the invariants I_1 and I_3 from [3, 4] in her class BgdET4 coinciding with the intersection class ShrID1 ∩ BgdET4.

Let's recall that in [12] the following theorem was proved for Bagderina's pseudoscalar field j_5^{Bgd} from her paper [11].

Theorem 8.1. *Within the intersection class* ShrID1 ∩ BgdET2*, i. e. if the conditions $F = 0$, $\boldsymbol{\alpha} \neq 0$, $M \neq 0$, $\Omega \neq 0$ are fulfilled, Bagderina's pseudoscalar field j_5^{Bgd} is expressed through M, through Ω, through the pseudoscalar fields Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} from* (8.3)*, and through covariant derivatives of them along the pseudovectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$.*

The field j_5^{Bgd} is defined for all cases of intermediate degeneration. In is introduced in [11] by means of the following formula:

$$j_5^{\text{Bgd}} = 5\,\bigl(2\,j_1^{\text{Bgd}}\,j_3^{\text{Bgd}} + (j_2^{\text{Bgd}} - j_0^{\text{Bgd}}/6)^2\bigr).\tag{8.25}$$

As it was shown in [12], the formula (8.25) is equivalent to the formula (6.1) (see (7.5) in [12]). In our present case $j_0^{\text{Bgd}} = -3\,\Omega = 0$ (see (7.3) and (7.4)). Therefore the formula (8.25) reduces to the following formula:

$$j_5^{\text{Bgd}} = 5\,\bigl(2\,j_1^{\text{Bgd}}\,j_3^{\text{Bgd}} + (j_2^{\text{Bgd}})^2\bigr).\tag{8.26}$$

The formula (8.26) is given in item 4 of Bagderina's Theorem 2 in [11]. This formula defines j_5^{Bgd} in our present class BgdET4 coinciding with the intersection class ShrID1 ∩ BgdET4. It is equivalent to the reduced formula (6.1):

$$j_5^{\text{Bgd}} = -125\,\det(R).\tag{8.27}$$

The formula (8.27) is produced from (6.1) by setting $\Omega = 0$ in it.

Looking through the proof of Theorem 8.1 in [12], one can see that it does not depend on Ω otherwise than through the entry of Ω^2 in (6.1). Therefore, repeating the arguments from [12], we can prove the following theorem for Bagderina's pseudoscalar field j_5^{Bgd} in (8.26).

Theorem 8.2. *Within the intersection class* ShrID1 ∩ BgdET4 *coinciding with* BgdET4*, i. e. if the conditions* (7.3) *are fulfilled, Bagderina's pseudoscalar field* j_5^{Bgd} *from* (8.26) *is expressed through* M*, through the pseudoscalar fields* Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} *from* (8.3)*, and through covariant derivatives of them along the pseudovectorial fields* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\gamma}$*.*

Bagderina's invariant I_2^{Bgd} is given by the formula (6.4). The field j_5^{Bgd} is in the numerator of this formula, while its denominator is defined by the field M due to (7.4) or (8.17). As for the pseudoscalar fields Γ^1_{11}, Γ^2_{11}, Γ^1_{12}, Γ^2_{12}, Γ^1_{21}, Γ^2_{21}, Γ^1_{22}, Γ^2_{22} from (8.3), due to (8.4), (8.5), (8.6), (8.7), (8.8), and (8.9) they are expressed through scalar invariants I_1, I_2, I_3, I_4, I_7 and through the fields N and M. According to Theorem 8.2, when expressing j_5^{Bgd} we might need to differentiate these fields, i. e. calculate their covariant derivatives along $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Doing it, we shall produce higher order invariants in the sequence given by (8.11) and some covariant derivatives of N and M along $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$. Covariant derivatives of N and M are described by the following lemma.

Lemma 8.2. *In the case where* $\Omega = 0$ *covariant derivatives of the pseudoscalar fields* M *and* N *along the pseudovectorial fields* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\gamma}$ *are expressed through the scalar invariants* I_1, I_2, I_3, I_4 *etc in the recurrent sequence given by* (8.11) *and through these two fields themselves.*

Lemma 8.2 in the present paper is similar to Lemma 8.2 in [12]. This lemma is proved by means of the following explicit formulas:

$$\begin{aligned}
&\nabla_{\boldsymbol{\alpha}} N = M, &&\nabla_{\boldsymbol{\gamma}} N = 0,\\
&\nabla_{\boldsymbol{\alpha}} M = I_4\,N^3 + 2\,I_1\,N\,M, &&\nabla_{\boldsymbol{\gamma}} M = \sqrt{N^3\,I_7}\,N^2.
\end{aligned} \tag{8.28}$$

The formulas (8.28) are derived from the formulas (8.15) and (8.17) in [12] by setting $\Omega = 0$ in them. They should be applied repeatedly in order to calculate higher order covariant derivatives of M and N.

Now, combining Lemma 8.2 with Theorem 8.2 and with the arguments given just after Theorem 8.2, we derive the following theorem.

Theorem 8.3. *Within the intersection class* ShrID1 ∩ BgdET4 *coinciding with* BgdET4*, i. e. if the conditions* (7.3) *are fulfilled, Bagderina's basic invariant* I_2^{Bgd} *from* (6.4) *can be expressed through* I_1, I_3 *and through higher order invariants* I_4, I_6, I_7 *etc in the recurrent sequence given by* (8.11)*.*

9. Conclusions.

Three classes of umbilical equations, zero Gaussian curvature equations, and zero mean curvature equations are defined in the present paper. They specify the equations of the form (1.1) in all cases of intermediate degeneration. Generally speaking, these geometric classes do not fit into particular subcases of both classifications from [3, 4] and/or from [11].

Being intersected with the class ShrID1, which corresponds to the first case of intermediate degeneration, the class of zero mean curvature equations produces a subclass coinciding with the intersection class ShrID1 ∩ BgdET4 and with Bagderina's class BgdET4 of type four equations. We have compared two classifications from [3, 4] and from [11] within this intersection class. As a result we have found that most basic structures and basic formulas from Bagderina's paper [11] do coincide or are very closely related to those in [3, 4], though they are given in different notations (see Lemma 7.1, Lemma 7.2, and Lemma 7.3). Similar results for the case of general position and for the other intersection class ShrID1 ∩ BgdET2 were obtained in [19] and [12].

For her type four equations class BgdET4 in [11] Yu. Yu. Bagderina introduces two basic invariants I_1^{Bgd} and I_2^{Bgd}. For our class ShrID1, which covers Bagderina's class BgdET4, three basic invariants I_1, I_2, I_3 were introduced in [3, 4]. However within the intersection class ShrID1 ∩ BgdET4 the invariant I_2 vanishes, so we have two Bagderina's invariants versus two ours. In [20] Yu. Yu. Bagderina managed to express the invariants I_1 and I_3 through her invariants I_1^{Bgd}, I_2^{Bgd} and through their derivatives. In the present paper we present the converse result, i. e. we have proved that the invariants I_1^{Bgd} and I_2^{Bgd} can be expressed through our invariants I_1, I_2 and through their derivatives (see Lemma 8.1 and Theorem 8.3). Thus both sets of basic invariants are equally applicable to solving the equivalence problem for the equations (1.1) within the intersection class ShrID1 ∩ BgdET4.

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