EVOLUTION PATTERNS IN COLLATZ PROBLEM.

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Abstract. The concept of evolution patterns is introduced for Collatz sequences and it is shown that any finite evolution pattern is implemented in some particular Collatz sequence.

1. Introduction.

Wikipedia [1] says that the Collatz conjecture also known as the $3n+1$ problem was formulated by Lothar Collatz in 1937. However Wikipedia gives a lot of different names of this conjecture associated with different persons and not only with persons: the Ulam conjecture, Kakutani's problem, the Thwaites conjecture, Hasse's algorithm, the Syracuse problem.

The statement of the Collatz conjecture is based on the mapping $f\colon\ \mathbb{N}\to\mathbb{N}$ which is defined as follows in the set of positive integers $\mathbb{N}$:

$$f(n)=\begin{cases} 3\,n+1 & \text{if } n \text{ is odd,}\\ n/2 & \text{if } n \text{ is even.}\end{cases} \tag{1.1}$$

Starting with an arbitrary number $m\in\mathbb{N}$, the sequence of numbers a_i is built such that $a_1=m$ and $a_{i+1}=f(a_i)$ for all $i\in\mathbb{N}$. For instance in the case of $m=1$, we get the sequence of numbers

$$1,\ 4,\ 2,\ 1,\ 4,\ 2,\ 1,\ \dots\ , \tag{1.2}$$

which repeats periodically with the period $T=3$. The cases $m\neq 1$ are described by the Collatz conjecture.

Conjecture 1.1 (Collatz). *For any positive integer $m\in\mathbb{N}$ the sequence of numbers a_i defined using the mapping (1.1) through $a_1=m$ and $a_{i+1}=f(a_i)$ for all $i\in\mathbb{N}$ reaches the number* 1 (i.e. $a_k=1$ for some $k\in\mathbb{N}$) and then repeats periodically as in (1.2).

There are many research works devoted to the Collatz conjecture 1.1, see [2] and [3]. The year of 2021 and early 2022 demonstrated rather high publication activity in this field, see [4–33]. We are not going to analyze all of these papers hear. Our goal is to show that the evolution of integers in Collatz sequences can be arbitrarily complicated in such a way that any predefined finite evolution pattern can be implemented in some Collatz sequence.

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2. Evolution patterns.

Lemma 2.1. *Any positive even integer n can be uniquely presented as $n = 2^s \cdot m$, where m and s are two positive integers and m is odd.*

The proof consists in iteratively dividing n by 2 until the odd number m is reached: $n \to n/2 \to n/2^2 \to \ldots \to n/2^s = m$.

Lemma 2.2. *Any positive odd integer n can be uniquely presented as $n = 2^s \cdot m - 1$, where m and s are two positive integers and m is odd.*

The proof consists in applying Lemma 2.1 to the even number $n + 1$.

Let's consider the Collatz evolution initiated by the odd number $a_1 = 2^s \cdot m - 1$. Due to (1.1) we get

$$a_2 = f(a_1) = 3\,a_1 + 1 = 3 \cdot 2^s \cdot m - 2. \tag{2.1}$$

The number a_2 is even, therefore

$$a_3 = f(a_2) = a_2/2 = 3^1 \cdot 2^{s-1} \cdot m - 1. \tag{2.2}$$

If $s - 1 \neq 0$, then a_3 is again odd and we can repeat the steps (2.1) and (2.2):

$$\begin{aligned} &a_4 = f(a_3) = 3\,a_3 + 1 = 3^2 \cdot 2^{s-1} \cdot m - 2,\\ &a_5 = f(a_4) = a_4/2 = 3^2 \cdot 2^{s-2} \cdot m - 1. \end{aligned} \tag{2.3}$$

Now due to (2.2) and (2.3) it is easy to see that

$$a_{1+2s} = 3^s \cdot m - 1. \tag{2.4}$$

The number (2.4) is even.

Definition 2.1. The Collatz evolution from an odd number $a_1 = 2^s \cdot m - 1$ to the even number $a_{1+2s} = 3^s \cdot m - 1$ is called the s-evolution.

Applying Lemma 2.1 to the even number (2.4), we get

$$a_{1+2s} = 3^s \cdot m - 1 = 2^q \cdot \tilde{n}, \tag{2.5}$$

where $q \geqslant 1$ and $\tilde{n}$ is a positive odd number. Applying the Collatz evolution to the even number (2.5), we find

$$a_{1+2s+q} = a_{1+2s}/2^q = \tilde{n}. \tag{2.6}$$

Definition 2.2. The Collatz evolution from an even number $a_{1+2s} = 2^q \cdot \tilde{n}$ to the odd number $a_{1+2s+q} = \tilde{n}$ is called the q-evolution.

Since the number (2.6) is again odd, we can apply the steps from (2.1) to (2.6) to it. As a result we see that we have proved the following theorem.

Theorem 2.1. *The Collatz evolution of any odd number is presented by an alternating sequence of s and q evolutions:*

$$s_1,\ q_1,\ s_2,\ q_2,\ \ldots\ ,\ s_r,\ q_r,\ s_{r+1},\ \ldots\ . \tag{2.7}$$

Definition 2.3. The infinite sequence of positive integers (2.7) is called the Collatz evolution pattern of a given odd number.

For instance the evolution pattern of the odd number 1 is given by the following trivial periodic sequence

$$1,\,1,\,1,\,\ldots\,,\,1\,\ldots\,. \tag{2.8}$$

The sequence (2.8) is easily derived from (1.2).

Definition 2.4. Any finite initial part of the Collatz evolution pattern (2.7) is called the finite Collatz evolution pattern of a given odd number:

$$s_1,\,q_1,\,s_2,\,q_2,\,\ldots\,,\,s_r,\,q_r,\,s_{r+1}. \tag{2.9}$$

Our further goal is to prove the following theorem.

Theorem 2.2. *For any finite sequence of positive integers (2.9) there is an odd number whose finite Collatz evolution pattern coincides with (2.9).*

3. Diophantine equations associated with *sqs*-patterns.

Assume that $r = 1$ in (2.9). Then we have the following finite Collatz evolution pattern: s_1, q_1, s_2. The actual evolution associated with this pattern looks like

$$2^{s_1}\cdot m_1 - 1 \xrightarrow{\;s_1\;} 3^{s_1}\cdot m_1 - 1 \xrightarrow{\;q_1\;} 2^{s_2}\cdot m_2 - 1. \tag{3.1}$$

Relying on (3.1), we can write the following equality:

$$3^{s_1}\cdot m_1 - 1 = 2^{q_1}\cdot(2^{s_2}\cdot m_2 - 1). \tag{3.2}$$

The equality (3.2) can be rewritten as

$$2^{q_1+s_2}\cdot m_2 - 3^{s_1}\,m_1 = 2^{q_1} - 1. \tag{3.3}$$

Since s_1, q_1, and s_2 are known, we can treat (3.3) as a linear Diophantine equation with respect to the variables m_1 and m_2.

The theory of linear Diophantine equations is given in Section 2.1 of Chapter I.2 in the book [34]. There one can find the following theorem.

Theorem 3.1. *Let a, b, c be integers, a and b nonzero. Consider the linear Diophantine equation*

$$a\,x + b\,y = c \tag{3.4}$$

1. *The equation (3.4) is solvable in integers if and only if the greatest common divisor $d = \mathrm{GCD}(a,b)$ divides c.*
2. *If $(x,y) = (x_0,y_0)$ is a particular solution to (3.4), then every integer solution is of the form*

$$x = x_0 + \frac{b}{d}\cdot t, \qquad\qquad y = y_0 - \frac{a}{d}\cdot t, \tag{3.5}$$

   where t is an integer.
3. *If $c = GCD(a,b)$ and if $|a|$ or $|b|$ is different from 1, then a particular solution $(x,y) = (x_0,y_0)$ in (3.5) can be found such that $|x_0| < |b|$ and $|y_0| < |a|$.*

Comparing (3.3) with (3.4), we find that in our particular case

$$a=2^{q_1+s_2},\qquad b=-3^{s_1},\qquad c=2^{q_1}-1,\tag{3.6}$$

i. e. the equation (3.3) is written as

$$2^{q_1+s_2}\cdot x-3^{s_1}y=2^{q_1}-1.\tag{3.7}$$

From (3.6) we derive

$$d=\operatorname{GCD}(a,b)=1.\tag{3.8}$$

Applying Item 1 from Theorem 3.1 to (3.8), we derive the following theorem.

Theorem 3.2. *For any sequence of positive integers* (2.9) *the Diophantine equation* (3.7) *is solvable.*

Then applying Item 2 from Theorem 3.1 to the equation (3.7), we find that the formulas (3.5) are written as

$$x=x_0-3^{s_1}\,t,\qquad y=y_0-2^{q_1+s_2}\,t,\tag{3.9}$$

The second equality (3.9) means that we can choose a unique particular solution (x_0,y_0) of the equation (3.3) such that

$$0\leqslant y_0<2^{q_1+s_2}.\tag{3.10}$$

The option $y_0=0$ is excluded since in this case we would have

$$x_0=\frac{2^{q_1}-1}{2^{q_1+s_2}},$$

where

$$0<\frac{2^{q_1}-1}{2^{q_1+s_2}}<1,\tag{3.11}$$

which would mean that x_0 is not integer. Therefore the inequalities in (3.10) are rewritten as

$$0<y_0<2^{q_1+s_2}.\tag{3.12}$$

Since (x_0,y_0) is a solution of the Diophantine equation (3.7), we have

$$x_0=\frac{3^{s_1}}{2^{q_1+s_2}}\,y_0+\frac{2^{q_1}-1}{2^{q_1+s_2}}.\tag{3.13}$$

Applying (3.12) to (3.13), we get

$$\frac{2^{q_1}-1}{2^{q_1+s_2}}<x_0<3^{s_1}+\frac{2^{q_1}-1}{2^{q_1+s_2}}.\tag{3.14}$$

Taking into account that x_0 is an integer number and taking into account the inequalities (3.11), from (3.14) we derive

$$0<x_0\leqslant 3^{s_1}.\tag{3.15}$$

Summarizing the above considerations, we can formulate the following theorem.

Theorem 3.3. *For any sequence of positive integers* (2.9) *the Diophantine equation* (3.7) *has a unique particular solution* (x_0, y_0) *such that* x_0 *and* y_0 *obey the inequalities* (3.15) *and* (3.12).

Theorem 3.3 is similar to Item 3 in Theorem 3.1. Relying on Theorem 3.3, we can introduce the following two functions

$$x_0 = X_0(s_1, q_1, s_2), \qquad\qquad y_0 = Y_0(s_1, q_1, s_2). \tag{3.16}$$

Substituting (3.16) into the equation (3.7), we get

$$2^{q_1+s_2} \cdot x_0 - 3^{s_1} y_0 = 2^{q_1} - 1. \tag{3.17}$$

For any sequence of positive integers (2.9) the first term in (3.17) is even, the coefficient 3^{s_1} is odd, and the right side $2^{q_1} - 1$ is also odd. Therefore y_0 in (3.17) should be odd. We have proved the following theorem.

Theorem 3.4. *If the arguments of the function* $Y_0(s_1, q_1, s_2)$ *are positive, then its value is positive and odd.*

The values of the function $X_0(s_1, q_1, s_2)$ can be either even or odd. However, solving the Diophantine equation (3.3), we need to get odd numbers m_1 and m_2. Therefore we should choose proper values of t in (3.9):

$$m_1 = \begin{cases} Y_0(s_1, q_1, s_2) + 2^{q_1+s_2} \cdot 2\,t & \text{if } X_0(s_1, q_1, s_2) \text{ is odd,} \\ Y_0(s_1, q_1, s_2) + 2^{q_1+s_2} \cdot (2\,t + 1) & \text{if } X_0(s_1, q_1, s_2) \text{ is even,} \end{cases} \tag{3.18}$$

$$m_2 = \begin{cases} X_0(s_1, q_1, s_2) + 3^{s_1} \cdot 2\,t & \text{if } X_0(s_1, q_1, s_2) \text{ is odd,} \\ X_0(s_1, q_1, s_2) + 3^{s_1} \cdot (2\,t + 1) & \text{if } X_0(s_1, q_1, s_2) \text{ is even,} \end{cases} \tag{3.19}$$

The formulas (3.18) and (3.19) provide the required odd numbers m_1 and m_2 solving the equation (3.3). Therefore the odd number $n = 2^{s_1} \cdot m_1 - 1$ proves Theorem 2.2 for the case $r = 1$ in (2.9).

The functions $X_0(s_1, q_1, s_2)$ and $Y_0(s_1, q_1, s_2)$ in (3.16) cannot be expressed by formulas. However they can be computed using Euclidean algorithm.

```
Dioph_solve:=proc(A,B,C) option remember:
 local AA,BB,XY,q:
 if B=0 then return [C/A,0]
 elif A=0 then return [0,-C/B]
 elif A<B
  then
   BB:=irem(B,A,'q'):
   XY:=procname(A,BB,C):
   return [XY[1]+q*XY[2],XY[2]]
 else
  AA:=irem(A,B,'q'):
  XY:=procname(AA,B,C):
  return [XY[1],XY[2]+q*XY[1]]
 end if
end proc:
```

The above code solves the Diophantine equation $A\,x - B\,y = C$ with $A > 0$ and $B > 0$. Below is the code for the function $Y_0(s_1, q_1, s_2)$.

```
Y0:=proc(s1,q1,s2) option remember:
 local A,B,C,XY,YY:
 A:=2^(q1+s2):
 B:=3^s1:
 C:=2^q1-1:
 XY:=Dioph_solve(A,B,C):
 YY:=XY[2]:
 if YY<0
  then return YY+(iquo(abs(YY),A)+1)*A
 elif YY>=A
  then return YY-iquo(YY,A)*A
 else
  return YY
 end if
end proc:
```

And finally we provide the code for the function $X_0(s_1, q_1, s_2)$.

```
X0:=proc(s1,q1,s2) option remember:
 local A,B,C,XY,XX,YY:
 A:=2^(q1+s2):
 B:=3^s1:
 C:=2^q1-1:
 XY:=Dioph_solve(A,B,C):
 XX:=XY[1]:
 YY:=XY[2]:
 if YY<0
  then return XX+(iquo(abs(YY),A)+1)*B
 elif YY>=A
  then return XX-iquo(YY,A)*B
 else
  return XX
 end if
end proc:
```

All of the above code is given using the programming language of the Maple package, version 9.01. Maple is a trademark of Waterloo Maple Inc.

4. Long evolution sequences.

Let's proceed to the case $r = 2$. In this case we have the long evolution sequence s_1, q_1, s_2, q_2, s_3 in (2.9) that subdivides into two short sequences

$$s_1,\ q_1,\ s_2, \qquad\qquad s_2,\ q_2,\ s_3. \tag{4.1}$$

The sequences (4.1) generate two Diophantine equations similar to (3.7). The solution of the first one is given by the formulas (3.18) and (3.19). Let's write these two formulas as follows:

$$m_1 = M_{10} + 2^{q_1+s_2}\cdot 2\,t_1, \qquad\qquad m_2 = \tilde{M}_{20} + 3^{s_1}\cdot 2\,t_1. \tag{4.2}$$

The solution of the second Diophantine equation is given by similar formulas

$$m_2 = M_{20} + 2^{q_2+s_3} \cdot 2\,t, \qquad m_3 = \tilde{M}_{30} + 3^{s_2} \cdot 2\,t. \tag{4.3}$$

The formulas (4.2) and (4.3) produce a new Diophantine equation

$$2^{q_2+s_3} \cdot t - 3^{s_1} \cdot t_1 = \frac{\tilde{M}_{20} - M_{20}}{2} \tag{4.4}$$

with respect to the variables t_1 and t. Note that $\tilde{M}_{20}$ and M_{20} are odd. Therefore the value of the fraction in the right hand side of the equation (4.4) is integer.

The equation (4.4) is very similar to (3.7). It is always solvable due to Item 1 of Theorem 3.1 since $\mathrm{GCD}(2^{q_2+s_3}, 3^{s_1}) = 1$. Its solution is written as

$$t_1 = T_{20} + 2^{q_2+s_3} \cdot t_2, \qquad t = \tilde{T}_{21} + 3^{s_1} \cdot t_2. \tag{4.5}$$

Substituting (4.5) into the first equality (4.2) and into the second equality (4.3), we derive

$$m_1 = M_{11} + 2^{q_1+q_2+s_2+s_3} \cdot 2\,t_2, \qquad m_3 = \tilde{M}_{31} + 3^{s_1+s_2} \cdot 2\,t_2, \tag{4.6}$$

where

$$M_{11} = M_{10} + 2^{q_1+s_2} \cdot 2\,T_{20}, \qquad \tilde{M}_{31} = \tilde{M}_{30} + 3^{s_2} \cdot 2\,\tilde{T}_{21}. \tag{4.7}$$

The formulas (4.6) and (4.7) serve the case $r = 2$ in (2.9). The odd number $n = 2^{s_1} \cdot m_1 - 1$ proves Theorem 2.2 for this case.

The formulas (4.6) are similar to (4.2). Therefore we can increment r by 1, complement the sequences (4.1) by s_3, q_3, s_4, and write the formulas

$$m_3 = M_{30} + 2^{q_3+s_4} \cdot 2\,t, \qquad m_4 = \tilde{M}_{40} + 3^{s_3} \cdot 2\,t. \tag{4.8}$$

The equalities (4.6) and (4.8) produce a new Diophantine equation

$$2^{q_3+s_4} \cdot t - 3^{s_1+s_2} \cdot t_2 = \frac{\tilde{M}_{31} - M_{30}}{2} \tag{4.9}$$

similar to (4.4). The equation (4.9) is similar to (3.7). It is always solvable due to Item 1 of Theorem 3.1 since $\mathrm{GCD}(2^{q_3+s_4}, 3^{s_1+s_2}) = 1$. Its solution is written as

$$t_2 = T_{30} + 2^{q_3+s_4} \cdot t_3, \qquad t = \tilde{T}_{31} + 3^{s_1+s_2} \cdot t_3. \tag{4.10}$$

We substitute (4.10) into the first equality (4.6) and into the second equality (4.8). As a result we get the equalities

$$m_1 = M_{12} + 2^{q_1+q_2+q_3+s_2+s_3+s_4} \cdot 2\,t_3, \qquad m_4 = \tilde{M}_{41} + 3^{s_1+s_2+s_3} \cdot 2\,t_3, \tag{4.11}$$

where

$$M_{12} = M_{11} + 2^{q_1+q_2+s_2+s_3} \cdot 2\,T_{30}, \qquad \tilde{M}_{41} = \tilde{M}_{40} + 3^{s_3} \cdot 2\,\tilde{T}_{31}. \tag{4.12}$$

Continuing the process we can go further by induction. Extending the sequence of formulas (4.2), (4.6), and (4.11), we write

$$m_1 = M_{1\,r-1} + 2^{\,Q_r} \cdot 2\,t_r, \qquad\qquad m_{r+1} = \tilde{M}_{r+1\,1} + 3^{S_r} \cdot 2\,t_r \tag{4.13}$$

The exponentials Q_r and S_r are given by the formulas

$$Q_r = \sum_{i=1}^{r} q_i + \sum_{i=2}^{r+1} s_i, \qquad\qquad S_r = \sum_{i=1}^{r} s_i. \tag{4.14}$$

They obey the relationships which are used below in proving the inductive step:

$$Q_r = Q_{r-1} + q_r + s_{r+1}, \qquad\qquad S_r = S_{r-1} + s_r. \tag{4.15}$$

The recurrent relationships (4.15) are immediate from (4.14).

For $r > 1$ the quantities $M_{1\,r-1}$ and $\tilde{M}_{r+1\,1}$ in (4.13) are defined inductively:

$$M_{1\,r-1} = M_{1\,r-2} + 2^{\,Q_{r-1}} \cdot 2\,T_{r0}. \tag{4.16}$$

$$\tilde{M}_{r+1\,1} = \tilde{M}_{r+1\,0} + 3^{s_r} \cdot 2\,\tilde{T}_{r1}, \tag{4.17}$$

where T_{r0} and $\tilde{T}_{r1}$ are defined by solving the Diophantine equation

$$2^{q_r+s_{r+1}} \cdot t - 3^{S_{r-1}} \cdot t_{r-1} = \frac{\tilde{M}_{r1} - M_{r0}}{2} \tag{4.18}$$

whose general solution is taken in the following form:

$$t_{r-1} = T_{r0} + 2^{q_r+s_{r+1}} \cdot t_r, \qquad\qquad t = \tilde{T}_{r1} + 3^{S_{r-1}} \cdot t_r. \tag{4.19}$$

In (4.18) and in (4.17) we see the quantities M_{r0} and $\tilde{M}_{r+1\,0}$ respectively, (4.12) being a particular case of (4.18) and (4.17) for $r = 3$. These quantities are determined by the short sequence of positive integers s_r, q_r, s_{r+1} through the functions X_0 and Y_0 defined in the previous section:

$$M_{r0} = \begin{cases} Y_0(s_r, q_r, s_{r+1}) & \text{if } \ X_0(s_r, q_r, s_{r+1}) \ \text{ is odd,} \\ Y_0(s_r, q_r, s_{r+1}) + 2^{q_r+s_{r+1}} & \text{if } \ X_0(s_r, q_r, s_{r+1}) \ \text{ is even,} \end{cases} \tag{4.20}$$

$$\tilde{M}_{r+1\,0} = \begin{cases} X_0(s_r, q_r, s_{r+1}) & \text{if } \ X_0(s_r, q_r, s_{r+1}) \ \text{ is odd,} \\ X_0(s_r, q_r, s_{r+1}) + 3^{s_r} & \text{if } \ X_0(s_r, q_r, s_{r+1}) \ \text{ is even.} \end{cases} \tag{4.21}$$

The formulas (4.20) and (4.21) mean that M_{r0} and $\tilde{M}_{r+1\,0}$ are always odd.

The quantities $M_{1\,r-1}$ and $\tilde{M}_{r+1\,1}$ in (4.13) are defined inductively by means of the formulas (4.16) and (4.17) for $r > 1$. The case $r = 1$ is the base of this induction. In this case $M_{1\,r-1}$ turns to M_{10} which is given by the formula (4.20). The case $r = 1$ is the base of induction for the formula (4.13) as well. In this case $\tilde{M}_{r+1\,1}$ turns to $\tilde{M}_{21}$, while (4.13) turns to (4.2). Therefore $\tilde{M}_{21} = \tilde{M}_{20}$ and $\tilde{M}_{20}$ is given by the formula (4.21).

In order to prove the formulas (4.13) now it is sufficient to prove the inductive step $r-1 \to r$. Let's replace r by $r-1$ in (4.13) and assume that the formulas obtained from (4.13) in such a way are valid:

$$m_1 = M_{1\,r-2} + 2^{\,Q_{r-1}} \cdot 2\,t_{r-1}, \qquad\qquad m_r = \tilde{M}_{r\,1} + 3^{S_{r-1}} \cdot 2\,t_{r-1} \tag{4.22}$$

We complement these formulas with the formulas similar to (4.2) and (4.3):

$$m_r = M_{r0} + 2^{q_r+s_{r+1}} \cdot 2\,t, \qquad\qquad m_{r+1} = \tilde{M}_{r+1\,0} + 3^{s_r} \cdot 2\,t. \tag{4.23}$$

These formulas are derived from the Diophantine equation similar to (3.7) and associated with the short sequence s_r, q_r, s_{r+1}.

The second formula (4.22) and the first formula (4.23) represent the same quantity m_r. Equating their right hand sides, we obtain a Diophantine equation with respect to t_{r-1} and t. This Diophantine equation coincides with (4.18). Its solution is given by the formulas (4.19). The formulas (4.13) then are derived by substituting (4.19) into the first formula (4.22) and into the second formula (4.23) if we take into account (4.15), (4.16), and (4.17).

Thus, the formulas (4.13) are proved. They serve the general case $r > 1$ in (2.9). The first formula (4.13) determines the positive odd number m_1 depending on an arbitrary positive integer parameter t_r. For any integer value of this parameter the positive odd number $n = 2^{s_1} \cdot m_1 - 1$ proves Theorem 2.2 in the case of an arbitrary finite sequence of positive integers (2.9).

5. Conclusions.

Theorem 2.2 is the main result of the present paper. It positively solves the problem of a predefined Collatz evolution for finite length evolution patterns (2.9). But this result cannot be easily transferred to the case of infinite patterns (2.7).

6. Dedicatory.

This paper is dedicated to my sister Svetlana Abdulovna Sharipova.

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