DYNAMICAL SYSTEMS ACCEPTING THE NORMAL SHIFT ON AN ARBITRARY RIEMANNIAN MANIFOLD.

Boldin A.Yu., Dmitrieva V.V., Safin S.S., Sharipov R.A.

August 6, 1993

Abstract. Newtonian dynamical systems which accept the normal shift on an arbitrary Riemannian manifold are considered. For them the determinating equations making the weak normality condition are derived. The expansion for the algebra of tensor fields is constructed.

1. Introduction.

The concept of dynamical systems accepting the normal shift was introduced in [1] (see also [2][1]). It appears as the result of transferring the classical Bonnet transformation from geometry to the field of dynamical systems. In [1] and [2] the Euclidean situation was studied i.e. the dynamical systems in $\mathbb{R}^n$ accepting the normal shift (we refer the reader to that papers for the detailed bibliography).

Apart from recent results of [1] and [2] one can find another purely geometric generalization of Bonnet transformation from [3] and [4] which is realized as a normal shift along the geodesics on some Riemannian manifold. The natural question here is how do these two generalizations relate each other. This question was investigated in [5]. There the special subclass of dynamical systems accepting the normal shift was separated for which the normal shift is equivalent to the geometrical generalization of the Bonnet transformation for some conformally-Euclidean metric in $\mathbb{R}^n$. Such systems are called metrizable. Comparing the explicit description of metrizable dynamical systems given in [5] with the examples of [1] and [2] one can conclude that non-metrizable systems do exist. Therefore the the concept of normal shift along the dynamical system is wider than the Bonnet transformation for conformally-Euclidean metrics. However it do not embrace the case of Bonnet transformation for arbitrary metric. In this paper below we consider the most general situation and study the dynamical systems accepting the normal shift on an arbitrary Riemannian manifold.

2. Newtonian dynamical systems on the Riemannian manifold.

The main object considered in [1] and [2] is the second order dynamical system in $\mathbb{R}^n$

$$\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) \tag{2.1}$$

reflecting the Newton's second law. Now let $\mathbf{r}$ be not radius-vector of a point in $\mathbb{R}^n$ but the vector of local coordinates for some manifold M. This case let us write the equations (2.1) in form of the system of the first order equations

$$\dot{x}^i = v^i \qquad\qquad \dot{v}^i = \Phi^i(\mathbf{r}, \mathbf{v}) \tag{2.2}$$

Systems of differential equations of the form (2.2) are traditionally connected with vector fields on manifolds. In this particular case the right hand sides of (2.2) form the components of the vector field not on M however but on the tangent bundle TM

$$\boldsymbol{\Phi} = v^1 \frac{\partial}{\partial x^1} + \dots + v^n \frac{\partial}{\partial x^n} + \Phi^1 \frac{\partial}{\partial v^1} + \dots + \Phi^n \frac{\partial}{\partial v^n} \tag{2.3}$$

[1] See also **chao-dyn/9403003** and **patt-sol/9404001**

First n components of the vector field (2.3) separately can be interpreted as components of the velocity vector

$$\mathbf{v} = v^1\frac{\partial}{\partial x^1} + \dots + v^n\frac{\partial}{\partial x^n} \tag{2.4}$$

tangent to M. Rest part of components in (2.3) do not admit such interpretation. But if the manifold M is equipped with Riemannian metric g_{ij} and with the metrical connection Γ^k_{ij} then using the components of (2.3) one can form the following quantities

$$F^i = \Phi^i + \Gamma^i_{jk}v^k v^j \tag{2.5}$$

that do behave like the components of some vector tangent to M when we change the local map on M

$$\mathbf{F} = F^1\frac{\partial}{\partial x^1} + \dots + F^n\frac{\partial}{\partial x^n} \tag{2.6}$$

Same indices on the different levels in (2.5) and everywhere below imply summation. Vector $\mathbf{F}$ from (2.6) with the components of the form (2.5) is natural to be considered as a vector of force. The analogy of (2.1) and (2.2) then becomes complete. The existence of the vector $\mathbf{F}$ lets us speak about the angle between force and velocity. It lets us also break the force $\mathbf{F}$ into two parts directed along the velocity and perpendicular to the velocity. When $\mathbf{F} = 0$ equations (2.2) become the equations of geodesic line.

3. Metric on the tangent bundle and the expansion of the algebra of tensor fields.

Let us consider again the manifold M with the Riemannian metric g_{ij}. Vector fields on the tangent bundle TM have the form

$$\mathbf{V} = X^1\frac{\partial}{\partial x^1} + \dots + X^n\frac{\partial}{\partial x^n} + W^1\frac{\partial}{\partial v^1} + \dots + W^n\frac{\partial}{\partial v^n} \tag{3.1}$$

First n components of $\mathbf{V}$ in (3.1) like in (2.4) are interpreted as the components of the vector $\mathbf{X}$ tangent to M. The whole set of components of (3.1) is transformed as follows

$$\tilde{X}^i = \frac{\partial\tilde{x}^i}{\partial x^j}X^j \qquad\qquad \tilde{W}^i = \frac{\partial^2\tilde{x}^i}{\partial x^j\,\partial x^k}v^k X^j + \frac{\partial\tilde{x}^i}{\partial x^s}W^s \tag{3.2}$$

when one change the local variables $x^1, \dots, x^n$ for $\tilde{x}^1, \dots, \tilde{x}^n$. The components of metric connection Γ^k_{ij} under the same action are transformed as follows

$$\tilde{\Gamma}^i_{pq}\frac{\partial\tilde{x}^p}{\partial x^j}\frac{\partial\tilde{x}^q}{\partial x^k} = \frac{\partial\tilde{x}^i}{\partial x^s}\Gamma^s_{jk} - \frac{\partial^2\tilde{x}^i}{\partial x^j\,\partial x^k} \tag{3.3}$$

Comparing (3.2) and (3.3) we conclude that the following quantities Z^i produced from the components of (3.1)

$$Z^i = W^i + \Gamma^i_{jk}v^k X^j \tag{3.4}$$

are transformed like the components of tangent vector to M under the change of local map on M. For the vector (2.3) the corresponding vector (3.4) coincides with (2.5). So the vector $\mathbf{V}$ tangent to TM can be replaced by two vectors $\mathbf{X}$ and $\mathbf{Z}$ tangent to M. This gives rise to the pair of linear maps π and ρ where π is canonical projection from the bundle TM to the base manifold M

$$\mathbf{X} = \pi(\mathbf{V}) \qquad\qquad \mathbf{Z} = \rho(\mathbf{V}) \tag{3.5}$$

The relationship (3.4) lets us introduce the Riemannian metric on the tangent bundle TM by forming the following quadratic form on the set of vectors (3.1)

$$\tilde{g}(\mathbf{V},\mathbf{V}) = g(\pi(V),\pi(V)) + g(\rho(V),\rho(V)) \tag{3.6}$$

In terms of differentials of local coordinates the metric (3.6) is written as follows

$$\tilde{g} = (g_{ij} + \Gamma^p_{ir} v^r g_{pq} \Gamma^q_{js} v^s) dx^i dx^j + 2(g_{iq} \Gamma^q_{js} v^s) dx^i dv^j + g_{ij} dv^i dv^j \tag{3.7}$$

Metric tensor for (3.7) is determined by the initial metric tensor g_{ij} of M. Its matrix has the natural block structure

$$\tilde{g}_{ij} = \begin{pmatrix} g_{ij} + \Gamma^p_{ir} v^r g_{pq} \Gamma^q_{js} v^s & \Gamma^p_{ir} v^r g_{pj} \\ g_{iq} \Gamma^q_{js} v^s & g_{ij} \end{pmatrix} \tag{3.8}$$

Upper left block corresponds to the coordinates on the base while the lower right block of (3.8) corresponds to the coordinates on the stalk. Explicit form of metric tensor makes possible the computation of the components of metric connection for it but we do not need them in what follows.

Much more interesting objects are the images of vector fields on TM under the action of maps π and ρ from (3.5). In general they could not be treated as vector fields on M. They are the vector-valued functions whose argument is the point of TM and whose value is the vector tangent to M at the point being the projection of the argument. Such function composes the algebra over the ring of scalar functions on TM. This algebra can easily be expanded up to the algebra of tensor-valued functions on M with the argument from TM. It is natural to call it the expanded algebra of tensor fields on M. The first example of the element of such algebra is the force field (2.6) for Newtonian dynamical system. This field has the real physical meaning when the manifold M is the configuration space for some real mechanical system restricted by inner bounds.

Expanded algebra of tensor fields on M is equipped with the natural operations of tensor product and contraction. Presence of metric on M adds two operations of covariant differentiation. First of them is the differentiation by velocity or the velocity gradient. For scalar, vectorial and covector fields it is defined by the following expressions

$$\tilde{\nabla}_i \varphi = \frac{\partial \varphi}{\partial v^i} \qquad \tilde{\nabla}_i X^m = \frac{\partial X^m}{\partial v^i} \qquad \tilde{\nabla}_i X_m = \frac{\partial X_m}{\partial v^i} \tag{3.9}$$

Second is the covariant differentiation by coordinate or the space gradient. It is the modification of the ordinary covariant differentiation. The result of its application to the scalar, vectorial and covector fields is as follows

$$\begin{aligned} \nabla_i \varphi &= \frac{\partial \varphi}{\partial x^i} - \Gamma^p_{ik} \frac{\partial \varphi}{\partial v^p} v^k \\ \nabla_i X^m &= \frac{\partial X^m}{\partial x^i} + \Gamma^m_{ip} X^p - \Gamma^p_{ik} \frac{\partial X^m}{\partial v^p} v^k \\ \nabla_i X_m &= \frac{\partial X_m}{\partial x^i} - \Gamma^p_{im} X_p - \Gamma^p_{ik} \frac{\partial X_m}{\partial v^p} v^k \end{aligned} \tag{3.10}$$

For other tensor fields the action of (3.9) and (3.10) is continued according to the condition of concordance with the operations of tensor product and contraction. Ordinary tensor fields are the part of expanded algebra. Velocity gradient for them is always zero while the space gradient coincides with ordinary covariant derivative. Particularly

$$\nabla_k g_{ij} = 0 \qquad \tilde{\nabla}_k g_{ij} = 0 \tag{3.11}$$

The relationships (3.11) express the compatibility of metric and metrical connection in terms of the above covariant derivatives.

One more element of the expanded algebra is the vector field of the velocity (2.4). For the velocity and space gradients of it we have

$$\nabla_k v^i = 0 \qquad \tilde{\nabla}_k v^i = \delta^i_k \tag{3.12}$$

Scalar field of modulus of velocity is defined by $\mathbf{v}$ according to the following formula

$$v^2 = |\mathbf{v}|^2 = g_{ij} v^i v^j \tag{3.13}$$

For the gradients of the scalar field defined by (3.13) we have

$$\nabla_k v = 0 \qquad \tilde{\nabla}_k v = N_k = g_{kq} N^q \tag{3.14}$$

The quantities N^q in (3.14) are the components of the unit vector field $\mathbf{N}$ directed along the velocity $\mathbf{v} = v\mathbf{N}$. Gradients for this vector field are the following

$$\nabla_k \mathbf{N}^i = 0 \qquad \tilde{\nabla}_k \mathbf{N}^i = v^{-1}(\delta^i_k - N_k N^i) \tag{3.15}$$

They are calculated on the base of (3.12) and (3.14). Components of the matrix $P^i_k = \delta^i_k - N_k N^i$ in (3.15) are the components of operator valued field $\mathbf{P}$ of normal projectors on a hyperplane that is perpendicular to $\mathbf{v}$. Covariant derivatives for $\mathbf{P}$ itself are

$$\nabla_k P^i_j = 0 \qquad \tilde{\nabla}_k P^i_j = -v^{-1}(g_{jq} P^q_k N^i + N_j P^i_k) \tag{3.16}$$

Along with $\mathbf{P}$ we define the additional projector-valued field with the components $Q^i_j = N_j N^i$. For it we have $\mathbf{P} + \mathbf{Q} = \mathbf{1}$. Covariant derivatives of $\mathbf{Q}$ are easily calculated from (3.16)

$$\nabla_k Q^i_j = 0 \qquad \tilde{\nabla}_k Q^i_j = v^{-1}(g_{jq} P^q_k N^i + N_j P^i_k) \tag{3.17}$$

In addition to the properties (3.16) and (3.17) we can see that projector-valued fields $\mathbf{P}$ and $\mathbf{Q}$ are symmetric respective to the metric g_{ij} on M.

Let the functions $x^1(t), \ldots, x^n(t)$ define the parametric curve on M in local map. The derivatives $\partial_t x^i$ define the tangent vector to M (it may be treated as velocity vector of a point moving along this curve). Suppose that at each point of this curve we have the tangent vector $\mathbf{U}$ to M (but possibly not tangent to the curve) with the components $U^i(t)$. In other words we have the vector-valued function on the curve. Let us produce another vector-valued function on that curve according to the formula

$$\nabla_t U^i = \partial_t U^i + \Gamma^i_{jk} U^j \partial_t x^k \tag{3.18}$$

Formula (3.18) defines the covariant derivative of the vector-valued function on the curve by the parameter t of it. It is well known that such derivative is zero by parallel displacement of the vector along the curve.

Let's add the functions $v^1(t), \ldots, v^n(t)$ to $x^1(t), \ldots, x^n(t)$ and consider all them as the curve lifted from M to TM. Such lifting is called natural if $v^i(t) = \partial_t x^i(t)$. However here we consider arbitrary (may be not natural) liftings. Let us define some vector field $\mathbf{U}$ of expanded algebra in some neighborhood of the lifted curve. Substituting $x^1(t), \ldots, x^n(t)$, $v^1(t), \ldots, v^n(t)$ for its argument we obtain the vector-valued function on the former curve. For this function we have

$$\nabla_t U^i = \nabla_k U^i \partial_t x^k + \tilde{\nabla}_k U^i \nabla_t v^k \tag{3.19}$$

Formula (3.19) approves the names velocity and space gradients for the covariant derivatives (3.9) and (3.10). It is easily modified for the case of scalar, covectorial and all other types of tensor fields of expanded algebra.

Let us find the relation between the ordinary covariant derivatives on TM and the modified covariant derivatives (3.9) and (3.10) on the manifold M itself. In order to do it consider the pair of vector fields $\mathbf{X}$ and $\mathbf{Y}$ on the tangent bundle TM. For the projections π and ρ from (3.5) applied to the commutator of these vector fields we derive

$$\begin{aligned} \pi([\mathbf{X}, \mathbf{Y}]) = \nabla_{\pi(\mathbf{X})} \pi(\mathbf{Y}) - \nabla_{\pi(\mathbf{Y})} \pi(\mathbf{X}) + \\ + \tilde{\nabla}_{\rho(\mathbf{X})} \pi(\mathbf{Y}) - \tilde{\nabla}_{\pi(\mathbf{Y})} \pi(\mathbf{X}) \end{aligned} \tag{3.20}$$

$$\begin{aligned} \rho([\mathbf{X}, \mathbf{Y}]) = \nabla_{\pi(\mathbf{X})} \rho(\mathbf{Y}) - \nabla_{\pi(\mathbf{Y})} \rho(\mathbf{X}) + \\ + \tilde{\nabla}_{\rho(\mathbf{X})} \rho(\mathbf{Y}) - \tilde{\nabla}_{\rho(\mathbf{Y})} \rho(\mathbf{X}) - R(\pi(\mathbf{X}), \pi(\mathbf{Y}))\mathbf{v} \end{aligned} \tag{3.21}$$

Here $R(\mathbf{A}, \mathbf{B})$ is the operator-valued skew-symmetric bilinear form defined by the curvature tensor of M. Formulae (3.20) and (3.21) are proved by the direct calculations in the coordinates.

Let $\mathbf{Z}$ be one more vector field on TM and φ be the scalar field on TM which also may be treated as a scalar field of the expanded algebra on M. Then

$$\nabla_{\mathbf{X}}\varphi = \partial_{\mathbf{X}}\varphi = \nabla_{\pi(\mathbf{X})}\varphi + \tilde{\nabla}_{\rho(\mathbf{X})}\varphi \tag{3.22}$$

Left hand side of (3.22) contain the ordinary covariant derivative on TM which coincides for the scalar function with the derivative along the vector X. Covariant derivatives in the right hand sides of (3.20), (3.21) and (3.22) are that of (3.9) and (3.10). For to calculate the covariant derivatives on TM we use the following formula from [6]

$$\begin{aligned}2\tilde{g}(\nabla_{\mathbf{X}}\mathbf{Y},\mathbf{Z}) &= \partial_{\mathbf{X}}\tilde{g}(\mathbf{Y},\mathbf{Z}) + \partial_{\mathbf{Y}}\tilde{g}(\mathbf{X},\mathbf{Z}) - \partial_{\mathbf{Z}}\tilde{g}(\mathbf{X},\mathbf{Y}) +\\ &+ \tilde{g}([\mathbf{Z},\mathbf{X}],\mathbf{Y}) + \tilde{g}([\mathbf{Z},\mathbf{Y}],\mathbf{X}) + \tilde{g}([\mathbf{X},\mathbf{Y}],\mathbf{Z})\end{aligned} \tag{3.23}$$

Covariant derivative on TM in (3.23) is ordinary one. Now let us use the relationship

$$\tilde{g}(\mathbf{X},\mathbf{Y}) = g(\pi(\mathbf{X}),\pi(\mathbf{Y})) + g(\rho(\mathbf{X}),\rho(\mathbf{Y})) \tag{3.24}$$

The relationship (3.24) is derived directly from (3.6). As the result of substitution of (3.24) and (3.20), (3.21), (3.22) into the identity (3.23) we get two formulae

$$\begin{aligned}\pi(\nabla_{\mathbf{X}}\mathbf{Y}) &= \nabla_{\pi(\mathbf{X})}\pi(\mathbf{Y}) + \tilde{\nabla}_{\rho(\mathbf{X})}\pi(\mathbf{Y}) -\\ &- \frac{1}{2}(R(\rho(\mathbf{Y},\mathbf{v})\pi(\mathbf{X}) + R(\rho(\mathbf{X}),\mathbf{v})\pi(\mathbf{Y}))\end{aligned} \tag{3.25}$$

$$\begin{aligned}\rho(\nabla_{\mathbf{X}}\mathbf{Y}) &= \nabla_{\pi(\mathbf{X})}\rho(\mathbf{Y}) + \tilde{\nabla}_{\rho(\mathbf{X})}\rho(\mathbf{Y}) -\\ &- \frac{1}{2}R(\pi(\mathbf{X}),\pi(\mathbf{Y}))\mathbf{v}\end{aligned} \tag{3.26}$$

Covariant derivatives in left hand sides of (3.25) and (3.26) are ordinary ones like in (3.23). Covariant derivatives in right hand sides of (3.25) and (3.26) are expanded ones. They should be treated as in (3.9) and (3.10).

4. Variations of trajectories and the equations of weak normality.

Let us consider the Newtonian dynamical system (2.2). In the second of the equations (2.2) we change the ordinary derivative of the velocity by its covariant derivative according to the formula (3.18). This gives us the equations

$$\partial_t x^i = v^i \qquad\qquad \nabla_t v^i = F^i \tag{4.1}$$

containing the force vector of (2.5). The equations (4.1) are more natural form for the Newton's second law on the manifold M.

For the Newtonian dynamical system (4.1) now we consider the Cauchy problem with the following initial data

$$x^i\big|_{t=0} = x^i(s) \qquad\qquad \partial_t x^i\big|_{t=0} = v^i(s) \tag{4.2}$$

depending on some parameter s. Because of (4.2) the trajectories of the dynamical system also depend on s. In local coordinates they are given by the functions $x^i = x^i(t,s)$. Their derivatives by s are the coordinates of some vector $\boldsymbol{\tau}$ tangent to M. It is the vector of variations of trajectories $\tau^i(t,s) = \partial_s x^i(t,s)$. The time derivative of $\boldsymbol{\tau}$ is the vector of variations of velocities

$$\nabla_t \tau^i = \partial_t \tau^i + \Gamma^i_{jk}\tau^j v^k \tag{4.3}$$

We shall use (4.3) to obtain the equations for the vector $\boldsymbol{\tau}$ from the equations (4.1) of the dynamical system itself. In order to do it let us differentiate (4.1) by s and combine the result with (4.3) and the time derivative of (4.3). Then we obtain

$$\nabla_t\nabla_t\tau^i + R^i_{jkq}v^j v^k \tau^q = \nabla_q F^i \tau^q + \tilde{\nabla}_q F^i \nabla_t \tau^q \tag{4.4}$$

Here R^i_{jqk} is the curvature tensor for M. So the variation vector $\boldsymbol{\tau}$ satisfies the ordinary differential equations of the second order (4.4).

The normal shift condition according to [1] and [2] consists in the orthogonality of $\boldsymbol{\tau}$ and the vector of velocity $\mathbf{v}$. It is convenient to express $\mathbf{v}$ via the unit vector $\mathbf{N}$ from (3.14) and (3.15). According to the method developed in [1] and [2] we introduce the function φ as a scalar product

$$\varphi = \langle \boldsymbol{\tau}, \mathbf{N} \rangle = \tau^i N_i = g_{ij} \tau^i N^j \tag{4.5}$$

For its time derivative by differentiating (4.5) we obtain

$$\partial_t \varphi = g_{ij} \nabla_t \tau^i N^j + g_{ij} \tau^i \nabla_t N^j$$

Because of (3.11) and (3.19) the metric tensor makes no contribution by covariant differentiation. To calculate $\nabla_t N^j$ we use the relationships (3.15) and (3.19). As a result we have

$$\partial_t \varphi = g_{ij} \nabla_t \tau^i N^j + v^{-1} g_{ij} \tau^i P^j_k F^k$$

Because $\mathbf{P}$ is the symmetric operator field we may rewrite this expression in the following form

$$\partial_t \varphi = g_{ij} \nabla_t \tau^i N^j + v^{-1} g_{ij} F^i P^j_q \tau^q \tag{4.6}$$

Differentiating (4.6) by t we obtain the second derivative for φ

$$\partial_{tt} \varphi = \partial_t (g_{ij} \nabla_t \tau^i N^j) + \partial_t (v^{-1} g_{ij} F^i P^j_q \tau^q) \tag{4.7}$$

Taking into account (3.15) and (3.19) we may write the first summand in (4.7) as

$$\partial_t (g_{ij} \nabla_t \tau^i N^j) = \nabla_{tt} \tau^i N_i + v^{-1} F_i P^i_q \nabla_t \tau^q \tag{4.8}$$

The first summand in (4.8) in turn is transformed by use of the equation (4.4) for the vector of variation of trajectory

$$\nabla_{tt} \tau^i N_i = N_i \nabla_q F^i \tau^q + N_i \tilde{\nabla}_q F^i \nabla_t \tau^q - N_i R^i_{\alpha q \beta} \tau^q v^\alpha v^\beta \tag{4.9}$$

Let us insert the projectors $\mathbf{P}$ and $\mathbf{Q}$ into all terms in the right hand side of (4.9). In order to do it we use the decomposition of identical operator as $\mathbf{1} = \mathbf{P} + \mathbf{Q}$. For the first summand we have

$$N_i \nabla_q F^i \tau^q = \nabla_i F^k N_k P^i_q \tau^q + \nabla_q F^k N^q N_k \tau^i N_i \tag{4.10}$$

For the second summand in (4.9) by the same way we derive

$$N_i \tilde{\nabla}_q F^i \nabla_t \tau^q = \tilde{\nabla}_i F^k N_k P^i_q \nabla_t \tau^q + \tilde{\nabla}_q F^k N^q N_k \nabla_t \tau^i N_i \tag{4.11}$$

The third summand in (4.9) vanishes because the vectors $\mathbf{N}$ and $\mathbf{v}$ are collinear $\mathbf{v} = |\mathbf{v}| \mathbf{N}$. Indeed

$$N_i R^i_{\alpha q \beta} \tau^q v^\alpha v^\beta = |\mathbf{v}|^2 R_{i \alpha q \beta} N^i N^\alpha N^\beta \tau^q \tag{4.12}$$

Curvature tensor is skew-symmetric in i and α. Therefore the result of contraction in (4.12) is zero.

Now let us transform the second summand in (4.7) using the relationships (3.14) and (3.19)

$$\partial (v^{-1} g_{ij} F^i P^j_k \tau^k) = -v^{-2} F^k N_k F_i P^i_q \tau^q + v^{-1} F_i P^i_q \nabla_t \tau^q + v^{-1} \nabla_t (F_i P^i_q) \tau^q \tag{4.13}$$

Then let us transform the last summand in (4.13) with the help of (3.16) and (3.19)

$$\begin{aligned} v^{-1} \nabla_t (F_i P^i_q) \tau^q &= \nabla_k F^i N^k P^i_q \tau^q + v^{-1} \tilde{\nabla}_k F_i F^k P^i_q \tau^q - \\ &- v^{-2} F^k N_k F_i P^i_q \tau^q - v^{-2} F_j P^j_k F^k \tau^i N_i \end{aligned} \tag{4.14}$$

As a result the second summand in (4.7) may be obtained by substituting (4.14) into (4.13) and the first one may be obtained by substitution of (4.10) and (4.11) into (4.9) followed by substitution of (4.9) into (4.8). By analyzing the obtained formulae (4.5), (4.6) and (4.7) for φ, $\partial_t\varphi$ and $\partial_{tt}\varphi$ we find that all they are the linear functionals with respect to vectors $boldsymbol\tau$ and $\nabla_t\boldsymbol{\tau}$ forming the phase space for the integral trajectories of the system of differential equations (4.4). Using φ, $\partial_t\varphi$ and $\partial_{tt}\varphi$ we construct another functional L of the form

$$\partial_{tt}\varphi - P\partial_t\varphi - Q\varphi = L \tag{4.15}$$

The coefficients for L in the formula (4.15) we define as

$$P = \tilde\nabla_q F^k N^q N_k \qquad\qquad Q = \tilde\nabla_q F^k N^q N_k - v^{-2} P^q_k F_q F^k$$

Then the functional L itself may be written in the form

$$L = \xi_i P^i_q \nabla_t \tau^q + \zeta_i P^i_q \tau^q \tag{4.16}$$

The coefficients ξ_i and ζ_i in (4.16) are defined by the above calculations. They are the following

$$\xi_i = \tilde\nabla_i F^k N_k + 2v^{-1} F_i \tag{4.17}$$

$$\begin{aligned}\zeta_i = (\nabla_i F_k &+ \nabla_k F_i - 2v^{-2} F_i F_k) N^k + \\ &+ v^{-1}(\tilde\nabla_k F_i F^k - \tilde\nabla_k F^q N^k N_q F_i)\end{aligned} \tag{4.18}$$

For the dynamical system (4.1) to accept the normal shift on M (see [1] and [2]) the functional L should identically vanish. This condition in [1] and [2] is called the condition of weak normality. In the present situation it gives us the following equations for ξ_i and ζ_i from (4.17) and (4.18)

$$\xi_i P^i_q = 0 \qquad\qquad \zeta_i P^i_q = 0 \tag{4.20}$$

The equations (4.20) may be rewritten in the following explicit form

$$(v^{-1} F_i + \tilde\nabla_i (F^k N_k)) P^i_q = 0 \tag{4.21}$$

$$\begin{aligned}(\nabla_i F_k + \nabla_k F_i - 2v^{-2} F_i F_k) N^k P^i_q &+ \\ v^{-1}(\tilde\nabla_k F_i F^k - \tilde\nabla_k F^r N^k N_r F_i) P^i_q &= 0\end{aligned} \tag{4.22}$$

The total number of the equations (4.21) and (4.22) is $2n$. It coincides with the twiced dimension of the manifold M. However because of presence of projector matrices P^i_q in them they are not independent. The number of independent equations in the system of (4.21) and (4.22) is $2n-1$ which is in concordance with the results of [1] and [2].

The equations (4.21) and (4.22) are the covariant form of the equations of weak normality on an arbitrary Riemannian manifold. In Euclidean case $M = \mathbb{R}^n$ they were derived in [1] and [2] by use of spherical coordinates in the space of velocities. The question of introducing the proper spherical coordinates here in the general situation is interesting but it is the subject for separate paper. Analyzing the equations (4.21) one can see that if the vector of force is decomposed into two parts first being along the velocity and second being perpendicular to it then the second part is defined by the first one. This fact was observed in [1] and [2]. It remains true for the general non-Euclidean situation.

This paper is written under the financial support of two of authors (Boldin A.Yu. and Sharipov R.A.) by Russian Fund for Fundamental Researches.

References

1. Boldin A.Yu., Sharipov R.A., *Dynamical systems accepting the normal shift.*, Theor. and Math. Phys. **97** (1993), no. 3, 386–395. (Russian)
2. Boldin A.Yu., Sharipov R.A., *Dynamical systems accepting the normal shift.*, Dikladi Akademii Nauk. **334** (1994), no. 2, 165–167. (Russian)
3. Bianchi L., *Sopra le deformazioni isogonali delle superficie a curvatura constante in geometria elliptica ed hiperbolica.*, Annali di Matem. **18** (1911), no. 3, 185–243.
4. Tenenblat K., *Bäcklund theorem for submanifolds of space forms and a generalized wave equation.*, Bol. Soc. Bras. Math. **18** (1985), no. 2, 67–92.
5. Sharipov R.A., *Problem of metrizability for the dynamical systems accepting the normal shift.*, Theor. and Math. Phys. (to appear). (Russain)
6. Kobayashy Sh., Nomidzu K., *Foundations of differential geometry.*, vol. 1, Intersc. Publ., New York - London, 1963.

Department of Mathematics, Bashkir State University, Frunze street 32, 450074 Ufa, Russia.
E-mail address: `root@bgua.bashkiria.su`