PSEUDO-HADAMARD MATRICES OF THE FIRST GENERATION AND AN ALGORITHM FOR PRODUCING THEM.

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ABSTRACT. Hadamard matrices in $\{0, 1\}$ presentation are square $m \times m$ matrices whose entries are zeros and ones and whose rows considered as vectors in \mathbb{R}^m produce the Gram matrix of a special form with respect to the standard scalar product in \mathbb{R}^m . The concept of Hadamard matrices is extended in the present paper. As a result pseudo-Hadamard matrices of the first generation are defined and investigated. An algorithm for generating these pseudo-Hadamard matrices is designed and is used for testing some conjectures.

1. INTRODUCTION.

Regular Hadamard matrices are defined in $\{1, -1\}$ presentation. They are square $n \times n$ matrices whose entries are ones and minus ones and whose rows are orthogonal to each other with respect to the standard scalar product in \mathbb{R}^n (see [1]). Hadamard matrices are associated with Hadamard's maximal determinant problem (see [2] and [3]). A simplified version of this problem was suggested in [4]. Using the well-known transformation from $\{1, -1\}$ to $\{0, 1\}$ presentation (see [2]), in [5] the concept of Hadamard matrices was transferred to the class of matrices whose entries are zeros and ones. A Hadamard matrix in $\{0, 1\}$ presentation is a special square $m \times m$ matrix, where m = 1 or m = 4q - 1 for some $q \in \mathbb{N}$ and where \mathbb{N} is the set of positive integers.

The case m = 1 is trivial. In this case we have exactly one Hadamard matrix which coincides with the identity matrix: H = ||1||. In the case m = 4q - 1 Hadamard matrices in $\{0, 1\}$ presentation can be defined as follows.

Definition 1.1. A Hadamard matrix is a square $m \times m$ matrix, where m = 4q-1 for some $q \in \mathbb{N}$, whose entries are zeros and ones and whose rows considered as vectors in \mathbb{R}^m produce the Gram¹ matrix of the form

$$G = \left\| \begin{array}{cccc} b & a & \dots & a \\ a & b & \dots & a \\ \vdots & \vdots & \ddots & \vdots \\ a & a & \dots & b \end{array} \right\|,$$
(1.1)

Typeset by \mathcal{AMS} -TEX

²⁰¹⁰ Mathematics Subject Classification. 05B20, 11D04, 11D09, 15B34, 15B36, 65-04.

Key words and phrases. Hadamard matrices, pseudo-Hadamard matrices.

¹ A Gram matrix is a matrix formed by pairwise mutual scalar products of a sequence of vectors in a space equipped with some scalar product.

where a = q and b = 2q, with respect to the standard scalar product in \mathbb{R}^m .

This definition is based on Theorems 2.1 and 2.2 from [5]. Below we omit the case m = 1 and consider the case m = 4 q - 1 with $q \in \mathbb{N}$ only.

Note that the restrictions m = 4q - 1, a = q, and b = 2q in Definition 1.1 are essential since there is the following matrix

M =	1	1	1	0			$\parallel 3$	2	2	$2 \mid$
	1	1	0	$1 \parallel$	i+la	G =	2	3	2	2
	1	0	1	$1 \parallel$	WIUII		2	2	3	2
	0	1	1	$1 \parallel$			2	2	2	3

which is not a Hadamard matrix, i. e. M is not produced from a regular Hadamard matrix by the $\{1, -1\}$ to $\{0, 1\}$ transformation.

Theorem 1.1. The transpose of a Hadamard matrix is again a Hadamard matrix.

Theorem 1.1 is immediate from Theorems 2.4 and 2.2 in [5]. In particular this theorem means that the Gram matrix associated with columns of a Hadamard matrix coincides with the Gram matrix (1.1) associated with its rows.

Below by analogy to Definition 1.1 we define pseudo-Hadamard matrices in $\{0, 1\}$ presentation and study a subclass of them. In particular, we design an algorithms for generating this subclass of pseudo-Hadamard matrices.

2. PSEUDO-HAGAMARD MATRICES OF THE FIRST GENERATION.

Relying upon Definition 1.1 and Theorem 1.1, it is easy to see that the set of Hadamard matrices is invariant under the following transformations:

- 1) permutation of rows;
- 2) permutation of columns.

Using these transformations, one can bring any Hadamard matrix to the form

Due to (1.1) with b = 2q the number of ones in the first row of the matrix (2.1) is equal to 2q. The number of zeros in this row is equal to 2q - 1. Due to Theorem 1.1 the same is valid for the first column of the matrix (2.1), i.e. its first column comprises 2q ones and 2q - 1 zeros.

Let's remove the first row and the first column of the matrix in (2.1) and denote through \tilde{H} the rest of the matrix H:

$$\tilde{H} = \begin{cases} \{0,1\} \end{cases} .$$
(2.2)

The matrix (2.2) coincides with the minor $M_{11}(H)$ in H associated with the top left entry of the matrix (2.1).

Definition 2.1. An $m \times m$ matrix \tilde{H} , where m = 4q - 2 and $q \in \mathbb{N}$, produced from some Hadamard matrix H of the form (2.1) according to (2.2) is called a pseudo-Hadamard matrix of the first generation.

Theorem 2.1. For any $m \times m$ pseudo-Hadamard matrix of the first generation \tilde{H} with m = 4q - 2, where $q \in \mathbb{N}$, its rows considered as vectors of \mathbb{R}^m with the standard scalar product produce the Gram matrix of the form

$$G = \begin{vmatrix} \tilde{b} & \tilde{a} & \dots & \tilde{a} & \tilde{a} & a & \dots & a \\ \tilde{b} & \tilde{b} & \dots & \tilde{a} & a & a & \dots & a \\ \tilde{a} & \tilde{b} & \dots & \tilde{b} & \tilde{a} & a & \dots & a \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \tilde{a} & \tilde{a} & \dots & \tilde{b} & a & a & \dots & a \\ a & a & \dots & a & b & a & \dots & a \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a & a & \dots & a & a & a & \dots & b \end{vmatrix} ,$$
(2.3)

where
$$a = (m+2)/4 = q$$
, $b = (m+2)/2 = 2q$, $\tilde{a} = a - 1$, and $b = b - 1$.

Proof. Let's denote through $\tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_m$ the rows of the matrix \tilde{H} in (2.2) and through $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_m$ the rows of the matrix H in (2.1), i. e. we denote the initial row of the matrix (2.1) through \mathbf{r}_0 . If we consider $\tilde{\mathbf{r}}_1, \ldots, \tilde{\mathbf{r}}_m$ as vectors in \mathbb{R}^m and $\mathbf{r}_0, \mathbf{r}_1, \ldots, \mathbf{r}_m$ as vectors in \mathbb{R}^{m+1} , then, applying the standard scalar products in \mathbb{R}^m and \mathbb{R}^{m+1} to them, we derive

$$(\mathbf{r}_{i}, \mathbf{r}_{j}) = \sum_{k=0}^{m} H_{ik} H_{jk} = H_{i0} H_{j0} + (\tilde{\mathbf{r}}_{i}, \tilde{\mathbf{r}}_{j}).$$
(2.4)

Looking at (2.1) and taking into account that m = 4q - 2, we see that

$$H_{i\,0} = \begin{cases} 1 & \text{for } 0 \leqslant i \leqslant 2\,q-1, \\ 0 & \text{for } 2\,q \leqslant i \leqslant m. \end{cases}$$
(2.5)

Since a = (m+2)/4 = q and b = (m+2)/2 = 2q, the formula (1.1) is equivalent to

$$(\mathbf{r}_i, \mathbf{r}_j) = q \, (\delta_{ij} + 1). \tag{2.6}$$

Since moreover $\tilde{a} = a - 1$ and $\tilde{b} = b - 1$, the formula (2.3) is equivalent to

$$(\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j) = q (\delta_{ij} + 1) - 1 \quad \text{for} \quad 1 \leq i, j \leq 2q - 1, \\ (\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j) = q (\delta_{ij} + 1) \quad \text{for} \quad 2q \leq i \leq m \quad \text{and/or} \quad 2q \leq j \leq m.$$

$$(2.7)$$

Applying (2.4) and (2.5) to (2.6), we easily derive (2.7). This means that (2.6) implies (2.7) and, hence, (1.1) implies (2.3). Theorem 2.1 is proved. \Box

A similar result is valid for columns of pseudo-Hadamard matrices. It is given by the following theorem.

Theorem 2.2. For any $m \times m$ pseudo-Hadamard matrix of the first generation \hat{H} with m = 4q - 2, where $q \in \mathbb{N}$, its columns considered as vectors of the space \mathbb{R}^m with the standard scalar product produce the Gram matrix of the form (2.3), where a = (m+2)/4 = q, b = (m+2)/2 = 2q, $\tilde{a} = a - 1$, and $\tilde{b} = b - 1$.

Theorem 2.2 follows from Theorem 2.1 due to Theorem 1.1. Theorems 2.1 and 2.2 are strengthened in the following theorem.

Theorem 2.3. A square $m \times m$ matrix \hat{H} whose entries are zeros and ones is a pseudo-Hadamard matrix of the first generation if and only if m = 4q-2 for some $q \in \mathbb{N}$ and if its rows and its columns considered as vectors of the space \mathbb{R}^m with the standard scalar product produce the same Gram matrix of the form (2.3), where a = (m+2)/4 = q, b = (m+2)/2 = 2q, $\tilde{a} = a-1$, and $\tilde{b} = b-1$.

Proof. The necessity part in the statement of Theorem 2.3 is proved by Theorems 2.1 and 2.2. Let's prove the sufficiency.

In proving Theorems 2.1 we have seen that (2.6) implies (2.7). However the converse is not true since the equalities (2.7) do not cover the cases with i = 0 and j = 0. Let's denote through $\tilde{\mathbf{r}}_0$ the initial row of the matrix (2.1) shortened by omitting the first entry of it. This row obeys the equalities

$$(\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_0) = 2 q - 1, \tag{2.8}$$

$$(\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j) = q - 1 \quad \text{for} \quad 1 \le j \le 2q - 1, \tag{2.9}$$

$$(\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j) = q \text{ for } 2q \leqslant j \leqslant m.$$
 (2.10)

The equalities (2.8), (2.9), and (2.10) follow from (2.6) due to (2.4) and (2.5) and moreover, if we adjoin (2.8), (2.9), and (2.10) to (2.7), the whole set of equalities (2.7), (2.8), (2.9), and (2.10) turns out to be equivalent to (2.6). Therefore, in order to complete our proof we need to derive (2.8), (2.9), and (2.10) from the premises of Theorem 2.3 being proved.

The equality (2.8) is trivial. It is fulfilled since the number of ones in the row $\tilde{\mathbf{r}}_0$ is equal to 2q - 1. In order to derive (2.9) and (2.10) we define the following row:

(compare with (2.3)). The scalar products of the row (2.11) with $\tilde{\mathbf{r}}_0$ and with the rows of the matrix \tilde{H} in the statement of Theorem 2.3 are easily calculated. They are equal to the number of ones in these rows:

$$(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}_0) = 2 q - 1,$$

$$(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}_j) = \tilde{b} = 2 q - 1 \text{ for } 1 \leq j \leq 2 q - 1,$$

(2.12)

$$(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}_j) = b = 2 q \text{ for } 2 q \leqslant j \leqslant m.$$
 (2.13)

Now let's consider the sum of all rows of the matrix H:

$$\tilde{\rho} = \sum_{i=1}^{m} \tilde{\mathbf{r}}_i. \tag{2.14}$$

Each entry of the row (2.14) is equal to the number of ones in the corresponding

column of the matrix \hat{H} . Since G in (2.3) is the Gram matrix not only for rows, but also for columns of the matrix \hat{H} , the numbers of ones in columns of \hat{H} are given by diagonal entries of the Gram matrix (2.3). As a result we get

$$\tilde{\boldsymbol{\rho}} = b \; \tilde{\mathbf{r}} + (\tilde{b} - b) \; \tilde{\mathbf{r}}_0 = 2 \, q \; \tilde{\mathbf{r}} - \tilde{\mathbf{r}}_0. \tag{2.15}$$

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Let's calculate scalar products of both sides of (2.15) with $\tilde{\mathbf{r}}_j$. In the case of the left hand side of (2.15) we have the following result:

$$(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{r}}_j) = \sum_{i=1}^m (\tilde{\mathbf{r}}_i, \tilde{\mathbf{r}}_j) = \sum_{i=1}^m G_{ij}.$$
(2.16)

The last sum in (2.16) is explicitly calculated using (2.3):

$$\sum_{i=1}^{m} G_{ij} = (2 q - 2) (a + \tilde{a}) + a + \tilde{b} \text{ for } 1 \leq j \leq 2 q - 1,$$
(2.17)

$$\sum_{i=1}^{m} G_{ij} = (2q-2)(a+a) + a + b \text{ for } 2q \leq j \leq m.$$
(2.18)

Since $a = q, b = 2q, \tilde{a} = a - 1$, and $\tilde{b} = b - 1$, (2.17) and (2.18) simplify to

$$(\tilde{\boldsymbol{\rho}}, \tilde{\mathbf{r}}_j) = \sum_{i=1}^m G_{ij} = \begin{cases} 4 q^2 - 3 q + 1 & \text{for } 1 \leq j \leq 2 q - 1, \\ 4 q^2 - q & \text{for } 2 q \leq j \leq m. \end{cases}$$
(2.19)

Now let's proceed to the right hand side of (2.15). In this case we have

$$(2q\,\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j) = 2q\,(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}_j) - (\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j), \qquad (2.20)$$

where

$$(\tilde{\mathbf{r}}, \tilde{\mathbf{r}}_j) = \begin{cases} \tilde{b} & \text{for } 1 \leq j \leq 2q - 1, \\ b & \text{for } 2q \leq j \leq m \end{cases}$$
(2.21)

(see (2.12) and (2.13)). Since b = 2q and $\tilde{b} = b - 1$, from (2.20) and (2.21) we get

$$(2q\,\tilde{\mathbf{r}} - \tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j) = -(\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j) + \begin{cases} 4q^2 - 2q & \text{for } 1 \leq j \leq 2q - 1, \\ 4q^2 & \text{for } 2q \leq j \leq m. \end{cases}$$
(2.22)

Note that (2.15) implies $(\tilde{\rho}, \tilde{\mathbf{r}}_j) = (2 q \, \tilde{\mathbf{r}} - \tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j)$. Substituting (2.19) and (2.22) into this equality, we get two expressions for $(\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j)$:

$$(\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j) = q - 1 \quad \text{for} \quad 1 \le j \le 2q - 1, \tag{2.23}$$

$$(\tilde{\mathbf{r}}_0, \tilde{\mathbf{r}}_j) = q \text{ for } 2q \leqslant j \leqslant m.$$
 (2.24)

Note that (2.23) and (2.24) do coincide with (2.9) and (2.10).

Thus, the formulas (2.8), (2.9), and (2.10) are derived from the premises of Theorem 2.3. Adjoining them to (2.7) and applying (2.4) and (2.5), we derive (2.6) for the rows of the matrix H. The matrix H now is produced backward from \tilde{H} by adjoining the initial row and the initial column according to (2.1) and (2.2). The equality (2.6) is equivalent to (1.1). Therefore we can apply either

Theorem 2.2 or Theorem 2.4 from [5]. Each of these two theorems means that the matrix H produced backward from \tilde{H} in (2.1) is a regular Hadamard matrix in $\{0,1\}$ presentation. Hence \tilde{H} is a pseudo-Hadamard matrix of the first generation according to Definition 1.1. The proof of Theorem 2.3 is over. \Box

3. PSEUDO-HADAMARD MATRICES OF HIGHER GENERATIONS.

Pseudo-Hadamard matrices of higher generations are defined recursively. A pseudo-Hadamard matrix of the second generation is produced from some pseudo-Hadamard matrix of the first generation upon rearranging its rows and columns in a way similar to (2.1) and then by removing the initial row and the initial column of it like in (2.2). Matrices of the third generation are produced in this way from matrices of the second generation etc, i.e. each next generation is produced from the previous one.

In this paper we shall not consider pseudo-Hadamard matrices of higher generations. They will be studied separately in forthcoming papers.

4. An Algorithm for generating pseudo-Hadamard Matrices of the first generation.

Like the algorithm for generating Hadamard matrices from [5], our present algorithm is based on partitioning of rows of matrices into groups (see Section 3 in [5]). We use the Maxima programming language (see [6]) for presenting its code. Almost all of the code coincide with the code in [5]. Below are those lines of the code that should be changed.

```
HM_size:m$
HM_guarter:(HM_size+1)/4$
HM_quarter:(HM_size+2)/4$
q:HM_quarter$
HM_row[1]:[[0,2*q],[1,2*q 1]]$
HM_row[1]:[[0,2*q-1],[1,2*q-1]]$
HM_row[2]:[[0,q],[1,q],[2,q],[3,q 1]]$
HM_row[2]:[[0,q-1],[1,q],[2,q],[3,q-1]]$
HM_matrix_num:1$
HM_stream:openw("output_file.txt")$
HM_make_row(3)$
close(HM_stream)$
```

The lines to be removed are shown with strikethrough text. The replacement lines are given in green. Like in [5], the whole job is practically done by the recursive function HM_make_row(). Here its code is also slightly changed.

```
HM_make_row(i):=block
([n,s,k,l,q,dummy,kk,y,dpnd,indp,nrd,nri,r,kr,qq,eq,eq_list,j,
LLL,RLL,RVV,RRV,subst_list],
if not integerp(HM_size) or HM_size<3 or mod(HM_size,4)#3
if not integerp(HM_size) or HM_size<2 or mod(HM_size,4)#2
then
(
print(printf(false,"Error: m=~a is incorrect size for</pre>
```

```
Hadamard matrices",HM_size)),
  generation one pseudo-Hadamard matrices", HM_size)),
  return(false)
 ).
if HM_size=3
if HM_size=2
 then
  (
  -<u>HM_row[2]:[[0,1],[1,1],[2,1]],</u>
  HM_row[2]:[[1,1],[2,1]],
  HM_row[3]:[[1,1],[2,1],[4,1]],
  HM_output_matrix(),
  return(false)
 ),
print(printf(false,"i=~a",i)),
/*-- prepare the equation list --*/
eq_list:[],
var_list:[],
eq_list:endcons(eq=2*HM_quarter,eq_list),
if i<2*HM_quarter
 then eq_list:endcons(eq=2*HM_quarter-1,eq_list)
 else eq_list:endcons(eq=2*HM_quarter,eq_list),
qq:1,
-eq_list:endcons(eq=HM_quarter,eq_list),
 if i<2*HM_quarter
  then eq_list:endcons(eq=HM_quarter-1,eq_list)
  else eq_list:endcons(eq=HM_quarter,eq_list),
 qq:qq*2
```

For the sake of brevity above we omit some unchanged portions of the code replacing them with dots. The lacking code can be taken from [5].

Apart from the function HM_make_row() the algorithm comprises two other functions HM_output_matrix() and HM_sc_prods_ok(i). Their code is unchanged. It can also be taken from [5].

Like in [5], the above code was run in Maxima, version 5.42.2, on Linux platform of Ubuntu 16.04 LTS using laptop computer DEXP Atlas H161 with the processor unit Intel Core i7-4710MQ. Below are performance data of the code.

The case m = 2 is trivial. In this case the algorithm terminated instantly and produced exactly one 2×2 pseudo-Hadamard matrix which coincides with the identity matrix.

The case m = 6 is less trivial. In this case the algorithm also terminated instantly, but produced 6 matrices.

The case m = 10. In this case the algorithm ran for 6 seconds and produced 1440 matrices. The matrix production rate is 14400 matrices/minute.

The case m = 14. In this case the algorithm did not terminate during observ-

ably short time. But setting timestamps upon each next 10000 matrices, I have found that the first 10000 matrices were produced for 56 seconds, i.e. the matrix production rate is 10714 matrices/minute.

The case m = 18. In this case the first 10000 matrices were produced for 1 minute and 29 seconds, i.e. the matrix production rate is 6742 matrices/minute.

The case m = 22 is different. In this case the algorithm becomes very slow. It has produced 10000 matrices upon running for 5 hours 1 minute and 17 seconds. The average matrix production rate is 33 matrices/minute. However this production rate is very unevenly distributed over the interval of running. In the beginning the algorithm does not produce matrices for about 3 hours.

As a conclusion we can say that m = 22 is a practical limit for the algorithm in its present version. Though theoretically the algorithm has no limits.

5. Analysis of output and some conjectures.

The above algorithm produces $m \times m$ matrices whose entries are zeros and ones and whose rows, when treated as vectors in \mathbb{R}^m , generate the Gram matrix of the form (2.3) with respect to the standard scalar product in \mathbb{R}^m . However we cannot apply Theorem 2.3 to these matrices since the Gramians of their columns are uncertain. Therefore the output matrices were additionally analyzed. Relying upon this analysis the following conjectures are formulated.

Conjecture 5.1. Let H be a square $m \times m$ matrix, where m = 4q - 2 for some $q \in \mathbb{N}$, whose entries are zeros and ones and whose rows considered as vectors of the space \mathbb{R}^m with the standard scalar product produce the Gram matrix of the form (2.3) with a = (m+2)/4 = q, b = (m+2)/2 = 2q, $\tilde{a} = a - 1$, and $\tilde{b} = b - 1$. Then \tilde{H} coincides with some pseudo-Hadamard matrix of the first generation upon some permutation of its columns.

Conjecture 5.2. Let \tilde{H} be a square $m \times m$ matrix, where m = 4q - 2 for some $q \in \mathbb{N}$, whose entries are zeros and ones and whose columns considered as vectors of the space \mathbb{R}^m with the standard scalar product produce the Gram matrix of the form (2.3) with a = (m+2)/4 = q, b = (m+2)/2 = 2q, $\tilde{a} = a - 1$, and $\tilde{b} = b - 1$. Then \tilde{H} coincides with some pseudo-Hadamard matrix of the first generation upon some permutation of its rows.

Conjectures 5.1 and 5.2 are dual to each other. They are either both valid or both invalid. I have tested these conjectures for all of my output matrices. They turned out to be valid

- 1) for six $m \times m$ matrices with m = 6;
- 2) for one thousand four hundred and forty 10×10 matrices;
- 3) for ten thousand 14×14 matrices;
- 4) for ten thousand 18×18 matrices;
- 5) for ten thousand 22×22 matrices.

This makes a good evidence in favor of these conjectures to be valid, though this does not prove them.

6. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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