# ON A SIMPLIFIED VERSION OF HADAMARD'S MAXIMAL DETERMINANT PROBLEM 

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#### Abstract

Hadamard's maximal determinant problem consists in finding the maximal value of the determinant of a square $n \times n$ matrix whose entries are plus or minus ones. This is a difficult mathematical problem which is not yet solved. In the present paper a simplified version of the problem is considered and studied numerically.


## 1. Introduction.

Hadamard's maximal determinant problem was first published in [1] by Jacques Salomon Hadamard ${ }^{1}$ in 1893. Let's denote

$$
M(n)=\max _{a_{i j}= \pm 1}\left|\begin{array}{ccc}
a_{11} & \ldots & a_{1 n}  \tag{1.1}\\
\vdots & \ddots & \vdots \\
a_{n 1} & \ldots & a_{n n}
\end{array}\right|
$$

The problem is to find $M(n)$ in (1.1) for each $n \in \mathbb{N}$. There is a closely related problem of finding $d_{n}$, where

$$
d_{n}=\max _{a_{i j} \in\{0,1\}}\left|\begin{array}{ccc}
a_{11} & \ldots & a_{1 n}  \tag{1.2}\\
\vdots & \ddots & \vdots \\
a_{n 1} & \ldots & a_{n n}
\end{array}\right|
$$

According to [5], the relation of two problems (1.1) and (1.2) is given by the formula

$$
\begin{equation*}
M(n)=2^{n-1} D(n), \text { where } D(n)=d_{n-1} . \tag{1.3}
\end{equation*}
$$

There are various estimates for $M(n), D(n)$, and $d_{n}$ (see [1], [5], and [6-12]). However the exact values of $M(n)$ and $D(n)$ in (1.3) are known only for $n \leqslant 21$. The exact values of $d_{n}$ are known for $n \leqslant 20$ (see [5]).

The goal of the present paper is neither to improve existing estimates not to give new ones. Here we consider a different problem which share some features of

[^0]Hadamard's problem, but is not dependent of it. Acting like in (1.2), let's denote through $A_{n}$ a square $n \times n$ matrix whose elements are zeros and ones:

$$
A_{n}=\left\|\begin{array}{ccc}
a_{11} & \ldots & a_{1 n}  \tag{1.4}\\
\vdots & \ddots & \vdots \\
a_{n 1} & \ldots & a_{n n}
\end{array}\right\|, \text { where } a_{i j} \in\{0,1\}
$$

The problem then is formulated as follows.
Problem 1.1. Starting from the unit matrix $A_{1}=\|1\|$ for $n=1$, find a sequence $A_{n}$ of square $n \times n$ matrices of the form (1.4) each next of which comprises the previous one as its upper left diagonal block and is of maximal determinant in the so described class.

We shall call this problem the simplified Hadamard maximal determinant problem until some more convenient name will be suggested.

## 2. Computer code for solving the problem.

The first three matrices $A_{1}, A_{3}, A_{3}$, in the series are easily written

$$
A_{1}=\|1\|, \quad A_{2}=\left\|\begin{array}{ll}
1 & 0  \tag{2.1}\\
1 & 1
\end{array}\right\|, \quad A_{3}=\left\|\begin{array}{lll}
1 & 0 & 1 \\
1 & 1 & 0 \\
0 & 1 & 1
\end{array}\right\| .
$$

The matrices (2.1) provide the following series of maximal determinants $b_{i}=\operatorname{det} A_{i}$ :

$$
\begin{equation*}
b_{1}=1, \quad b_{2}=1, \quad b_{3}=2 \tag{2.2}
\end{equation*}
$$

Further terms in (2.1) and (2.2) are produced computationally. Since the definition of the matrices $A_{n}$ in Problem 1.1 is by induction, the matrix $A_{n+1}$ looks like

$$
A_{n+1}=\left\|\begin{array}{cccc}
a_{11} & \ldots & a_{1 n} & y_{1}  \tag{2.3}\\
\vdots & \ddots & \vdots & \vdots \\
a_{n 1} & \ldots & a_{n n} & y_{n} \\
x_{1} & \ldots & x_{n} & z
\end{array}\right\|
$$

When computing $A_{n+1}$ its upper left diagonal block is already known, it coincides with $A_{n}$. The determinant of the matrix (2.3) is given by the formula

$$
\begin{equation*}
\operatorname{det} A_{n+1}=z \operatorname{det} A_{n}+\sum_{i=1}^{n} \sum_{j=1}^{n} b_{i j} x_{i} y_{j} \tag{2.4}
\end{equation*}
$$

Since $\operatorname{det} A_{n}>0$, the choice of $z$ in the formula (2.4) is obvious from the maximal determinant condition in Problem 1.1:

$$
\begin{equation*}
z=1 \tag{2.5}
\end{equation*}
$$

Substituting (2.5) into (2.4), we derive

$$
\begin{equation*}
\operatorname{det} A_{n+1}=\operatorname{det} A_{n}+\sum_{i=1}^{n} \sum_{j=1}^{n} b_{i j} x_{i} y_{j} \tag{2.6}
\end{equation*}
$$

The double sums in (2.6) constitute a bilinear form with respect to the variables $x_{1}, \ldots, x_{n}$ and $y_{1}, \ldots, y_{n}$. The coefficients $b_{i j}$ of this bilinear form are expressed through the entries of the matrix $A_{n}$. The following code was used for computing the determinant $\operatorname{det}\left(A_{n+1}\right)$ in (2.6).

```
function_to_maximize(PM):=block
    ([n,M,i,j,D],
        n:length(PM)+1,
        M:zeromatrix(n,n),
        for i:1 step 1 thru n-1 do
        for j:1 step 1 thru n-1 do
            M[i,j]:PM[i,j],
        for i:1 step 1 thru n-1 do
        (
            M[n,i]:x[i],
            M[i,n]:y[i]
        )
        M[n,n]:1,
        M_max:copy(M)
        D:determinant(M),
        D:ratexpand(D),
        return(D)
)$
```

This code defines a function with the name function_to_maximize, where [ $n, M$, $i, j, D]$ is the list of its local variables. The function function_to_maximize uses one global variable M_max in order to output the matrix (2.3) with $z=1$. The argument PM of this function is used in order to load the input matrix $A_{n}$ of the form (1.4). The result of this function is the expression of the form (2.6) that should be maximized by choosing proper values for the variables $x_{i}$ and $y_{i}$.

The above code is written in Maxima programming language. Maxima is a Computer Algebra System available for Linux, Windows, and MacOS. I run Maxima version 5.42.2 on the Linux platform Ubuntu 16.04 LTS.

Another code is used in order to maximize the expression (2.6). It is as follows.

```
compute_max_det(n,F):=block
    ([i,j,P,r,rr],
        r:0,
        if n=0
        then
            (
                    if F>D_max
                    then
                    (
                        D_max:F,
                        r:1
                )
        )
        else
            (
```

```
        for i:0 step 1 thru 1 do
        for j:0 step 1 thru 1 do
            (
                P:psubst([x[n]=i,y[n]=j],F),
                rr:compute_max_det(n-1,P),
                if rr=1
                then
                    (
                        x_max[n]:i,
                y_max[n]:j,
                    r:1
                )
        )
    ),
    return(r)
)$
```

This code defines the function compute_max_det with local variables i, j, $\mathrm{P}, \mathrm{r}, \mathrm{rr}$ and with two arguments $n, F$. The argument $F$ is used in order to load the expression (2.6), the argument n admits the value of the number $n$ in (2.6). The function compute_max_det uses three global variables D_max, x_max,y_max. The variable D_max is used in order to output the maximal value of expression (2.6), i.e. the maximal value of the determinant. The global variables $x \_m a x, y \_m a x ~ a r e ~ u s e d ~ a s ~ u n d e c l a r e d ~$ arrays. Through them we output the values of $x_{1}, \ldots, x_{n}$ and $y_{1}, \ldots, y_{n}$ at which the maximum of the determinant (2.6) is attained. Substituting these values along with (2.5) into (2.4), we get the matrix $A_{n+1}$ at which the maximum of the determinant is reached.

The main code for computing several matrices $A_{n}$ looks like a loop with respect to the integer variable $n$. We start with the matrix $A_{3}$ in (2.1).

```
file_output_append:true$
MM:matrix([1,0,1],[1, 1, 0],[0,1,1])$
for n:3 step 1 thru 14 do
    (
        FF:function_to_maximize(MM),
        N_max:length(listofvars(FF))/2,
        D_max:0,
        compute_max_det(N_max,FF),
        MM: copy(M_max),
        for i:1 step 1 thru N_max do
        MM:psubst([x[i]=x_max[i], y[i]=y_max[i]],MM),
    stringout("output_file",[n+1,D_max,MM])
    )$
```

The matrices $A_{n}$ along with their determinants are written to the output_file.

## 3. The result of computations and conclusions.

The above code was run once in a loop for $n$ from $n=3$ through $n=14$. As a result the matrices $A_{4}, A_{5}, A_{6}, A_{7}, A_{8}, A_{9}, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}$ were computed in addition to the matrices (2.1). According to the statement of

Problem 1.1, these matrices are enclosed in each other like "matryoshkas" (nesting dolls) in the form of upper left diagonal blocks. Therefore it is sufficient to typeset only the last one of them, i.e. the matrix $A_{15}$ :

$$
A_{15}=\left\|\begin{array}{lllllllllllllll}
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right\| .
$$

Along with the matrices we obtain their determinants which are maximal in their classes. Indeed, according to the statement of Problem 1.1, we have

$$
b_{n+1}=\operatorname{det}\left(A_{n+1}\right)=\max _{\substack{x_{1}, \ldots, x_{n} \in\{0,1\}  \tag{3.1}\\
y_{1}, \ldots, y_{n} \in\{0,1\}}}\left|\begin{array}{cccc}
a_{11} & \ldots & a_{1 n} & y_{1} \\
\vdots & \ddots & \vdots & \vdots \\
a_{n 1} & \ldots & a_{n n} & y_{n} \\
x_{1} & \ldots & x_{n} & 1
\end{array}\right|
$$

For the purposes of comparison with the original Hadamard's problem we provide the quantities $b_{n}$ along with the quantities $d_{n}$ from (1.3):

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{n}$ | 1 | 1 | 2 | 3 | 5 | 9 | 18 | 40 | 96 |
| $d_{n}$ | 1 | 1 | 2 | 3 | 5 | 9 | 32 | 56 | 144 |


| $n$ | 10 | 11 | 12 | 13 | 14 | 15 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $b_{n}$ | 220 | 604 | 1608 | 4734 | 14898 | 45034 |
| $d_{n}$ | 320 | 1458 | 3645 | 9477 | 25515 | 131073 |

The inequality $b_{n} \leqslant d_{n}$ observed in the above tables is not surprising. The quantities $d_{n}$ in (1.2) are defined as total maxima with respect to all entries of the matrices, while $b_{n}$ in (3.1) are partial maxima calculated for the case where entries of the upper left blocks of the matrices are fixed.

Note that the matrix $A_{2}$ in the sequence (2.1) is not unique. There are two other equivalent options for choosing this matrix:

$$
A_{2}=\left\|\begin{array}{ll}
1 & 1  \tag{3.2}\\
0 & 1
\end{array}\right\|, \quad A_{2}=\left\|\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right\|
$$

These two options (3.2) are associated with the same value of $b_{2}$. Such a nonuniqueness can happen for $n>2$ as well. Therefore the solution of the simplified Hadamard maximal determinant problem is a collection of sequences rather than a single sequence of matrices. Each two sequences of this collection share some initial part thus producing a tree structure in the collection as a whole.

The above code produces only one sequence of the collection. This means that the research of the simplified Hadamard maximal determinant problem should be continued. Probably this would contribute to the solution of the original Hadamard's maximal determinant problem as well.

## 4. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

## 5. Acknowledgments.

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    ${ }^{1}$ Mark Kac in [2] supposed that the problem belongs to Maurice René Fréchet. Fréchet was a student of Hadamard at secondary school Lycée Buffon in Paris, so the version has some ground. This version is supported in [3]. With the reference to [3] in the Russian segment of Wikipedia the problem is called Fréchet's problem of maximal determinant (see [4]).

