ON A SIMPLIFIED VERSION OF HADAMARD'S MAXIMAL DETERMINANT PROBLEM

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ABSTRACT. Hadamard's maximal determinant problem consists in finding the maximal value of the determinant of a square $n \times n$ matrix whose entries are plus or minus ones. This is a difficult mathematical problem which is not yet solved. In the present paper a simplified version of the problem is considered and studied numerically.

1. INTRODUCTION.

Hadamard's maximal determinant problem was first published in [1] by Jacques Salomon Hadamard¹ in 1893. Let's denote

$$M(n) = \max_{a_{ij} = \pm 1} \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$
 (1.1)

The problem is to find M(n) in (1.1) for each $n \in \mathbb{N}$. There is a closely related problem of finding d_n , where

$$d_{n} = \max_{a_{ij} \in \{0,1\}} \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}.$$
 (1.2)

According to [5], the relation of two problems (1.1) and (1.2) is given by the formula

$$M(n) = 2^{n-1} D(n)$$
, where $D(n) = d_{n-1}$. (1.3)

There are various estimates for M(n), D(n), and d_n (see [1], [5], and [6–12]). However the exact values of M(n) and D(n) in (1.3) are known only for $n \leq 21$. The exact values of d_n are known for $n \leq 20$ (see [5]).

The goal of the present paper is neither to improve existing estimates not to give new ones. Here we consider a different problem which share some features of

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¹ Mark Kac in [2] supposed that the problem belongs to Maurice René Fréchet. Fréchet was a student of Hadamard at secondary school Lycée Buffon in Paris, so the version has some ground. This version is supported in [3]. With the reference to [3] in the Russian segment of Wikipedia the problem is called Fréchet's problem of maximal determinant (see [4]).

RUSLAN SHARIPOV

Hadamard's problem, but is not dependent of it. Acting like in (1.2), let's denote through A_n a square $n \times n$ matrix whose elements are zeros and ones:

$$A_{n} = \left\| \begin{array}{ccc} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{array} \right\|, \text{ where } a_{ij} \in \{0, 1\}.$$
(1.4)

The problem then is formulated as follows.

Problem 1.1. Starting from the unit matrix $A_1 = ||1||$ for n = 1, find a sequence A_n of square $n \times n$ matrices of the form (1.4) each next of which comprises the previous one as its upper left diagonal block and is of maximal determinant in the so described class.

We shall call this problem the simplified Hadamard maximal determinant problem until some more convenient name will be suggested.

2. Computer code for solving the problem.

The first three matrices A_1 , A_3 , A_3 , in the series are easily written

$$A_1 = \|1\|, \qquad A_2 = \| \begin{array}{ccc} 1 & 0 \\ 1 & 1 \\ \end{array} \|, \qquad A_3 = \| \begin{array}{ccc} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ \end{array} \|.$$
(2.1)

The matrices (2.1) provide the following series of maximal determinants $b_i = \det A_i$:

$$b_1 = 1,$$
 $b_2 = 1,$ $b_3 = 2.$ (2.2)

Further terms in (2.1) and (2.2) are produced computationally. Since the definition of the matrices A_n in Problem 1.1 is by induction, the matrix A_{n+1} looks like

$$A_{n+1} = \begin{vmatrix} a_{11} & \dots & a_{1n} & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} & y_n \\ x_1 & \dots & x_n & z \end{vmatrix} .$$
(2.3)

When computing A_{n+1} its upper left diagonal block is already known, it coincides with A_n . The determinant of the matrix (2.3) is given by the formula

$$\det A_{n+1} = z \, \det A_n + \sum_{i=1}^n \sum_{j=1}^n b_{ij} \, x_i \, y_j.$$
(2.4)

Since det $A_n > 0$, the choice of z in the formula (2.4) is obvious from the maximal determinant condition in Problem 1.1:

$$z = 1. \tag{2.5}$$

Substituting (2.5) into (2.4), we derive

$$\det A_{n+1} = \det A_n + \sum_{i=1}^n \sum_{j=1}^n b_{ij} \, x_i \, y_j.$$
(2.6)

The double sums in (2.6) constitute a bilinear form with respect to the variables x_1, \ldots, x_n and y_1, \ldots, y_n . The coefficients b_{ij} of this bilinear form are expressed through the entries of the matrix A_n . The following code was used for computing the determinant det (A_{n+1}) in (2.6).

```
function_to_maximize(PM):=block
 ([n,M,i,j,D],
  n:length(PM)+1,
  M:zeromatrix(n,n),
  for i:1 step 1 thru n-1 do
   for j:1 step 1 thru n-1 do
    M[i,j]:PM[i,j],
  for i:1 step 1 thru n-1 do
   (
    M[n,i]:x[i],
    M[i,n]:y[i]
   )
  M[n,n]:1,
  M_max:copy(M),
  D:determinant(M),
  D:ratexpand(D),
  return(D)
 )$
```

This code defines a function with the name function_to_maximize, where [n,M, i,j,D] is the list of its local variables. The function function_to_maximize uses one global variable M_max in order to output the matrix (2.3) with z = 1. The argument PM of this function is used in order to load the input matrix A_n of the form (1.4). The result of this function is the expression of the form (2.6) that should be maximized by choosing proper values for the variables x_i and y_i .

The above code is written in Maxima programming language. Maxima is a Computer Algebra System available for Linux, Windows, and MacOS. I run Maxima version 5.42.2 on the Linux platform Ubuntu 16.04 LTS.

Another code is used in order to maximize the expression (2.6). It is as follows.

```
compute_max_det(n,F):=block
```

```
([i,j,P,r,rr],
r:0,
if n=0
then
   (
    if F>D_max
    then
        (
            D_max:F,
            r:1
        )
   )
else
   (
```

```
for i:0 step 1 thru 1 do
     for j:0 step 1 thru 1 do
      (
       P:psubst([x[n]=i,y[n]=j],F),
       rr:compute_max_det(n-1,P),
       if rr=1
        then
         (
          x_max[n]:i,
          y_max[n]:j,
          r:1
         )
      )
   ),
return(r)
)$
```

This code defines the function compute_max_det with local variables i, j, P, r, rrand with two arguments n, F. The argument F is used in order to load the expression (2.6), the argument n admits the value of the number n in (2.6). The function compute_max_det uses three global variables D_max, x_max, y_max . The variable D_max is used in order to output the maximal value of expression (2.6), i.e. the maximal value of the determinant. The global variables x_max, y_max are used as undeclared arrays. Through them we output the values of x_1, \ldots, x_n and y_1, \ldots, y_n at which the maximum of the determinant (2.6) is attained. Substituting these values along with (2.5) into (2.4), we get the matrix A_{n+1} at which the maximum of the determinant is reached.

The main code for computing several matrices A_n looks like a loop with respect to the integer variable n. We start with the matrix A_3 in (2.1).

```
file_output_append:true$
MM:matrix([1,0,1],[1,1,0],[0,1,1])$
for n:3 step 1 thru 14 do
  (
    FF:function_to_maximize(MM),
    N_max:length(listofvars(FF))/2,
    D_max:0,
    compute_max_det(N_max,FF),
    MM:copy(M_max),
    for i:1 step 1 thru N_max do
        MM:psubst([x[i]=x_max[i], y[i]=y_max[i]],MM),
        stringout("output_file",[n+1,D_max,MM])
)$
```

The matrices A_n along with their determinants are written to the **output_file**.

3. The result of computations and conclusions.

The above code was run once in a loop for n from n = 3 through n = 14. As a result the matrices A_4 , A_5 , A_6 , A_7 , A_8 , A_9 , A_{10} , A_{11} , A_{12} , A_{13} , A_{14} , A_{15} were computed in addition to the matrices (2.1). According to the statement of

4

Problem 1.1, these matrices are enclosed in each other like "matryoshkas" (nesting dolls) in the form of upper left diagonal blocks. Therefore it is sufficient to typeset only the last one of them, i.e. the matrix A_{15} :

	1	0	1	1	0	0	0	0	0	0	1	0	0	1	$1 \parallel$	
	1	1	0	0	0	1	0	1	1	1	1	0	0	0	0	
	0	1	1	0	1	0	0	1	0	0	1	1	0	1	0	
	0	1	0	1	1	1	0	0	0	0	0	0	1	0	1	
	1	0	0	0	1	1	1	1	0	0	0	0	0	1	0	
	0	0	1	0	0	1	1	0	1	0	0	0	1	1	0	
	0	1	0	1	0	0	1	1	0	0	1	0	1	1	0	
$A_{15} =$	0	0	1	1	0	1	0	1	0	1	0	1	1	0	0	.
	0	0	0	1	1	0	0	1	1	1	0	0	0	1	1	
	1	1	1	0	1	0	1	0	0	1	0	0	1	0	0	
	0	0	0	0	1	1	1	0	0	1	1	1	0	0	1	
	1	1	0	1	0	0	1	0	1	0	0	1	0	0	0	
	1	0	0	0	1	0	0	1	1	0	1	1	1	0	1	
	1	1	0	0	0	1	0	0	0	1	0	1	1	1	1	
	0	1	1	0	0	0	1	1	0	0	0	0	0	0	$1 \mid$	

Along with the matrices we obtain their determinants which are maximal in their classes. Indeed, according to the statement of Problem 1.1, we have

$$b_{n+1} = \det(A_{n+1}) = \max_{\substack{x_1, \dots, x_n \in \{0,1\}\\y_1, \dots, y_n \in \{0,1\}}} \begin{vmatrix} a_{11} & \dots & a_{1n} & y_1\\ \vdots & \ddots & \vdots & \vdots\\ a_{n1} & \dots & a_{nn} & y_n\\ x_1 & \dots & x_n & 1 \end{vmatrix}.$$
 (3.1)

For the purposes of comparison with the original Hadamard's problem we provide the quantities b_n along with the quantities d_n from (1.3):

n	1	2	3	4	5	6	7	8	3	9	
b_n	1	1	2	3	5	9	18	4	0	96	
d_n	1	1	2	3	5	9	32	5	6	144	
n	10		11	12		13	14		15		
b_n	220	0 6	604	160	8 4	1734	14898		45034		
d_n	320) 1	458	364	5 9	9477	25515		131073		

The inequality $b_n \leq d_n$ observed in the above tables is not surprising. The quantities d_n in (1.2) are defined as total maxima with respect to all entries of the matrices, while b_n in (3.1) are partial maxima calculated for the case where entries of the upper left blocks of the matrices are fixed.

Note that the matrix A_2 in the sequence (2.1) is not unique. There are two other equivalent options for choosing this matrix:

$$A_{2} = \left\| \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right\|, \qquad \qquad A_{2} = \left\| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right\|. \tag{3.2}$$

RUSLAN SHARIPOV

These two options (3.2) are associated with the same value of b_2 . Such a nonuniqueness can happen for n > 2 as well. Therefore the solution of the simplified Hadamard maximal determinant problem is a collection of sequences rather than a single sequence of matrices. Each two sequences of this collection share some initial part thus producing a tree structure in the collection as a whole.

The above code produces only one sequence of the collection. This means that the research of the simplified Hadamard maximal determinant problem should be continued. Probably this would contribute to the solution of the original Hadamard's maximal determinant problem as well.

4. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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References

- Hadamard J., Résolution d'une question relative aux determinants, Bulletin des Sciences Mathématiques 17 (1893), 240–246.
- 2. Kac M., *Probability and related topics in physical sciences*, Lectures in applied mathematics, proceedings of the summer seminar, Volume I, Boulder, Colorado, 1957.
- Yadrenko M. I., Leonenko N. N., On some unsolved problems of analysis, combinatorics, and probability, Mathematics today, collection of scientific papers (A. Ya. Dorogovtsev, ed.), Vishcha shkola publishers, Kiev, 1982, pp. 94–111.
- 4. List of unsolved problems in mathematics (Rissian), Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
- 5. *Hadamard's maximal determinant problem*, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
- 6. Sylvester J. J., Thoughts on inverse orthogonal matrices, simultaneous sign successions, and tessellated pavements in two or more colours, with applications to Newton's rule, ornamental tile-work, and the theory of numbers, London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science 34 (1867), 461–475.
- Barba G., Intorno al teorema di Hadamard sui determinanti a valore massimo, Giornale di Matematiche di Battaglini 71 (1933), 70–86.
- Ehlich H., Determinantenabschätzungen für binäre Matrizen, Mathematische Zeitschrift 83 (1964), 123–132, doi: 10.1007/BF01111249.
- Wojtas M., On Hadamard's inequality for the determinants of order non-divisible by 4, Colloquium Mathematicum 12 (1964), 73–83.
- 10. Ehlich H., Determinantenabschatzungen fur binare Matrizen mit $n \equiv 3 \mod 4$, Mathematische Zeitschrift **84** (1964), 438–447, doi: 10.1007/BF01109911.
- 11. Cohn J. H. E., Almost D-optimal designs, Utilitas Mathematica 57 (2000), 121-128.
- Tamura H., D-optimal designs and group divisible designs, Journal of Combinatorial Designs 14 (2006), 451–462, doi: 10.1002/jcd.20103.

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