# HADAMARD MATRICES IN $\{0,1\}$ PRESENTATION AND AN ALGORITHM FOR GENERATING THEM. 

Ruslan Sharipov


#### Abstract

Hadamard matrices are square $n \times n$ matrices whose entries are ones and minus ones and whose rows are orthogonal to each other with respect to the standard scalar product in $\mathbb{R}^{n}$. Each Hadamard matrix can be transformed to a matrix whose entries are zeros and ones. This presentation of Hadamard matrices is investigated in the paper and based on it an algorithm for generating them is designed.


## 1. Introduction.

Hadamard matrices are known for $n=1$, for $n=2$, and for $n=4 q$, where $q \in \mathbb{N}$ and $\mathbb{N}$ is the set of positive integers. However it is not yet known if they do exist for all $q \in \mathbb{N}$ (see [1]). Hadamard matrices are associated with Hadamard's maximal determinant problem (see [2] and [3]) and solve this problem for $n=4 q$, where $q \in \mathbb{N}$. For the general case $n \in \mathbb{N}$ Hadamard's maximal determinant problem is yet unsolved. Its simplified version is suggested in [4].

Let $H$ be an $n \times n$ Hadamard matrix. By definition its rows considered as vectors in $\mathbb{R}^{n}$ are orthogonal to each other with respect to the standard scalar product in $\mathbb{R}^{n}$. The same is valid for its columns. The proof is elementary. Indeed, since $\left|\mathbf{r}_{i}\right|^{2}=n$ for each row $\mathbf{r}_{i}$ treated as a vector, the orthogonality of rows implies

$$
\begin{equation*}
\sum_{k=1}^{n} H_{i k} H_{j k}=n \delta_{i j} \tag{1.1}
\end{equation*}
$$

where $\delta_{i j}$ is the Kronecker delta (see $\S 23$ in Chapter I of [5]). The equality (1.1) means $H H^{\top}=n I$, where $H^{\top}$ is the transpose of $H$ and $I$ is the identity matrix. Then $H\left(H^{\top} / n\right)=I$ and $H^{-1}=H^{\top} / n$, which yields the equalities $\left(H^{\top} / n\right) H=I$ and $H^{\top} H=n I$. The latter one is written as

$$
\begin{equation*}
\sum_{k=1}^{n} H_{k i} H_{k j}=n \delta_{i j} \tag{1.2}
\end{equation*}
$$

The equality (1.2) is exactly the orthogonality condition for the columns of $H$.
For each particular $n \in \mathbb{N}$ the set of $n \times n$ Hadamard matrices is invariant under the following transformations (see [6]):

1) permutation of rows/columns;
2) multiplication of any row/column by -1 .
[^0]Using these transformations each Hadamard matrix can be brought to a form where its first row and its first column both are filled with ones only (see [6]):

$$
H=\left\|\begin{array}{ccccc}
1 & 1 & 1 & \ldots & 1  \tag{1.3}\\
1 & & & & \\
1 & & & \\
\vdots & & \{-1,1\} & \\
1 & & &
\end{array}\right\|
$$

Almost all Hadamard matrices in Sloan's library [7] are presented in the form (1.3).
By writing $\{-1,1\}$ in (1.3) we indicate that the lower right block of the matrix is built by ones and minus ones. The next trick is to subtract the first row from each of the other rows in (1.3). As a result we get the matrix

$$
M=\left\|\begin{array}{ccccc}
1 & 1 & 1 & \ldots & 1  \tag{1.4}\\
0 & & & & \\
0 & & & \\
\vdots & & -2,0\} & \\
0 & & &
\end{array}\right\|
$$

Let's denote through $\tilde{H}$ the lower right block of the matrix (1.4) divided by -2 :

$$
\begin{equation*}
\tilde{H}=\frac{1}{-2} \cdot\{-2,0\}=\{0,1\} \tag{1.5}
\end{equation*}
$$

The transformation $H \longrightarrow \tilde{H}$ given by (1.3), (1.4), (1.5) is well-known (see [2]).
Definition 1.1. The matrix $\tilde{H}$ produced from a Hadamard matrix $H$ of the form (1.3) according to (1.4) and (1.5) is called the $\{0,1\}$ presentation of the matrix $H$.

It is obvious that the transformation $H \longrightarrow \tilde{H}$ is invertible, i. e. each matrix $H$ of the form (1.3) is associated with a unique matrix $\tilde{H}$ of the form (1.5) and vice versa. Our goal in this paper is to study $\{0,1\}$ presentation of Hadamard matrices and to design an algorithm for generating them in this presentation.

## 2. Gram matrices.

Definition 2.1. For any ordered list of vectors $\mathbf{e}_{1}, \ldots, \mathbf{e}_{s}$ in a Euclidean space $\mathbb{E}$ their Gram matrix is the following matrix composed by their mutual scalar products (see [8] or $\S 1$ in Chapter V of [9]):

$$
G=\left\|\begin{array}{ccc}
\left(\mathbf{e}_{1}, \mathbf{e}_{1}\right) & \ldots & \left(\mathbf{e}_{1}, \mathbf{e}_{s}\right) \\
\vdots & \ddots & \vdots \\
\left(\mathbf{e}_{s}, \mathbf{e}_{1}\right) & \ldots & \left(\mathbf{e}_{s}, \mathbf{e}_{s}\right)
\end{array}\right\|
$$

Let $\tilde{\mathbf{r}}_{1}, \ldots, \tilde{\mathbf{r}}_{n-1}$ be rows of the matrix $\tilde{H}$ in (1.5) considered as vectors in $\mathbb{R}^{n-1}$. If we enumerate the entries of $H$ in (1.3) starting from zero, then we can write

$$
\begin{equation*}
\tilde{\mathbf{r}}_{i}=f\left(\mathbf{r}_{i}\right), \quad i=1, \ldots, n-1 \tag{2.1}
\end{equation*}
$$

where $\mathbf{r}_{0}, \mathbf{r}_{1}, \ldots, \mathbf{r}_{n-1}$ are the rows of the matrix (1.3) and $f: H \longrightarrow \tilde{H}$ is the mapping given by (1.3), (1.4), and (1.5). From (2.1) we derive

$$
\begin{equation*}
\left(\tilde{\mathbf{r}}_{i}, \tilde{\mathbf{r}}_{j}\right)=\left(\frac{\mathbf{r}_{i}-\mathbf{r}_{0}}{-2}, \frac{\mathbf{r}_{j}-\mathbf{r}_{0}}{-2}\right) . \tag{2.2}
\end{equation*}
$$

The equality (2.2) holds since the initial entry in almost all rows of the matrix (1.4) is zero. Expanding the right hand side of (2.2), we get

$$
\begin{equation*}
\left(\tilde{\mathbf{r}}_{i}, \tilde{\mathbf{r}}_{j}\right)=\frac{\left(\mathbf{r}_{i}, \mathbf{r}_{j}\right)}{4}-\frac{\left(\mathbf{r}_{0}, \mathbf{r}_{i}\right)}{4}-\frac{\left(\mathbf{r}_{0}, \mathbf{r}_{j}\right)}{4}+\frac{\left(\mathbf{r}_{0}, \mathbf{r}_{0}\right)}{4} \tag{2.3}
\end{equation*}
$$

Since $H$ is a Hadamard matrix, taking into account the renumeration of the entries of $H$, from (1.1) we derive the following four equalities:

$$
\begin{array}{ll}
\left(\mathbf{r}_{i}, \mathbf{r}_{j}\right)=n \delta_{i j}, & \left(\mathbf{r}_{i}, \mathbf{r}_{0}\right)=0  \tag{2.4}\\
\left(\mathbf{r}_{j}, \mathbf{r}_{0}\right)=0, & \left(\mathbf{r}_{0}, \mathbf{r}_{0}\right)=n
\end{array}
$$

Applying (2.4) to (2.3) yields

$$
\begin{equation*}
\left(\tilde{\mathbf{r}}_{i}, \tilde{\mathbf{r}}_{j}\right)=\frac{n\left(\delta_{i j}+1\right)}{4} \tag{2.5}
\end{equation*}
$$

The equality (2.5) means that we have proved the following theorem.
Theorem 2.1. For any $m \times m$ Hadamard matrix in $\{0,1\}$ presentation with $m>1$ its size $m=4 q-1$, where $q \in \mathbb{N}$, and the Gram matrix associated with the rows of this Hadamard matrix looks like

$$
G=\left\|\begin{array}{cccc}
b & a & \ldots & a  \tag{2.6}\\
a & b & \ldots & a \\
\vdots & \vdots & \ddots & \vdots \\
a & a & \ldots & b
\end{array}\right\|,
$$

where $a=(m+1) / 4=q$ and $b=(m+1) / 2=2 q$.
The calculations (2.2), (2.3), (2.4), and (2.5) are invertible. Therefore Theorem 2.1 can be strengthened as follows.

Theorem 2.2. A square $m \times m$ matrix with $m=4 q-1$ whose entries are zeros and ones is a Hadamard matrix in $\{0,1\}$ presentation if an a only if the Gram matrix associated with its rows is of the form (2.6), where $a=(m+1) / 4=q$ and $b=(m+1) / 2=2 q$.
Proof. The necessity part in the statement of Theorem 2.2 is proved by Theorem 2.1. Let's prove the sufficiency.

Going backward from (1.5) to (1.4), we insert the initial column of zeros to $\tilde{H}$. Since $\left|\tilde{\mathbf{r}}_{i}\right|^{2}=b=(m+1) / 2=2 q=n / 2$ in (2.6), upon this step we get an $m \times n$ matrix with equal number of zeros and ones in each row. Then we multiply this matrix by -2 , insert the initial row composed by ones, and add this initial row to each of the other rows. As a result we get a matrix of the form (1.3). Each of its rows, except for the initial one, comprises equal number of ones and minus
ones. This means that the equalities $\left(\mathbf{r}_{i}, \mathbf{r}_{0}\right)=0$ and $\left(\mathbf{r}_{j}, \mathbf{r}_{0}\right)=0$ in (2.4) are fulfilled. The equality $\left(\mathbf{r}_{0}, \mathbf{r}_{0}\right)=n$ in (2.4) holds since the initial row $\tilde{r}_{0}$ in (1.3) is composed by ones only. The equality $\left(\mathbf{r}_{i}, \mathbf{r}_{j}\right)=n \delta_{i j}$ in (2.4) is derived from (2.5) using (2.3), while (2.5) itself follows from (2.6) since $a=(m+1) / 4=q=n / 4$ and $b=(m+1) / 2=2 q=n / 2$. The whole set of the equalities (2.4) is equivalent to (1.1) upon passing to the standard enumeration of the entries of $H$, thus proving that the matrix $H$ in (1.3) produced backward from (1.5) through (1.4) is a regular Hadamard matrix. Theorem 2.2 is proved.

Now let's consider the columns of the matrix $\tilde{H}$ in (1.5). We denote them through $\tilde{\mathbf{c}}_{1}, \ldots, \tilde{\mathbf{c}}_{n-1}$. If we enumerate the entries of the matrix $H$ in (1.3) starting from zero, then for $\tilde{\mathbf{c}}_{1}, \ldots, \tilde{\mathbf{c}}_{n-1}$ we can write

$$
\begin{equation*}
\tilde{\mathbf{c}}_{i}=f\left(\mathbf{c}_{i}\right), \quad i=1, \ldots, n-1 \tag{2.7}
\end{equation*}
$$

where $\mathbf{c}_{0}, \mathbf{c}_{1}, \ldots, \mathbf{c}_{n-1}$ are the columns of the matrix (1.3) and $f: H \longrightarrow \tilde{H}$ is the mapping given by (1.3), (1.4), and (1.5). From (2.7) we derive

$$
\left(\tilde{\mathbf{c}}_{i}, \tilde{\mathbf{c}}_{j}\right)=\sum_{k=1}^{n-1} \frac{\left(H_{k i}-H_{0 i}\right)}{-2} \frac{\left(H_{k j}-H_{0 j}\right)}{-2}=\sum_{k=1}^{n-1} \frac{\left(H_{k i}-H_{0 i}\right)\left(H_{k j}-H_{0 j}\right)}{4}
$$

The right hand side of this equality can be transformed as

$$
\sum_{k=1}^{n-1} \frac{\left(H_{k i}-H_{0 i}\right)\left(H_{k j}-H_{0 j}\right)}{4}=\sum_{k=0}^{n-1} \frac{\left(H_{k i}-H_{0 i}\right)\left(H_{k j}-H_{0 j}\right)}{4}
$$

since the term with $k=0$ in the sum does vanish. As a result we get

$$
\begin{equation*}
\left(\tilde{\mathbf{c}}_{i}, \tilde{\mathbf{c}}_{j}\right)=\sum_{k=0}^{n-1} \frac{\left(H_{k i}-H_{0 i}\right)\left(H_{k j}-H_{0 j}\right)}{4} \tag{2.8}
\end{equation*}
$$

But $H_{0 i}=H_{k 0}=1$ and $H_{0 j}=H_{k 0}=1$ due to (1.3). Therefore from (2.8) we get

$$
\begin{equation*}
\left(\tilde{\mathbf{c}}_{i}, \tilde{\mathbf{c}}_{j}\right)=\sum_{k=0}^{n-1} \frac{\left(H_{k i}-H_{k 0}\right)\left(H_{k j}-H_{k 0}\right)}{4} \tag{2.9}
\end{equation*}
$$

Expanding the right hand side of (2.9), we write

$$
\begin{gather*}
\left(\tilde{\mathbf{c}}_{i}, \tilde{\mathbf{c}}_{j}\right)=\sum_{k=0}^{n-1} \frac{H_{k i} H_{k j}}{4}-\sum_{k=0}^{n-1} \frac{H_{k i} H_{k 0}}{4}- \\
\quad-\sum_{k=0}^{n-1} \frac{H_{k j} H_{k 0}}{4}+\sum_{k=0}^{n-1} \frac{H_{k 0} H_{k 0}}{4} \tag{2.10}
\end{gather*}
$$

Each sum in (2.10) is expressed through the scalar product of some definite pair of columns of the matrix (1.3). Indeed we have

$$
\begin{equation*}
\left(\tilde{\mathbf{c}}_{i}, \tilde{\mathbf{c}}_{j}\right)=\frac{\left(\mathbf{c}_{i}, \mathbf{c}_{j}\right)}{4}-\frac{\left(\mathbf{c}_{i}, \mathbf{c}_{0}\right)}{4}-\frac{\left(\mathbf{c}_{j}, \mathbf{c}_{0}\right)}{4}+\frac{\left(\mathbf{c}_{0}, \mathbf{c}_{0}\right)}{4} \tag{2.11}
\end{equation*}
$$

The equality (2.11) is similar to (2.3). Since $H$ is a Hadamard matrix, taking into account the renumeration of the entries of $H$, from (1.2) we derive

$$
\begin{array}{ll}
\left(\mathbf{c}_{i}, \mathbf{c}_{j}\right)=n \delta_{i j}, & \left(\mathbf{c}_{i}, \mathbf{c}_{0}\right)=0  \tag{2.12}\\
\left(\mathbf{c}_{j}, \mathbf{c}_{0}\right)=0, & \left(\mathbf{c}_{0}, \mathbf{c}_{0}\right)=n .
\end{array}
$$

Applying (2.12) to (2.11), we derive an equality which is similar to (2.5):

$$
\begin{equation*}
\left(\tilde{\mathbf{c}}_{i}, \tilde{\mathbf{c}}_{j}\right)=\frac{n\left(\delta_{i j}+1\right)}{4} . \tag{2.13}
\end{equation*}
$$

The equality (2.13) means that we have proved a theorem similar to Theorem 2.1.
Theorem 2.3. For any $m \times m$ Hadamard matrix in $\{0,1\}$ presentation with $m>1$ its size $m=4 q-1$, where $q \in \mathbb{N}$, and the Gram matrix associated with the columns of this Hadamard matrix looks like

$$
G=\left\|\begin{array}{cccc}
b & a & \ldots & a  \tag{2.14}\\
a & b & \ldots & a \\
\vdots & \vdots & \ddots & \vdots \\
a & a & \ldots & b
\end{array}\right\|,
$$

where $a=(m+1) / 4=q$ and $b=(m+1) / 2=2 q$.
Note that the Gram matrices in (2.14) and (2.6) do coincide though their entries are defined differently. A theorem similar to Theorem 2.2 is formulated as follows.

Theorem 2.4. A square $m \times m$ matrix with $m=4 q-1$ whose entries are zeros and ones is a Hadamard matrix in $\{0,1\}$ presentation if an a only if the Gram matrix associated with its columns is of the form (2.6), where $a=(m+1) / 4=q$ and $b=(m+1) / 2=2 q$.

Proof. The necessity part in the statement of Theorem 2.4 is proved by Theorem 2.3. Let's prove the sufficiency.

When producing the matrix (1.3) backward from the matrix (1.5) each one in the matrix (1.5) becomes minus one in the matrix (1.3) and each zero in the matrix (1.5) becomes one in the matrix (1.3). Extra ones in the initial row and in the initial column of the matrix (1.5) are inserted independently. Therefore the equality $\left|\tilde{\mathbf{c}}_{i}\right|^{2}=b=(m+1) / 2=2 q=n / 2$ for the diagonal entries in (2.14) means that the number of ones is equal to the number of minus ones in each column of the matrix (1.5), except for the initial column $\mathbf{c}_{0}$. This yields the equalities $\left(\mathbf{c}_{i}, \mathbf{c}_{0}\right)=0$ and $\left(\mathbf{c}_{j}, \mathbf{c}_{0}\right)=0$ in (2.12). The equality $\left(\mathbf{c}_{0}, \mathbf{c}_{0}\right)=n$ in (2.12) holds since the initial column of the matrix (1.3) is composed by ones only. Then the equality $\left(\mathbf{c}_{i}, \mathbf{c}_{j}\right)=n \delta_{i j}$ in (2.12) is derived from (2.13) using (2.11), while (2.13) itself follows from (2.14) since $a=(m+1) / 4=q=n / 4$ and $b=(m+1) / 2=2 q=n / 2$. The whole set of the equalities (2.12) is equivalent to (1.2) upon passing to the standard enumeration of the entries of $H$, thus proving that the matrix $H$ in (1.3) produced backward from (1.5) through (1.4) is a regular Hadamard matrix.

## 3. Partitions and groupings <br> in rows of $\{0,1\}$ Hadamard matrices.

Due to Theorem (2.2) and Theorem (2.4) the whole set of Hadamard matrices in $\{0,1\}$ presentation is invariant under permutation of rows and columns of the matrices. Any two matrices produced from each other by means of these transformations are called equivalent. Below they are threated as inessentially different.

Let $H$ be some particular $15 \times 15$ Hadamard matrix in $\{0,1\}$ presentation. We choose the following one as an example:

$$
H=\left\|\begin{array}{lllllllllllllll}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0  \tag{3.1}\\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1
\end{array}\right\|
$$

The first row of the matrix (3.1) is partitioned into two groups - the group of ones and the group of zeros next to it:


We use the Maxima programming language (see [10]) in order to present the partition structure (3.2). Each of the two groups (3.2) is presented by two numbers. The first number is the ordering number of the group in the row. The second number is the number of elements in the group. As a result the first row of the matrix (3.1) is presented through the following code:

$$
\text { r1: }[[0,8],[1,7]] \$
$$

Square brackets in Maxima programming language delimit lists. Therefore each row of the matrix (3.1) is presented as a list of lists.

In the second row of the matrix (3.1) we see the following four groups:


The first two groups in (3.3) are subordinate to the first group in (3.1), the last two groups of (3.3) are subordinate to the second group in (3.1). Groups with
even numbers correspond to ones in the matrix and the groups with odd numbers correspond to zeros. As a result we have the following code for the second row of the matrix (3.1):

```
r2:[[0,4],[1, 4],[2, 4], [3,3]]$
```

Upon defining each row, the matrix $H$ from (3.1) is presented as the list of its rows:

```
H:[r1,r2,r3,r4,r5,r6,r7,r8,r9,r10,r11,r12,r13,r14,r15]$
```

The upper estimate for the number of groups in the $n$-th row of a matrix is $2^{n}$. However the actual number of groups in the row description is much smaller since the groups with zero elements are not explicitly listed, though they implicitly assumed. Here is the explicit group lists presentation for the matrix (3.1):

```
H:[[[0, 8] , [1,7]],
    [[0,4],[1,4], [2,4], [3,3]],
    [[0,3], [1, 1], [2, 1], [3,3], [4,1], [5,3], [6,3]],
    [[0, 1], [1, 2], [2, 1] , [5, 1] , [6, 2], [7, 1] , [8, 1], [10, 1] , [11, 2],
        [12, 2], [13,1]],
        [[1, 1], [2, 1], [3, 1], [5, 1], [10, 1], [12, 1], [13, 1], [14, 1], [16, 1],
        [20,1], [22, 1], [23,1], [24, 1], [25,1], [27,1]],
        [[2,1], [5, 1], [7, 1], [11, 1], [20, 1], [25,1], [26,1], [28,1], [32, 1],
        [41, 1], [44, 1], [46, 1], [49, 1], [50, 1], [55, 1]],
        [[4, 1], [10, 1], [15, 1] , [23, 1], [41, 1], [50, 1], [52, 1], [57, 1] , [65, 1],
        [83,1], [88, 1], [92, 1], [98,1], [101, 1], [110,1]],
        [[9, 1], [21, 1], [30,1], [46, 1], [83, 1], [101, 1], [104, 1], [114, 1],
        [130, 1], [167,1], [176,1], [185,1], [196,1], [203,1], [220, 1]],
        [[18, 1], [43, 1], [61, 1], [92, 1], [166, 1], [203,1], [209, 1], [228,1],
        [261, 1], [334, 1], [353, 1], [370, 1], [392, 1], [407, 1], [440, 1]],
        [[37, 1], [87, 1], [122 , 1] , [185, 1], [332 , 1] , [406,1], [418, 1], [457, 1],
        [522, 1], [668,1], [707, 1], [740, 1], [785, 1], [815, 1], [880, 1]],
        [[75,1], [174, 1], [245,1], [370,1], [664,1], [813,1], [836,1],
        [915, 1], [1045, 1], [1336, 1], [1414, 1], [1481,1], [1571, 1],
        [1630,1], [1760,1]],
        [[151, 1], [349, 1], [490, 1], [740, 1], [1328, 1], [1626,1], [1673,1],
        [1831, 1], [2091, 1], [2673,1], [2828,1], [2962, 1] , [3142,1],
        [3260,1], [3521, 1]],
    [[303, 1], [698, 1], [980, 1], [1481, 1], [2657, 1], [3253, 1], [3346, 1],
        [3662, 1], [4183, 1], [5346, 1], [5657, 1], [5924, 1] , [6284, 1],
        [6520,1], [7043,1]],
    [[607, 1], [1396, 1], [1961, 1], [2962, 1], [5315,1], [6506,1], [6693,1],
        [7324, 1], [8366,1], [10693,1], [11315,1], [11848,1], [12569,1],
        [13040, 1], [14086, 1]],
        [[1214, 1], [2793,1], [3922, 1], [5925, 1], [10631,1], [13012, 1],
        [13387, 1] , [14648,1], [16733, 1], [21386,1], [22630,1], [23697,1],
        [25139,1], [26080, 1], [28172, 1]]]$
```

The above presentation of the matrix (3.1) looks more complicated than the regular presentation of matrices in Maxima. However this presentation is more convenient from the algorithmic point of view. Below we shall call it the "group lists presentation" or the "partition lists presentation.

## 4. An algorithm for generating Hadamard matrices.

The partition lists presentation of Hadamard matrices is the basis for the algorithm suggested below. We shall not describe the algorithm in full details verbally. Instead we provide the source code of it using Maxima programming language (see [10]). Hadamard matrices are generated by the following code:

```
HM_size:m$
HM_quarter:(HM_size+1)/4$
q:HM_quarter$
HM_row[1]:[[0,2*q],[1,2*q-1]]$
HM_row [2]: [[0,q], [1,q], [2,q] , [3, q-1]]$
HM_matrix_num:1$
HM_stream:openw("output_file.txt")$
HM_make_row(3)$
close(HM_stream)$
```

Most of this code initializes global variables. The first variable HM_size defines the size of Hadamard matrices to be generated. It is given by $m$ which is any positive integer greater than or equal to 3 and obeying the condition

$$
m=3 \bmod 4
$$

Otherwise an error message is generated.
The partition lists of the first two rows of the matrix are predefined. They are stored in the variables HM_row [1] and HM_row [2] (compare with (3.2) and (3.3)). The third row and all other rows of Hadamard matrices are produced by calling the recursive function HM_make_row () with the argument 3. The whole job is practically done by this function. Below is its code:

```
HM_make_row(i):=block
    ([n, s, k,l, q, dummy, kk,y,dpnd,indp,nrd,nri, r,kr,qq, eq, eq_list, j,
    LLL,RLL,RVV,RRV,subst_list],
    if not integerp(HM_size) or HM_size<3 or mod(HM_size,4)#3
        then
            (
            print(printf(false,"Error: m=^a is incorrect size for
            Hadamard matrices",HM_size)),
            return(false)
        ),
    if HM_size=3
        then
            (
            HM_row[2]:[[0, 1], [1, 1], [2,1]],
            HM_row[3]:[[1, 1], [2, 1], [4,1]],
            HM_output_matrix(),
            return(false)
        ),
    print(printf(false,"i=~a",i)),
    n:length(HM_row[i-1]),
    k:makelist(100,y,n),
```

```
l:makelist(100,y,n),
q:makelist(100,y,n),
for s:1 step 1 thru n do
    (
        l[s]:HM_row[i-1][s][1],
        q[s]:HM_row[i-1][s] [2]
    ),
/*-- prepare the equation list --*/
eq_list:[],
var_list:[],
eq:0,
for s:1 step 1 thru n do
    (
        eq:eq+HM_V[i] [s],
        var_list:endcons(HM_V [i][s],var_list)
    ),
eq_list:endcons(eq=2*HM_quarter,eq_list),
qq:1,
for j:i-1 step -1 thru 1 do
    (
        eq:0,
        for s:1 step 1 thru n do
            if evenp(floor(l[s]/qq)) then eq:eq+HM_V[i][s],
        eq_list:endcons(eq=HM_quarter,eq_list),
        qq:qq*2
    ),
/*----- solve the equations -----*/
linsolve_params:false,
HM_SOL[i]:linsolve(eq_list,var_list),
LLL:map(lhs,HM_SOL[i]),
dpnd:[],
for r:1 step 1 thru length(LLL) do
    dpnd:endcons(args(LLL[r])[1],dpnd),
RLL:map(rhs,HM_SOL[i]),
RVV:map(listofvars,RLL),
RRV:{},
for r:1 step 1 thru length(RVV) do
    RRV:union(RRV,setify(RVV[r])),
RRV:listify(RRV),
indp:[],
for r:1 step 1 thru length(RRV) do
    indp:endcons(args(RRV[r])[1],indp),
/*-- initiate the multiindex loop --*/
nrd:length(dpnd),
nri:length(indp),
kr::makelist(0,y,nri+1),
for dummy:1 step 1 while kr[nri+1]=0 do
    (
        subst_list:[],
```

```
        for r:1 step 1 thru nri do
            (
                s:indp[r],
                k[s]:kr[r],
                kk[s]:q[s]-k[s],
                subst_list:endcons(HM_V [i] [s]=k[s],subst_list)
            ),
        for r:1 step 1 thru nrd do
            (
            s:dpnd[r],
            k[s]:psubst(subst_list,RLL[r]),
            kk[s]:q[s]-k[s]
            ),
            /*----- create a new row -----**/
            HM_row[i]:[],
            for s:1 step 1 thru n do
                (
                    if k[s]#0 then HM_row[i]:endcons([2*l[s],k[s]],HM_row[i]),
            if kk[s]#O then HM_row[i]:endcons([2*l[s]+1,kk[s]],HM_row[i])
            ),
        if HM_sc_prods_ok(i)
            then
            if i=n
                then HM_output_matrix()
                else HM_make_row(i+1),
            /*--- increment the multiindex ---*/
            if nri=0 then kr[nri+1]:1,
            for r:1 step 1 thru nri do
                (
            if r=1 then kr[1]:kr[1]+1,
            s:indp[r],
            if kr[r]>q[s] then (kr[r]:0, kr[r+1]:kr[r+1]+1)
        )
        )
)$
```

There are two auxiliary functions which are called from within the above code. One of them is used in order to output generated matrices.

```
HM_output_matrix():=block
    ([s,LL],
    LL: [],
    for s:1 step 1 thru HM_size do LL:endcons(HM_row[s],LL),
    printf(HM_stream, "HM_~ a_~ a: ~a$~%" , HM_size, HM_matrix_num, LL),
    HM_matrix_num:HM_matrix_num+1
)$
```

The second function verifies if the row data prepared for output are correct.

```
HM_sc_prods_ok(i):=block
    ([n,result,ss,s,j,qq],
```

```
n:length(HM_row[i]),
result:true,
ss:0,
for s:1 step 1 thru n do
    (
        result:result and (HM_row[i][s][2]>0)
    ),
return(result)
)$
```


## 5. Running the code and Results.

The above code was run in Maxima, version 5.42.2, on Linux platform of Ubuntu 16.04 LTS using laptop computer DEXP Atlas H161 with the processor unit Intel Core i7-4710MQ. Here we focus on performance of the algorithm.

The case $m=3$ is trivial. In this case the algorithm terminated instantly and produced exactly one Hadamard matrix.

The case $m=7$ is less trivial. In this case the algorithm also terminated instantly and produced 25 matrices. All of them were tested and turned out to be correct $7 \times 7$ Hadamard matrices in $\{0,1\}$ presentation.

The case $m=11$. In this case the algorithm terminated upon running for 3 minutes and 45 seconds. It produced 60481 matrices. Ten of these matrices randomly chosen were tested and turned out to be correct $11 \times 11$ Hadamard matrices in $\{0,1\}$ presentation. The matrix production rate is

$$
v=16128 \text { matrices } / \text { minute. }
$$

The case $m=15$. In this case the algorithm did not terminate during observably short time. Upon running for 16 minutes and 49 seconds it produced 162500 matrices. Ten of these matrices randomly chosen were tested and turned out to be correct $15 \times 15$ Hadamard matrices in $\{0,1\}$ presentation. The production rate is

$$
v=9663 \text { matrices/minute. }
$$

The case $m=19$. Again the algorithm did not terminate during observably short time. Upon running for 1 minute and 40 seconds it produced 10000 matrices. Ten randomly chosen matrices were tested and passed the test. The rate is

$$
v=6000 \text { matrices } / \text { minute }
$$

The case $m=23$ is similar to the previous one. The algorithm did not terminate during observably short time. 10000 matrices were generated for 2 minutes and 43 seconds. Ten randomly chosen matrices were tested. They turned out to be correct $23 \times 23$ Hadamard matrices in $\{0,1\}$ presentation. The production rate is

$$
v=3680 \text { matrices/minute. }
$$

The case $m=27$ is absolutely different. In this case the algorithm ran overnight for several hours but did not produce any matrices. So $m=27$ is a practical limit for the algorithm.

One of the features of the present algorithm is that it solves linear equations arising from the form of the Gram matrix (2.6) using Maxima's linsolve function instead of scanning over the ranges of variables. However it does not solve inequalities in this manner (see HM_row [i] [s] [2]>0 in the code of the function HM_sc_prods_ok above). Probably using some module for solving linear inequalities along with linear equations could improve the algorithm and $m=27$ would not be a limit for it any more.

## 4. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

## References

1. Hadamard conjecture, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
2. Hadamard's maximal determinant problem, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
3. Hadamard J., Résolution d'une question relative aux determinants, Bulletin des Sciences Mathématiques 17 (1893), 240-246.
4. Sharipov R. A., On a simplified version of Hadamard's maximal determinant problem, e-print arXiv:2104.01749 [math.NT].
5. Sharipov R. A., Course of analytical geometry, Bashkir State University, Ufa, 2010; see also arXiv:1111.6521 [math.HO].
6. Cherowitzo B., Hadamard matrices and designs, online resource hadamard.pdf, course of Combinatorial Structures, University of Colorado Denver.
7. Sloane N. J. A., A library of Hadamard matrices, online resourse neilsloane.com/hadamard, Personal home page of Neil Sloan.
8. Gramian matrix, Wikipedia, Wikimedia Foundation Inc., San Francisco, USA.
9. Sharipov R. A., Course of linear algebra and multidimensional geometry, Bashkir State University, Ufa, 1996; see also arXiv:math/0405323 [math.HO].
10. Maxima manual, version 5.44.0, online resourse maxima.pdf at sourceforge.io.

Bashkir State University, 32 Zaki Validi street, 450074 Ufa, Russia
E-mail address: r-sharipov@mail.ru


[^0]:    2010 Mathematics Subject Classification. 05B20, 11D04, 11D09, 15B10, 15B34, 15B36, 65-04.
    Key words and phrases. Hadamard matrices.

