# ON LINEAR REGRESSION IN THREE-DIMENSIONAL EUCLIDEAN SPACE. 

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#### Abstract

The three-dimensional linear regression problem is a problem of finding a spacial straight line best fitting a group of points in three-dimensional Euclidean space. This problem is considered in the present paper and a solution to it is given in a coordinate-free form.


## 1. Introduction.

The linear regression problem in two-dimensional case (i.e. on a plane) typically arises when approximating experimental data with a linear function (see [1]). Its solution using least squares method was first published by Legendre in 1805 (see [2]). In an unpublished form the least squares method is attributed to Carl Friedrich Gauss 1795. His work was published only in 1809 (see [3]).

There are various fitting problems in three-dimensional Euclidean space (see plane, circle and ellipse fitting problems in [4] and [5], see ellipsoid fitting problem in [6] and [7]). The linear regression problem in three-dimensional case is the problem of best fitting some straight line to a group of points in three-dimensional Euclidean space. A solution of this problem is given by Jean Jacquelin in [8]. His method is substantially based on direct calculations using coordinates. Our goal in the present paper is to give a coordinate-free solution to the problem.

## 2. Parametric and non-Parametric <br> VECTORIAL EQUATIONS OF A STRAIGHT LINE.

Let's consider the straight line $A X$ in Fig 2.1. The point $A$ is a fixed point of this line, its radius-vector is $\mathbf{r}_{0}$. The point $X$ is a variable point, its radius-vector is $\mathbf{r}$. These two radius-vectors are related


Fig. 2.1
to each other by means of the equation

$$
\begin{equation*}
\mathbf{r}=\mathbf{r}_{0}+\mathbf{a} t \tag{2.1}
\end{equation*}
$$

where a is some non-zero vector on the line and $t$ is a scalar parameter. The equality (2.1) is called the vectorial parametric equation of the line in the space (see [9]).

The choice of the point $A$ on the line is not unique. Therefore the equation (2.1) has some extent of ambiguity. In order to avoid this ambiguity non-parametric

[^0]equations are used. Let's multiply both sides of the equality (2.1) by the vector a using the vector product ${ }^{1}$ operation. As a result we get
\[

$$
\begin{equation*}
[\mathbf{r}, \mathbf{a}]=\left[\mathbf{r}_{0}, \mathbf{a}\right] . \tag{2.2}
\end{equation*}
$$

\]

The product of two constant vectors in the right hand side of (2.2) is a constant vector. If we denote it through $\mathbf{b}$, we get the equality

$$
\begin{equation*}
[\mathbf{r}, \mathbf{a}]=\mathbf{b}, \quad \text { where } \mathbf{b} \perp \mathbf{a} . \tag{2.3}
\end{equation*}
$$

The equality (2.3) is known as the non-parametric vectorial equation of the line in the space (see [9]). Note that the vector $\mathbf{b}=\left[\mathbf{r}_{0}, \mathbf{a}\right]$ has no ambiguity arising from the uncertainty in choosing the initial point $A$ on the line. Indeed, it is easy to see that $\mathbf{b}$ is invariant with respect to the transformation $\mathbf{r}_{0} \rightarrow \mathbf{r}_{0}+\mathbf{a} t$.

## 3. The statement of the problem.

Let $X_{1}, \ldots, X_{n}$ be a group of points in the space given by their radius-vectors $\mathbf{r}_{1}, \ldots, \mathbf{r}_{n}$. The linear regression problem consists in finding a line given by the equation (2.2) such that the root mean square of the distances $d_{1}, \ldots, d_{n}$ from the points $X_{1}, \ldots, X_{n}$ to the line (2.2) takes its minimal value:

$$
\begin{equation*}
\bar{d}^{2}=\frac{1}{n} \sum_{i=1}^{n} d_{i}^{2} \tag{3.1}
\end{equation*}
$$

## 4. The solution of the problem.

The distance from the point $X_{i}$ to the line (2.1) is given by the formula

$$
\begin{equation*}
d_{i}=\frac{\left|\left[\mathbf{r}_{i}-\mathbf{r}_{0}, \mathbf{a}\right]\right|}{|\mathbf{a}|} \tag{4.1}
\end{equation*}
$$

Without loss of generality we can assume that

$$
\begin{equation*}
|\mathbf{a}|=1 \tag{4.2}
\end{equation*}
$$

Then, taking into account $\mathbf{b}=\left[\mathbf{r}_{0}, \mathbf{a}\right]$ and (4.2), from (4.1) we derive

$$
\begin{equation*}
d_{i}=\left|\left[\mathbf{r}_{i}, \mathbf{a}\right]-\mathbf{b}\right| . \tag{4.3}
\end{equation*}
$$

Now we substitute (4.3) into (3.1). As a result we obtain

$$
\begin{equation*}
\bar{d}^{2}=|\mathbf{b}|^{2}-\frac{2}{n} \sum_{i=1}^{n}\left(\left[\mathbf{r}_{i}, \mathbf{a}\right], \mathbf{b}\right)+\frac{1}{n} \sum_{i=1}^{n}\left|\left[\mathbf{r}_{i}, \mathbf{a}\right]\right|^{2} \tag{4.4}
\end{equation*}
$$

The formula (4.4) is an analog of the formula (2.3) in [4]. The round brackets in (4.4) denote the scalar product ${ }^{2}$ operation.

Definition 4.1. A line given by the equation (2.3) with $|\mathbf{a}|=1$ is called an optimal root mean square line if the quantity (4.4) takes its minimal value.

[^1]The right hand side of (4.4) is a quadratic polynomial with respect to the components of the vector $\mathbf{b}$. It takes its minimal value if $\mathbf{b}$ is given by the formula

$$
\begin{equation*}
\mathbf{b}=\frac{1}{n} \sum_{i=1}^{n}\left[\mathbf{r}_{i}, \mathbf{a}\right] \tag{4.5}
\end{equation*}
$$

Substituting (4.5) back into (4.3), we derive

$$
\begin{equation*}
\bar{d}^{2}=\frac{1}{n} \sum_{i=1}^{n}\left|\left[\mathbf{r}_{i}, \mathbf{a}\right]\right|^{2}-\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{j=1}^{n}\left(\left[\mathbf{r}_{i}, \mathbf{a}\right],\left[\mathbf{r}_{j}, \mathbf{a}\right]\right) \tag{4.6}
\end{equation*}
$$

The formula (4.6) is an analog of the formula (2.5) in [4]. Its right hand side is a quadratic form wit respect to the vector $\mathbf{a}$. We denote it through $Q(\mathbf{a}, \mathbf{a})$ :

$$
\begin{equation*}
Q(\mathbf{a}, \mathbf{a})=\frac{1}{n} \sum_{i=1}^{n}\left|\left[\mathbf{r}_{i}, \mathbf{a}\right]\right|^{2}-\frac{1}{n^{2}} \sum_{i=1}^{n} \sum_{j=1}^{n}\left(\left[\mathbf{r}_{i}, \mathbf{a}\right],\left[\mathbf{r}_{j}, \mathbf{a}\right]\right) \tag{4.7}
\end{equation*}
$$

and call the non-linearity form for a group of points in three-dimensional Euclidean space. Like the non-flatness form (2.14) in [4], the non-linearity form (4.7) is positive in the sense of the following inequality:

$$
Q(\mathbf{a}, \mathbf{a}) \geqslant 0 \text { for } \mathbf{a} \neq 0
$$

Like in [4] one can draw some analogy to mechanics using the inertia tensor. However, we shall not do it now. We just note that like any quadratic form $Q(\mathbf{a}, \mathbf{a})$ diagonalizes in some orthonormal basis associated with its primary axes.

Let's introduce the following notation analogous to (2.6) in [4]:

$$
\begin{equation*}
\mathbf{r}_{\mathrm{cm}}=\frac{1}{n} \sum_{i=1}^{n} \mathbf{r}_{i} \tag{4.8}
\end{equation*}
$$

The vector $\mathbf{r}_{\mathrm{cm}}$ in (4.8) is the radius-vector of the center of mass of a group of points $X_{1}, \ldots, X_{n}$ if assume that unit masses are placed at each of these points. In terms of (4.8) the formula (4.5) is written as

$$
\begin{equation*}
\mathbf{b}=\left[\mathbf{r}_{\mathrm{cm}}, \mathbf{a}\right] . \tag{4.9}
\end{equation*}
$$

Comparing (4.9) with $\mathbf{b}=\left[\mathbf{r}_{0}, \mathbf{a}\right]$, we conclude that the optimal line should pass through the center of mass of a group of points. Its direction is determined by the non-linearity form $Q(\mathbf{a}, \mathbf{a})$ according to the following theorem.

Theorem 4.1. A line is an optimal root mean square line for a group of points if and only if it passes through the center of mass of these points and if its direction vector $\mathbf{a}$ is directed along the primary axis of the non-linearity form $Q$ of these points corresponding to its minimal eigenvalue.

## 5. Conclusion.

Theorem 4.1 solves the linear regression problem formulated in Section 3. Its proof is obvious from the consideration preceding it. Practically this theorem means that in order to find a line best fitting a group of points in three-dimensional Euclidean space one should find their center of mass and diagonalize the symmetric matrix associated with their non-linearity form (4.7). In some cases this matrix can have two minimal eigenvalues $\lambda_{1}=\lambda_{2}<\lambda_{3}$. In these cases the shape of the group of points resembles a disc and hence there is no preferable direction for the optimal line within the plane of this disc.

If $\lambda_{1}=\lambda_{2}=\lambda_{3}$, the shape of the group of points resembles a ball. In this case we have no preferable direction for the optimal line at all.

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[^0]:    2010 Mathematics Subject Classification. 51N20, 68W25.

[^1]:    ${ }^{1}$ It is also called the cross product, i. e. $[\mathbf{x}, \mathbf{y}]=\mathbf{x} \times \mathbf{y}$.
    ${ }^{2}$ It is also called the dot product, i. e. $(\mathbf{x}, \mathbf{y})=\mathbf{x} \cdot \mathbf{y}$.

