# COMPLETE NORMALITY CONDITIONS FOR THE DYNAMICAL SYSTEMS ON RIEMANNIAN MANIFOLDS.

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Abstract. New additional equations for the Newtonian dynamical systems on Riemannian manifolds are found. They supplement the previously found weak normality conditions up to the complete normality conditions for Newtonian dynamical systems.

## 1. Introduction.

Let $M$ be the Riemannian manifold with metric tensor $g_{ij}$. The Newtonian dynamical system on M is the law of motion of particles on $M$ given by the following equations in local coordinates

$$\dot{x}^i = v^i \qquad \dot{v}^i = \Phi^i(x^1, \dots, x^n, v^1, \dots, v^n) \tag{1.1}$$

The equations (1.1) realize the Newton's second law for the mass point with unit mass. The concept of normality for Newtonian dynamical systems was introduced and then investigated in the series of papers [1 - 5]. It is based on the study of some hypersurface $S$ in $M$. For some point $P$ on $S$ we define the initial data

$$x^i\big|_{t=0} = x^i(P) \qquad v^i\big|_{t=o} = v(P)n^i(P) \tag{1.2}$$

for the equations (1.1). Here $n^i(P)$ are the components of unit normal vector at the point $P$ and $v(P)$ in (1.2) is some scalar function on $S$. The equations (1.1) and initial data (1.2) determine the particle flow starting at $t = 0$ from $S$ along the normal vector $\mathbf{n}(P)$ with the initial velocity $v(P)$. For $t > 0$ these particles form another hypersurface $S_t$ and determine the family of diffeomorphisms $f_t : S \longrightarrow S_t$.

**Definition 1.** The family of diffeomorphisms $f_t$ is called the normal shift along the trajectories of dynamical system if the trajectories of (1.1) cross all hypersurfaces $S_t$ along their normal vectors.

**Definition 2.** Newtonian dynamical system (1.1) is called the dynamical system accepting the normal shift of hypersurfaces if for any hypersurface $S \subset M$ one can find the function $v(P)$ on $S$ such that the family of diffeomorphisms $f_t$ given by (1.1) and (1.2) is the normal shift of $S$.

The normality condition from definition 2 was first stated in [1] and [2]. By the analysis of this condition in [3] for Euclidean case $M = \mathbb{R}^n$ two relatively independent normality conditions were derived: weak normality condition and additional normality condition. Each of them is written in the form of the system of partial differential equations for the functions $\Phi^i$ from (1.1). They two both form the sufficient condition for the normality condition from definition 2 to be fulfilled for the dynamical system (1.1).

In [5] the weak normality condition was generalized for the non-Euclidean case of an arbitrary Riemannian manifold $M$. In this paper we generalize the additional normality condition from [3] for the case of Riemannian manifold and we get the complete normality condition for (1.1) in form of aggregate of weak and additional normality conditions. Here as in the Euclidean case of $\mathbb{R}^n$ the complete normality is sufficient but not necessary for the normality condition from the definition 2.

Since it seems difficult to obtain the conditions equivalent to definition 2 and written in form of differential equations for the functions $\Phi^i$ we shall replace the definition 2 by more strict definition 3.

**Definition 3.** We say that dynamical system (1.1) satisfies the strong normality condition if for any hypersurface $S \subset M$, for any choice of the point $P_0 \in S$ and for any real number $v_0 \neq 0$ one can find the function $v(P)$ such that it is normalized by $v(P_0) = v_0$ and the family of diffeomorphisms defined by this function is the normal shift of hypersurface $S$.

Note here that for to avoid the multivalued functions $v(P)$ one should always consider only connected and simply connected hypersurfaces in definitions 2 and 3.

## 2. On the expansion of the algebra of tensor fields.

Systems of differential equations of the form (1.1) are usually connected with the vector fields on the manifolds. In our case the right hand sides of (1.1) form the vector field on the tangent bundle $TM$ for the manifold $M$. Digressing for a while from the particular vector field given by (1.1) let us consider some vector $\mathbf{V}$ tangent to the tangent bundle

$$\mathbf{V} = X^1 \frac{\partial}{\partial x^1} + \dots + X^n \frac{\partial}{\partial x^n} + W^1 \frac{\partial}{\partial v^1} + \dots + W^n \frac{\partial}{\partial x^n} \tag{2.1}$$

First $n$ components of the vector (2.1) form the tangent vector $\mathbf{X} = \pi(\mathbf{V})$ to $M$. Other components of (2.1) after some modification form another tangent vector $\mathbf{Z} = \rho(\mathbf{V})$ to $M$ whose components are $Z^i = W^i + \Gamma^i_{jk} v^k X^j$. Two linear maps $\pi$ and $\rho$ project the vector $\mathbf{V}$ onto the pair of vectors $\mathbf{X}$ and $\mathbf{Z}$ tangent to the manifold $M$.

Projections $\pi$ and $\rho$ applied to the vector fields on $TM$ however will not give the vector fields on $M$. They give the elements of quite other set: the expanded algebra of tensor fields on $M$. Exact definition of tensor field from such algebra is the following.

**Definition 4.** Tensor field of expanded algebra is a map argument of which is a point of tangent bundle $TM$ and the value of which is in the tensor algebra build over the tangent space to $M$ at the point being the projection of argument from $TM$ to $M$.

Expanded algebra of tensor fields is equipped with the natural operations of tensor product and contraction. It is also equipped with two operations of covariant differentiation $\nabla_i$ and $\tilde{\nabla}_i$. The detailed discussions of these features of expanded algebra of tensor fields can be found in [5].

Each point of $TM$ is the pair of point $P \in M$ and tangent vector $\mathbf{v}$ at this point (vector of velocity). If we map each point of $TM$ onto the corresponding vector $\mathbf{v}$ then we get the vector field $\mathbf{v}$ of expanded algebra. From $\mathbf{v}$ we construct the scalar field $v = |\mathbf{v}|$ of expanded algebra being the modulus of velocity. In addition to these two fields we define the vector field $\mathbf{N} = |\mathbf{v}|^{-1}\mathbf{v}$ that consists of unit vectors directed along the vector of velocity $\mathbf{v}$. It is defined only at that points of $TM$ where $\mathbf{v} \neq 0$. By means of the components of $\mathbf{N}$ we construct two projector valued fields

$$Q^i_k = N_k N^i \qquad\qquad P^i_k = (\delta^i_k - N_k N^i) \tag{2.2}$$

from the expanded algebra. Various relationships with projectors $\mathbf{Q}$ and $\mathbf{P}$ from (2.2) can be found in [5].

Replacing time derivatives of velocity in (1.1) by its covariant time derivatives we can rewrite (1.1) as follows

$$\dot{x}^i = v^i \qquad\qquad \nabla_t v^i = F^i(x^1, \dots, x^n, v^1, \dots, v^n) \tag{2.3}$$

where $F^i = \Phi^i + \Gamma^i_{jk} v^k v^j$ are the components of vector field $\mathbf{F}$ of expanded algebra known as a force field of Newtonian dynamical system (2.3).

## 3. Weak and complete normality conditions.

For the dynamical system (2.3) we consider the Cauchy problem with initial data (1.2) on some hypersurface $S$. Let us choose the local coordinates $u^1, \dots, u^{n-1}$ on $S$. Solving the above Cauchy problem we obtain the family of trajectories of the dynamical system (2.3) on $M$ parameterized by $u^1, \dots, u^{n-1}$. Variation of these variables defines the following vectors $\boldsymbol{\tau}_i$ tangent to $M$

$$\boldsymbol{\tau}_i = \frac{\partial x^1}{\partial u^i} \frac{\partial}{\partial x^1} + \dots + \frac{\partial x^n}{\partial u^i} \frac{\partial}{\partial x^n} \tag{3.1}$$

on the trajectories of (2.3). Scalar products of $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_{n-1}$ and $\mathbf{N}$ are the scalar functions $\varphi_i = \langle \boldsymbol{\tau}_i, \mathbf{N} \rangle = g_{kq} \tau^k_i N^q$. These functions define the mutual orientation of trajectories and the hypersurfaces $S_t$.

**Definition 5.** Say that Newtonian dynamical system (2.2) satisfies the weak normality condition if each function $\varphi_i = \varphi_i(t)$ is the solution of linear homogeneous ordinary differential equation of second order for any choice of parametric family of trajectories of it.

The main result of [5] is that the weak normality condition is equivalent to the following system of nonlinear partial differential equations for the force field $\mathbf{F}$ of dynamical system

$$
\begin{aligned}
&(v^{-1}F_i + \tilde{\nabla}_i(F^k N_k))P^i_k = 0\\
&(\nabla_i F_k + \nabla_k F_i - 2v^{-2}F_i F_k)N^k P^i_q +\\
&\qquad + v^{-1}(\tilde{\nabla}_k F_i F^k - \tilde{\nabla}_k F^r N^k N_r F_i)P^i_q = 0
\end{aligned}
\tag{3.2}
$$

Let the weak normality condition in form of the equations (3.2) be fulfilled. Then for to get normality in the sense of definitions 2 and 3 we should provide the following initial data

$$
\varphi_i\big|_{t=0} = 0 \qquad\qquad \dot{\varphi}_i\big|_{t=0} = 0
\tag{3.3}
$$

for the functions $\varphi_i$ by the proper choice of function $v(P) = v(u^1, \ldots, u^{n-1})$ in (1.2). There $\mathbf{n}(P)$ is the unit normal vector for the hypersurface $S$. From (1.2) we obtain

$$
\mathbf{N}\big|_{t=0} = \mathbf{n}(u^1, \ldots, u^{n-1})
\tag{3.4}
$$

Because of (3.4) first part of the initial conditions (3.3) is satisfied for any choice of $v(P)$. Now let us proceed with second part of initial conditions in (3.3)

$$
\dot{\varphi}_i\big|_{t=0} = \nabla_t \tau^j_i N_j\big|_{t=0} + g_{jk}\tau^k_i \nabla_t N^j\big|_{t=0}
\tag{3.5}
$$

For $\nabla_t N^j$ we use (3.15) and (3.19) from [5] and then we obtain $\nabla_t N^j = v^{-1}P^j_q F^q$. For the covariant derivatives of the vectors (3.1) we use (3.18) from [5]

$$
\nabla_t \tau^j_i\big|_{t=0} = \frac{\partial^2 x^j}{\partial t \partial u^i}\big|_{t=0} + \Gamma^j_{pq}\frac{\partial x^p}{\partial u^i} v^q\big|_{t=0}
$$

Taking into account the initial data (1.2) we may transform this relationship into the following form

$$
\nabla_t \tau^j_i\big|_{t=0} = \frac{\partial v}{\partial u^i} n^j\big|_{t=0} + v\frac{\partial n^j}{\partial u^i}\big|_{t=0} + \Gamma^j_{pq} n^q \tau^p_i\big|_{t=0}
\tag{3.6}
$$

For the further transformations of the equations (3.6) we should recall some facts concerning submanifolds of Riemannian spaces

$$
\frac{\partial \tau^j_k}{\partial u^i} + \Gamma^j_{pq}\tau^q_k \tau^p_i = \hat{\Gamma}^q_{ik}\tau^j_q + b_{ik} n^j
\tag{3.7}
$$

$$
\frac{\partial n^j}{\partial u^i} + \Gamma^j_{pq} n^q \tau^p_i = -b^q_i \tau^j_q
\tag{3.8}
$$

Here $b_{ik}$ and $b^q_i$ are components of tensor of second quadratic form and $\hat{\Gamma}^q_{ik}$ are components of metric connection on $S$ defined by the metric $\hat{g}_{ik} = g_{pq}\tau^p_i \tau^q_k$ induced from $M$ to the hypersurface $S$. The equations (3.7) and (3.8) are known as Gauss and Weingarten formulae (see [6] and [7]). Comparing (3.6) with (3.8) we get

$$
\nabla_t \tau^j_i\big|_{t=0} = \frac{\partial v}{\partial u^i} n^j\big|_{t=0} - v b^q_i \tau^j_q\big|_{t=0}
\tag{3.9}
$$

Now we substitute the above obtained formula $\nabla_t N^j = v^{-1}P^j_q F^q$ and (3.9) into (3.5). Then from (3.3) we obtain the following equation for still unknown function $v = v(u^1, \ldots, u^{n-1})$

$$
\frac{\partial v}{\partial u^i} = -v^{-1} g_{jk} F^j \tau^k_i
\tag{3.10}
$$

Components $F^j$ of the force field in (3.10) depend on the velocity vector therefore they contain the dependence on the unknown function $v$ in the form of $v^p = v(u^1, \dots, u^{n-1})n^p(u^1, \dots, u^{n-1})$.

The equations (3.10) form the overdetermined system of differential equations. For to have common solution $v$ they should satisfy the compatibility conditions. We obtain these conditions when consider the following derivatives

$$\frac{\partial^2 v}{\partial u^i \partial u^j} = \frac{\partial}{\partial u^i}\left(-v^{-1} g_{pq} F^p \tau^q_j\right) \tag{3.11}$$

calculated according to the equations (3.10). To make shorter all further calculations we define covariant derivatives $D_i$ as covariant derivatives with respect to $u^i$ given by the formula (3.18) from [5]. These covariant derivatives are applicable to the tensor-valued functions whose domain of definition is $S$. For the tensor fields of $M$ restricted to the hypersurface $S$ these covariant derivatives are calculated as $D_i = \tau^k_i \nabla_k$. Note that $D_i$ are not applicable to tensor fields of expanded algebra unless some lifting of $S$ from $M$ to tangent bundle $TM$ is defined. For the force field $\mathbf{F}$ in (3.10) and (3.11) such lifting is given by the second part of initial conditions in (1.2). Therefore

$$D_i F^p = \tau^k_i \nabla_k F^p + (n^k D_i v + v D_i n^k)\tilde{\nabla}_k F^p \tag{3.12}$$

The results of applying $D_i$ to vector fields $\tau^j_k$ and $n^j$ are defined by Gauss and Weingarten formulae (3.7) and (3.8)

$$D_i \tau^j_k = \hat{\Gamma}^q_{ik} \tau^j_q + b_{ik} n^j \qquad\qquad D_i n^j = -b^q_i \tau^j_q \tag{3.13}$$

So the derivatives $D_i$ establish the link between inner geometry of $S$ and the geometry of outer space $M$ itself.

The derivatives $D_i v$ for the scalar function $v$ on $S$ are given by the equations (3.10) the compatibility condition for which we are going to derive now. Let's combine (3.12) and (3.13)

$$\begin{aligned} \frac{\partial^2 v}{\partial u^i \partial u^j} &= -v^{-3} F_p \tau^p_i F_q \tau^q_j - v^{-1} \nabla_p F_q \tau^p_i \tau^q_j + \\ &+ v^{-2} F_p \tau^p_i \tilde{\nabla}_r F_q n^r \tau^q_j + \tilde{\nabla}_p F_q b^r_i \tau^p_r \tau^q_j - v^{-1} \Gamma^k_{ij} \tau^q_k F_q - v^{-1} b_{ij} n^q F_q \end{aligned} \tag{3.14}$$

After exchanging indices $i$ and $j$ in (3.14) we obtain another expression for the same derivative in left hand side of (3.14). The difference of these two expressions should be zero. This gives us the compatibility condition for (3.10) in the following form

$$\begin{aligned} &\tau^k_i \tau^q_j \left( \frac{\nabla_q F_k - \nabla_k F_q}{v} + N^r \frac{\tilde{\nabla}_r F_q F_k - \tilde{\nabla}_r F_k F_q}{v^2} \right) + \\ &+ b^r_i \tau^k_r \tau^q_j \tilde{\nabla}_k F_q - b^r_j \tau^k_r \tau^q_i \tilde{\nabla}_k F_q = 0 \end{aligned} \tag{3.15}$$

In order to simplify the equations (3.15) we need to recall the following relationships due to (3.4)

$$(g_{kr} \tau^r_i \hat{g}^{ij}) \tau^q_j = P^q_k \tag{3.16}$$

By means of contracting (3.15) with the quantities enclosed in brackets in (3.16) we obtain

$$\begin{aligned} &P^k_i P^q_j \left( \frac{\nabla_q F_k - \nabla_k F_q}{v} + N^r \frac{\tilde{\nabla}_r F_q F_k - \tilde{\nabla}_r F_k F_q}{v^2} \right) + \\ &+ H^k_i P^q_j \tilde{\nabla}_k F_q - H^k_j P^q_i \tilde{\nabla}_k F_q = 0 \end{aligned} \tag{3.17}$$

where $H^k_i$ are the components of the symmetric operator $\mathbf{H}$ determined by the second quadratic form of hypersurface $S$

$$H^k_i = g_{ir} \tau^r_q \hat{g}^{qj} b^p_j \tau^k_p \tag{3.18}$$

Symmetric operator $\mathbf{H}$ with the matrix (3.18) satisfies the following relationships

$$\mathbf{HP} = \mathbf{PH} = \mathbf{H} \qquad\qquad \operatorname{rank}(\mathbf{H}) \leq n - 1 \tag{3.19}$$

**Lemma 1.** *For any point $P$ in $M$ and any operator $\mathbf{H}$ in tangent space to $M$ at this point satisfying the properties* (3.19) *one can find the hypersurface $S$ passing trough the point $P$ such that the matrix of the operator $\mathbf{H}$ is given by formula* (3.18) *at this point.*

Proof of this lemma technical. We omit it noting only that one can choose $S$ in the class of quadrics for some local coordinates $x^1, \dots, x^n$ in the neighborhood of $P$.

Lemma 1 shows that the matrix of the operator $\mathbf{H}$ in (3.17) is rather unrestricted. This let us replace (3.17) by two separate equations of the following form

$$\begin{aligned}&P_i^k P_j^q \left(\nabla_q F_k - \nabla_k F_q + N^r \frac{\tilde{\nabla}_r F_q F_k - \tilde{\nabla}_r F_k F_q}{v}\right) = 0\\ &H_i^k P_j^q \tilde{\nabla}_k F_q - H_j^k P_i^q \tilde{\nabla}_k F_q = 0\end{aligned} \tag{3.20}$$

Now consider the operator $\mathbf{K}$ with the matrix $K_i^j = P_i^k \tilde{\nabla}_k F_q P_r^q g^{rj}$. Its properties are similar to (3.19) i.e.

$$\mathbf{KP} = \mathbf{PK} = \mathbf{K} \qquad\qquad \operatorname{rank}(\mathbf{K}) \leq n-1 \tag{3.21}$$

Second equation (3.20) then means that the product $\mathbf{KH}$ is symmetric operator. Because of lemma 1 the operator $\mathbf{H}$ is arbitrary symmetric operator. Taking $\mathbf{H} = \mathbf{P}$ and using (3.21) we get that $\mathbf{K}$ is also the symmetric operator. The product of two symmetric operators $\mathbf{KH}$ is symmetric if and only if they are commutating. Thus we have $\mathbf{KH} = \mathbf{HK}$ for arbitrary symmetric operator $\mathbf{H}$ satisfying the relationships (3.19). This is possible only if $\mathbf{K}$ is proportional to $\mathbf{P}$ with some scalar factor $\mathbf{K} = \lambda\mathbf{P}$. Scalar factor $\lambda$ is easily calculated as $\lambda = \operatorname{tr}(\mathbf{K})/\operatorname{tr}(\mathbf{P})$. Now we can write

$$\begin{aligned}&P_i^k P_j^q \left(N^r \frac{\tilde{\nabla}_r F_k}{v} F_q - \nabla_q F_k\right) = P_i^k P_j^q \left(N^r \frac{\tilde{\nabla}_r F_q}{v} F_k - \nabla_k F_q\right)\\ &P_i^k \tilde{\nabla}_k F^q P_q^j = \frac{P_r^k \tilde{\nabla}_k F^q P_q^r}{n-1} P_i^j\end{aligned} \tag{3.22}$$

excluding the matrix $\mathbf{H}$ from (3.20) at all. The equations (3.22) just above combined with (3.2) form the complete normality conditions which are sufficient for the definitions 2 and 3 to be fulfilled for the dynamical system (2.2).

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