NEWTONIAN NORMAL SHIFT IN MULTIDIMENSIONAL RIEMANNIAN GEOMETRY.

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Abstract. Explicit description for arbitrary Newtonian dynamical system admitting the normal shift in Riemannian manifold of the dimension $n \geqslant 3$ is found. On the base of this result the kinematics of normal shift of hypersurfaces along trajectories of such system is studied.

1. Introduction.

In series of papers [1–16] written in 1993–1996 a theory was constructed that determines and describes special class of **Newtonian dynamical systems admitting the normal shift** of hypersurfaces in Riemannian and Finslerian manifolds. On the base of these papers two theses were prepared: thesis for the degree of **Doctor of Sciences in Russia** [17] and thesis for the degree of **Candidate of Sciences in Russia** [18]. However, some results included in thesis [17] are still not published in journals (see [10] and [11]).

Moreover, when preparing thesis [17], in paper [16] an error was found. Eliminating this error led to new result that consists in complete and exhausting description of all Newtonian dynamical systems admitting the normal shift in Riemannian[1] manifolds of the dimension $n \geqslant 3$. The goal of this paper is to explain this new result, and to give a description for kinematics of normal shift of hypersurfaces, more detailed than it was possible before now. On the base of the same result one can get new (more simple) proof for the main theorem from unpublished paper [11] (see also § 7 in Chapter VI of thesis [17]), and one can answer the question by A. V. Bolsinov and A. T. Fomenko, which they asked when author reported thesis [17] in the seminar of the Chair of Differential Geometry and its Applications at Moscow State University.

Classical construction of normal shift of hypersurfaces in Riemannian manifold M is well-known. In its original form it arises in the case when M is $\mathbb{R}^3$ with standard flat Euclidean metric. Let S be two-dimensional surface in $\mathbb{R}^3$. From each point p on S we draw a segment of straight line in the direction of normal vector $\mathbf{n} = \mathbf{n}(p)$. Denote by p_t the second end of this segment. When p runs over S, point p_t sweeps some other surface S_t as shown on Fig. 1.1 below. So we have the map $f_t\colon S \to S_t$ known as *classical normal shift* or as *Bonnet transformation*.

When transferring from $M = \mathbb{R}^3$ to the case of arbitrary Riemannian manifold M, we replace surfaces by hypersurfaces, and straight line segments, connecting

[1]This result has no direct generalization for the case of dynamical systems in Finslerian manifolds (this case was considered in [10], see also Chapter VIII in thesis [17]). But, nevertheless, it possibly has some analog in Finslerian geometry. This problem is not yet studied.

p and p_t, by the segments of geodesic lines. In this form classical construction of normal shift of hypersurfaces is known as *geodesic normal shift*. Construction of geodesic normal shift contains a numeric parameter t, i. e. it determines the whole family of hypersurfaces S_t. When parameter t is varied, point p_t moves along geodesic lines (here they are called *trajectories of the shift*). In local coordinates $x^1, \ldots, x^n$ on M they are described by ordinary differential equations

$$\ddot{x}^k + \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma^k_{ij}\, \dot{x}^i\, \dot{x}^j = 0, \tag{1.1}$$

where $k = 1, \ldots, n$. By $\Gamma^k_{ij} = \Gamma^k_{ij}(x^1, \ldots, x^n)$ in (1.1) we denote components of standard metric connection Γ for Riemannian metric $\mathbf{g}$ on M. The property of normality of geodesic shift is expressed by the following well-known fact.

Theorem 1.1. *All hypersurfaces S_t in the construction of geodesic normal shift are perpendicular to trajectories of shift.*

In other words, trajectories of shift described by the equations (1.1) cross each hypersurface S_t transversally; at the points of intersection they pass along normal vectors to S_t.

The idea of generalizing the construction of normal shift, which was realized in papers [1–16], is very simple. It consists in replacing (1.1) by slightly more complicated ordinary differential equations in t:

$$\ddot{x}^k + \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma^k_{ij}\, \dot{x}^i\, \dot{x}^j = F^k(x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n). \tag{1.2}$$

When $M = \mathbb{R}^3$ and $\Gamma^k_{ij} = 0$, the equations (1.2) express Newton's second law: they describe the motion of a mass point with unit mass in the force field determined by right hand sides of these equations. In the case of arbitrary Riemannian manifold M these equations, as appears, also have physical interpretation. They describe the dynamics of complex mechanical systems with holonomic constraints. Manifold M arises as configuration space of such systems, its dimension is determined by actual number of degrees of freedom (upon resolving all constraints). Thereby M is canonically equipped with the structure of Riemannian manifold, its metric is given by quadratic form of kinetic energy:

$$K = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}\, \dot{x}^i\, \dot{x}^j$$

(see details in Chapter II of thesis [17]). Due to the analogy with Newton's second law the equations (1.2) are called the equations *of Newtonian dynamical system* on Riemannian manifold. Vector $\mathbf{F}$, whose components are given by right hand sides of the equations (1.2), is called a **force vector**. It determines **force field** of Newtonian dynamical system (1.2).

Note that the choice of local coordinates in defining Newtonian dynamical system (1.2) is of no importance. By the change of local coordinates the shape of the equations remains unchanged, thought the components of connections and components of force vector are transformed according to standard formulas, which are well-known from course of differential geometry (see [19–22]). This property expresses **coordinate covariance** of differential equations (1.2).

2. Newtonian normal shift of hypersurfaces.

Having formulated the idea of generalizing the construction of geodesic normal shift, we shall describe how it was realized in papers [1–16]. Let S be some hypersurface in M and let p be some point on S. In local coordinates $x^1, \ldots, x^n$ on M such point p is characterized by its coordinates $x^1(p), \ldots, x^n(p)$ and by normal vector $\mathbf{n}(p)$ at this point. Let's use Newtonian dynamical system (1.2) in order to define a shift of hypersurface S. With this aim let's associate each point $p \in S$ with the following initial data for the system of differential equations (1.2):

$$x^k\Big|_{t=0} = x^k(p), \qquad\qquad \dot{x}^k\Big|_{t=0} = \nu(p)\cdot n^k(p). \tag{2.1}$$

Here $n^k(p)$ is k-th component of normal vector $\mathbf{n}(p)$, and $\nu(p)$ is some scalar quantity depending on the point $p \in S$. Solving Cauchy problem with initial data (2.1) for the equations (1.2), we obtain a set of functions

$$\begin{cases} x^1 = x^1(t, p),\\ \dots\dots\dots\\ x^n = x^n(t, p).\end{cases} \tag{2.2}$$

These functions define in parametric form the trajectory $r = r(t, p)$ of Newtonian dynamical system with force field $\mathbf{F}$. This trajectory at initial instant of time $t = 0$ crosses hypersurface S at the point p, passing in the direction of unitary normal vector $\mathbf{n}(p)$. Parameter $\nu(p)$ in initial data (2.1) determines the modulus of initial velocity for this trajectory:

$$\mathbf{v}\Big|_{t=0} = \nu(p)\cdot\mathbf{n}(p). \tag{2.3}$$

Choice of local coordinates in defining trajectory $r = r(t, p)$ is of no importance. Change of local coordinates changes functions (2.2), but it doesn't change the curve $r = r(t, p)$. This is due to coordinate covariance of differential equations (1.2) and coordinate covariance of initial data (2.1).

Drawing trajectories $r = r(t, p)$ outgoing from all points $p \in S$ and taking points $p_t = r(t, p)$ that corresponds to some fixed value of parameter t, we obtain the hypersurface S_t and displacement map $f_t\colon S \to S_t$. However, we should remember two nuances. Parameter t for trajectory $r = r(t, p)$ of dynamical system (1.2) do not coincide with its length. The range of this parameter t always includes initial point $t = 0$, but it can be a restricted interval

$$t_1(p) < t < t_2(p).$$

Upper and lower bounds of this interval in general case depend on the point $p \in S$. Hence for a fixed value of t the displacement map $f_t\colon S \to S_t$ can be defined not for all points $p \in S$.

Second nuance is due to singular points (caustics) that may appear on the hypersurface S_t for large enough values of parameter t. This imposes one more restriction onto the range of parameter t. Note that this restriction is present in classical construction of geodesic normal shift as well.

The above two nuances restrict possible range of parameter t. However, if we are interested in small values of t only (as below), we can use the following lemma.

Lemma 2.1. *If parameter $\nu(p)$ in* (2.3) *is a smooth nonzero function on the hypersurface S, then for each $p \in S$ there exists some neighborhood $S' = O_S(p)$ on S and there exists a number ε such that displacement maps $f_t\colon S' \to S'_t$ are defined for all $t \in (-\varepsilon, +\varepsilon)$. They form smooth one-parametric family of diffeomorphisms.*

This lemma is an immediate consequence of theorem on existence, uniqueness, and smooth dependence of initial data for the solution of Cauchy problem for systems of ODE's (see [23] and [24]). Taking into account lemma 2.1, we can consider displacement maps $f_t : S \to S_t$, which are possibly defined only locally on S, as a construction of **shift** of hypersurface S **along trajectories** of Newtonian dynamical system with force field $\mathbf{F}$. Function $\nu(p)$ on S is a parameter in such construction of shift.

Shift of hypersurface S along trajectories of dynamical system (1.2), as it was constructed above, possess the property of normality at the initial instant of time $t = 0$. This means that trajectories of shift are passing through initial hypersurface S along normal vectors on it. Does this property persist for $t \neq 0$, i. e. can we prove theorem similar to theorem 1.1 ? The answer to this question in general case is negative (see examples in [18]). But there are special cases, when the property of normality persists for all instants of time. We describe them by formulating the following definition.

Definition 2.1. Shift $f_t\colon S \to S_t$ of hypersurface S along trajectories of Newtonian dynamical system with force field $\mathbf{F}$ is called a **normal shift** if all hypersurfaces S_t (for all permissible values of parameter t) are orthogonal to trajectories of shift.

3. Dynamical systems admitting the normal shift.

What does the property of normality for shift $f_t\colon S \to S_t$ depend on ? On the choice of hypersurface S ? On the force field $\mathbf{F}$ of Newtonian dynamical system ? We also have the opportunity to choose the function $\nu(p)$ on S. In the case of identically zero force field $\mathbf{F} = 0$ (which corresponds to geodesic flow on M) the choice $\nu(p) = 1$ provides normality condition from definition 2.1 for arbitrary initial hypersurface $S \subset M$. Are there some other force fields with similar property ? The aim to know this was the motivation for writing preprint [1]. In this preprint we introduced the concept of **Newtonian dynamical system admitting the normal shift**. This concept has become a central point for later investigations.

Definition 3.1. Newtonian dynamical system on Riemannian manifold M is called a system **admitting the normal shift** if for any hypersurface S in M, and for any point $p_0 \in S$, there is a neighborhood $S' = O_S(p_0)$ of the point p_0 on S, and there is a smooth function $\nu(p)$ in S', such that the shift $f_t\colon S' \to S'_t$ defined by the function $\nu(p)$ is a normal shift along trajectories of considered dynamical in the sense of definition 2.1.

The condition stated in definition 3.1 was called **the normality condition** for Newtonian dynamical system with force field $\mathbf{F}$. First we considered the case $M = \mathbb{R}^2$ (see [1]). In [1] we derived partial differential equations for the components of force field $\mathbf{F}$ which, when being fulfilled, are sufficient to provide normality condition from definition 3.1. These equations were called the **normality equations** or, more exactly, **weak normality equations**. In preprint [1] we also constructed first non-trivial examples of dynamical systems that admit normal shift. When generalizing these results from $M = \mathbb{R}^2$ to multidimensional case $M = \mathbb{R}^n$ in preprint [1], we have found that weak normality equations should be supplemented by so called additional normality equations. All above results in brief form were announced in [4]. Their full version were published in [2] and [3]. Later in papers [6] and [7] they were generalized for the case of arbitrary Riemannian manifold M. Main result of paper [6] is the derivation of weak normality equations for this more complicated geometric situation. We write these equations without comments so far:

$$\begin{cases} \displaystyle\sum_{i=1}^{n}\left(v^{-1}\,F_i+\sum_{j=1}^{n}\tilde{\nabla}_i\left(N^j\,F_j\right)\right)P^i_k=0,\\[2ex] \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j+\nabla_jF_i-2\,v^{-1}\,F_i\,F_j\right)N^j\,P^i_k+\\[2ex] \displaystyle+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde{\nabla}_jF_i}{v}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde{\nabla}_jF_r}{v}\,F_i\right)P^i_k=0.\end{cases}\tag{3.1}$$

Additional normality equations for the force field $\mathbf{F}$ of Newtonian dynamical systems on Riemannian manifolds were derived in [7]. They look like

$$\begin{cases} \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_i\,\tilde{\nabla}_mF_j}{v}-\nabla_iF_j\right)=\\[2ex] \displaystyle\qquad=\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_j\,\tilde{\nabla}_mF_i}{v}-\nabla_jF_i\right),\\[2ex] \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}P^j_\sigma\,\tilde{\nabla}_jF^i\,P^\varepsilon_i=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}\frac{P^j_m\,\tilde{\nabla}_jF^i\,P^m_i}{n-1}\,P^\varepsilon_\sigma.\end{cases}\tag{3.2}$$

Results of papers [6] and [7] were announced in [12]. In deriving the equations (3.2) we have found that it is convenient to make slight modification of definition 3.2. We added normalizing condition for the function $\nu(p)$:

$$\nu(p_0)=\nu_0.\tag{3.3}$$

As a result we obtained **strong normality condition**. It is formulated as follows.

Definition 3.2. Newtonian dynamical system on Riemannian manifold M is called a system admitting the normal shift **in strong sense** if for any hypersurface S in M, for any point $p_0 \in S$, and for any real number $\nu_0 \neq 0$ there is a neighborhood $S' = O_S(p_0)$ of the point p_0 on S, and there is a smooth function $\nu(p)$ in S' normalized by the condition (3.3), such that the shift $f_t\colon S' \to S'_t$ defined by the function $\nu(p)$ is a normal shift along trajectories of considered dynamical in the sense of the definition 2.1.

Strong normality condition from definition 3.2 implies normality condition formulated in definition 3.1. The relation of strong normality condition with normality equations (3.1) and (3.2) is described by the following theorem.

Theorem 3.1. *Newtonian dynamical system on Riemannian manifold M admits the normal shift in strong sense if and only if its force field $\mathbf{F}$ satisfies normality equations* (3.1) *and* (3.2) *simultaneously.*

Theorem 3.1 was proved in [13]. Detailed version of this proof can be found in thesis [17]. We shall not give this proof here, since this would require to reproduce many details of derivation of the normality equations (3.1) and (3.2), and hence this would be doubling for the papers [6] and [7]. Instead, we shall give comments to normality equations (3.1) and (3.2).

4. Extended algebra of tensor fields.

Normality equations (3.1) and (3.2), as well as the equations (1.2), possess the property of **coordinate covariance**. Writing these equations, we assume that some local coordinates $x^1, \dots, x^n$ in M are chosen. Under the change of local coordinates all quantities, which are contained in the equations (3.1) and (3.2), do change according some definite rules. However, this do not change the shape of these equations. Components of force vector $\mathbf{F}$ in (3.1) and (3.2) depend on $x^1, \dots, x^n$, and on components of velocity vector $\mathbf{v}$; the latter is a tangent vector at the point p with coordinates $x^1, \dots, x^n$. This means that vector $\mathbf{F}$ depend on the point $q = (p, \mathbf{v})$ of the tangent bundle TM.

Definition 4.1. Vector-function $\mathbf{F}$ that for each point $q = (p, \mathbf{v})$ of tangent bundle TM puts into the correspondence some vector from tangent space $T_p(M)$ at the point $p = \pi(q)$ is called an **extended vector field** on the manifold M.

Here $\pi : TM \to M$ is a map of canonical projection from TM to the base manifold M. Let's consider the following tensor product:

$$T^r_s(p, M) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r \text{ times}} \otimes \underbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}_{s \text{ times}}.$$

Linear space $T^r_s(p, M)$ is called a space of tensors of the type (r, s) at the point p of the manifold M. Elements of this space are called *r-times contravariant and s-times covariant tensors*, or simply tensors of type (r, s) at the point $p \in M$.

Definition 4.2. Tensor-valued function $\mathbf{X}$ that for each point q of tangent bundle TM puts into the correspondence some tensor from the space $T^r_s(p, M)$ at the point $p = \pi(q)$ is called an **extended tensor field** of type (r, s) on the manifold M.

In local coordinates $x^1, \ldots, x^n$ on the manifold M extended tensor fields are expressed by the functions of double-set of arguments:

$$X^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x^1, \ldots, x^n, v^1, \ldots, v^n). \tag{4.1}$$

In the normality equations (3.1) and (3.2) we can see components of several extended tensor fields. Velocity vector $\mathbf{v}$ by itself can be considered as extended vector field on M. Its modulus $v = |\mathbf{v}|$ is an extended scalar. Extended vector field $\mathbf{N}$ with components $N^1, \ldots, N^n$, which are contained in the equations (3.1) and (3.2), is defined as the following quotient:

$$\mathbf{N} = \frac{\mathbf{v}}{v} = \frac{\mathbf{v}}{|\mathbf{v}|}. \tag{4.2}$$

This is the field of unitary vectors directed along the vector of velocity. And finally, in equations (3.1) and (3.2) we have components of operator field $\mathbf{P}$. This is the field of orthogonal projectors onto the hyperplane perpendicular to the velocity vector. Components P^i_j of this field are given by the formula

$$P^i_j = \delta^i_j - N^i\, N_j. \tag{4.3}$$

Let $T^r_s(M)$ be the set of smooth extended tensor fields of type (r, s). This set has the structure of module over the ring of extended scalar fields. The following sum is a graded algebra over this ring with respect to tensorial multiplication:

$$\mathbf{T}(M) = \bigoplus_{r=0}^{\infty} \bigoplus_{s=0}^{\infty} T^r_s(M). \tag{4.4}$$

Algebra (4.4) is called an **extended algebra of tensor fields** on the manifold M. In extended algebra of tensor fields $\mathbf{T}(M)$ one can define two operations of covariant differentiation, we denote them by ∇ and $\tilde{\nabla}$:

$$\nabla\colon T^r_s(M) \to T^r_{s+1}(M), \qquad\qquad \tilde{\nabla}\colon T^r_s(M) \to T^r_{s+1}(M).$$

In local coordinates the result of applying covariant differentiation ∇ to a tensor field $\mathbf{X}$ with components (4.1) is expressed by the following formula:

$$\begin{aligned} \nabla_m X^{i_1 \ldots i_r}_{j_1 \ldots j_s} &= \frac{\partial X^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial x^m} - \sum_{a=1}^{n} \sum_{b=1}^{n} v^a\, \Gamma^b_{ma}\, \frac{\partial X^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial v^b} + \\ &+ \sum_{k=1}^{r} \sum_{a_k=1}^{n} \Gamma^{i_k}_{m\, a_k}\, X^{i_1 \ldots a_k \ldots i_r}_{j_1 \ldots \ldots \ldots j_s} - \sum_{k=1}^{s} \sum_{b_k=1}^{n} \Gamma^{b_k}_{m\, j_k} X^{i_1 \ldots \ldots \ldots i_r}_{j_1 \ldots b_k \ldots j_s}. \end{aligned} \tag{4.5}$$

The result of applying $\tilde{\nabla}$ to $\mathbf{X}$ is expressed by less complicated formula:

$$\tilde{\nabla}_m X^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial X^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial v^m}. \tag{4.6}$$

Formula (4.6) for the components of the field $\tilde{\nabla}\mathbf{X}$ contains only derivatives with respect to components of velocity vector. Therefore $\tilde{\nabla}$ is called **velocity gradient**. Covariant differentiation ∇ defined by formula (4.5) is called **spatial gradient**.

Defining operators ∇ and $\tilde{\nabla}$ by means of formulas (4.5) and (4.6), we assume that some local coordinates are chosen. This way of defining ∇ and $\tilde{\nabla}$ is quite sufficient for our purposes in the theory of newtonian dynamical systems admitting the normal shift. But there is another (invariant) way of defining these operators. It is based on the analysis of differentiations in the extended algebra of tensor fields.

Definition 4.3. The map $D:\mathbf{T}(M)\to\mathbf{T}(M)$ is called a **differentiation** in extended algebra of tensor fields if the following conditions are fulfilled:

1. compatibility with grading: $D(T^r_s(M))\subset T^r_s(M)$;
2. $\mathbb{R}$-linearity: $D(\mathbf{X}+\mathbf{Y})=D(\mathbf{X})+D(\mathbf{Y})$ and $D(\lambda\mathbf{X})=\lambda D(\mathbf{X})$ for $\lambda\in\mathbb{R}$;
3. permutability with contractions: $D(C(\mathbf{X}))=C(D(\mathbf{X}))$;
4. Leibniz rule: $D(\mathbf{X}\otimes\mathbf{Y})=D(\mathbf{X})\otimes\mathbf{Y}+\mathbf{X}\otimes D(\mathbf{Y})$.

Among the results of thesis [17] the following structural theorem is worth to mention here. It describes the structure of all differentiations in extended algebra of tensor fields $\mathbf{T}(M)$.

Theorem 4.1. *Let M be smooth real manifold equipped with some extended affine connection Γ. Then each differentiation D in extended algebra of tensor fields $\mathbf{T}(M)$ on this manifold breaks into the sum*

$$D=\nabla_{\mathbf{X}}+\tilde{\nabla}_{\mathbf{Y}}+\mathbf{S},$$

where $\mathbf{X}$ and $\mathbf{Y}$ are some extended vector fields, and $\mathbf{S}$ is a degenerate differentiation defined by some extended tensor field $\mathbf{S}$ of type $(1,1)$ in M.

Theorem 4.1 is an analog of structural theorem for differentiations in the algebra of ordinary (not extended) tensor fields (see [19]).

5. Reduction of normality equations in the dimension $n\geqslant 3$.

If we take into account formulas (4.2), (4.3), (4.5), and (4.6), we see that normality equations (3.1) and (3.2) form strongly overdetermined system of partial differential equations with respect to components of force field $\mathbf{F}$ of Newtonian dynamical system. Analysis of this system (see [16]), is based on scalar ansatz

$$F_k=A\,N_k-|\mathbf{v}|\sum_{i=1}^{n}P^i_k\,\tilde{\nabla}_iA. \tag{5.1}$$

Formula (5.1) expresses components of force vector through one scalar field A, which is interpreted as the projection of $\mathbf{F}$ onto the direction of velocity vector. This formula follows from first part of equations in the system (3.1). Therefore, when substituting (5.1) into weak normality equations (3.1), first part of these equations appears to be identically fulfilled. While second part is brought to

$$\begin{aligned}\sum_{s=1}^{n}\Biggl(\nabla_sA+|\mathbf{v}|\sum_{q=1}^{n}\sum_{r=1}^{n}P^{qr}\,\tilde{\nabla}_qA\,\tilde{\nabla}_r\tilde{\nabla}_sA\,-\\-\sum_{r=1}^{n}N^r\,A\,\tilde{\nabla}_r\tilde{\nabla}_sA-|\mathbf{v}|\sum_{r=1}^{n}N^r\,\nabla_r\tilde{\nabla}_sA\Biggr)\,P^s_k=0.\end{aligned} \tag{5.2}$$

For $n = 2$ the equations (5.2) exhaust whole list of reduced normality equations. The matter is that in two-dimensional case, as we mentioned above, additional normality equations do not arise at all. While the equations (5.2) are reduced to the only one nonlinear partial differential equation for the function $A(x^1, x^2, v^1, v^2)$. Detailed study of this equation is given in the thesis by A. Yu. Boldin [18].

In multidimensional case $n \geqslant 3$ the process of reducing normality equations can be moved much further. Substituting (5.1) into the first part of additional normality equations brings them to the following form:

$$\begin{aligned}\sum_{s=1}^{n}\sum_{r=1}^{n} P^r_\sigma\,P^s_\varepsilon\left(\nabla_r\tilde{\nabla}_s A+\sum_{q=1}^{n}\tilde{\nabla}_r A\,N^q\,\tilde{\nabla}_q\tilde{\nabla}_s A\right)=\\ =\sum_{s=1}^{n}\sum_{r=1}^{n} P^r_\sigma\,P^s_\varepsilon\left(\nabla_s\tilde{\nabla}_r A+\sum_{q=1}^{n}\tilde{\nabla}_s A\,N^q\,\tilde{\nabla}_q\tilde{\nabla}_r A\right).\end{aligned}\tag{5.3}$$

Similarly, substituting (5.1) into second part of the equations (3.2) gives

$$\sum_{r=1}^{n}\sum_{s=1}^{n} P^r_\sigma\,\tilde{\nabla}_r\tilde{\nabla}_s A\,P^{s\varepsilon}=\lambda\,P^\varepsilon_\sigma.\tag{5.4}$$

Here λ is a scalar quantity, the value of which is uniquely determined by the equations (5.4) even if we do not know it a priori:

$$\lambda=\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{P^{rs}\,\tilde{\nabla}_r\tilde{\nabla}_s A}{n-1}.$$

The equations (5.4) are most remarkable. According to the formula (4.6) they contain only the derivatives with respect to the variables $v^1, \ldots, v^n$. This corresponds to varying the function $A(x^1, \ldots, x^n, v^1, \ldots, v^n)$ within the fiber of TM over the fixed point with coordinates $x^1, \ldots, x^n$ in the base manifold M.

Definition 5.1. Extended tensor field $\mathbf{X}$ on Riemannian manifold M is called **fiberwise spherically symmetric** if $|\mathbf{v}_1| = |\mathbf{v}_2|$ implies $\mathbf{X}(p, \mathbf{v}_1) = \mathbf{X}(p, \mathbf{v}_2)$.

In other words, fiberwise spherically symmetric extended tensor fields depend only on modulus of velocity vector within fibers of tangent bundle TM. Such fields naturally arises in the analysis of the equations (5.4). Here we have the following theorem proved in paper [16].

Theorem 5.1. *Extended Scalar field A on Riemannian manifold M satisfies equations* (5.4) *if and only if it is given by formula*

$$A=a+\sum_{i=1}^{n} b_i\,v^i,\tag{5.5}$$

where a is some fiberwise spherically symmetric scalar field, and b_i are components of some fiberwise spherically symmetric covectorial field $\mathbf{b}$.

Further substitution of (5.5) into the equations (5.2) and (5.3) yields the following equations with respect to fields a and $\mathbf{b}$:

$$\left(\frac{\partial}{\partial x^s}+b_s\,\frac{\partial}{\partial v}\right)a=\left(a\,\frac{\partial}{\partial v}\right)b_s,\tag{5.6}$$

$$\left(\frac{\partial}{\partial x^s}+b_s\,\frac{\partial}{\partial v}\right)b_r=\left(\frac{\partial}{\partial x^r}+b_r\,\frac{\partial}{\partial v}\right)b_s. \tag{5.7}$$

Here $a=a(x^1,\ldots,x^n,v)$ and $b_i=b_i(x^1,\ldots,x^n,v)$. Variable v denotes the modulus of velocity vector: $v=|\mathbf{v}|$.

Equations (5.6) and (5.7) is bound with most dramatic instant in constructing theory of dynamical systems admitting the normal shift. By deriving these equations in paper [16] the mistake was made, nonlinear terms in (5.7) were omitted. As a result (5.7) looked like $\partial b_r/\partial x^s=\partial b_s/\partial x^r$. Further analysis of erroneous equations has led to the ordinary differential equation

$$y''=H_y(y'+1)+H_x \tag{5.8}$$

where $H=H(x,y)$, $H_y=\partial H/\partial y$, $H_x=\partial H/\partial x$. With the aim to find as more functions $H(x,y)$, for which the equation (5.8) is explicitly solvable, as possible we considered the following change of variables:

$$\begin{cases}\tilde x=\tilde x(x,y),\\ \tilde y=\tilde y(x,y).\end{cases} \tag{5.9}$$

The equations that could be brought to the form (5.8) by means of change of variables (5.9) belong to the following class of equations:

$$y''=P(x,y)+3\,Q(x,y)\,y'+3\,R(x,y)\,{y'}^2+S(x,y)\,{y'}^3. \tag{5.10}$$

Study of point transformations in the class of equations (5.10) has the long history (see [25–46]). However, we couldn't find an answer to the question: how to extract the equations (5.10) that could be brought to the form (5.8) by means of point transformation (5.9). This stimulated our own investigations (see [47–50]). We managed to get some results in describing classes of point equivalence for the equations (5.10). But now, since the error in [16] is found, these results have separate value, which is not related to the theory of dynamical systems admitting the normal shift. And we are to return to the equations (5.6) and (5.7).

6. Derivation of reduced normality equations.

For the beginning let's derive the (5.6) and (5.7) by substituting (5.5) into the equations (5.2) and (5.3). Denote by a' and b_i' the derivatives

$$a'=\frac{\partial a}{\partial v},\qquad\qquad b_i'=\frac{\partial b_i}{\partial v}.$$

Let's do the calculations necessary for substituting (5.5) into (5.2) and (5.3):

$$\nabla_s A=\nabla_s a+\sum_{i=1}^{n}\nabla_s b_i\,v^i, \tag{6.1}$$

$$\nabla_r\tilde\nabla_s A=\left(\nabla_r a'+\sum_{i=1}^{n}\nabla_r b_i'\,v^i\right)N_s+\nabla_r b_s. \tag{6.2}$$

$$\tilde{\nabla}_s A = \left(a' + \sum_{i=1}^{n} b'_i\, v^i\right) N_s + b_s, \tag{6.3}$$

$$\begin{aligned} \tilde{\nabla}_r \tilde{\nabla}_s A &= \left(a'' + \sum_{i=1}^{n} b''_i\, v^i\right) N_r\, N_s\, + \\ &+ b'_s\, N_r + b'_r\, N_s + \left(\frac{a'}{v} + \sum_{i=1}^{n} b'_i\, N^i\right) P_{rs}. \end{aligned} \tag{6.4}$$

From formulas (6.3) and (6.4) for derivatives we obtain the following relations:

$$\sum_{r=1}^{n} P^r_\sigma\, \tilde{\nabla}_r A = \sum_{r=1}^{n} P^r_\sigma\, b_r, \tag{6.5}$$

$$\sum_{s=1}^{n} \sum_{q=1}^{n} P^s_\varepsilon\, N^q\, \tilde{\nabla}_q\, \tilde{\nabla}_s A = \sum_{s=1}^{n} P^s_\varepsilon\, b'_s. \tag{6.6}$$

Let's combine (6.5) and (6.6). As a result we get the relationship

$$\sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{q=1}^{n} P^r_\sigma\, P^s_\varepsilon\, \tilde{\nabla}_r A\, N^q\, \tilde{\nabla}_q \tilde{\nabla}_s A = \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\, b_r\, b'_s.$$

Then let's multiply both sides of the relationship (6.2) by $P^r_\sigma\, P^s_\varepsilon$ and contract with respect to pair of indices r and s. This yields one more relationship:

$$\sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\, \nabla_r \tilde{\nabla}_s A = \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\, \nabla_r b_s.$$

Now, if we add above two relationships, we get the result of substituting (5.5) into the left hand side of the equation (5.3):

$$\begin{aligned} \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon \left(\nabla_r \tilde{\nabla}_s A + \sum_{q=1}^{n} \tilde{\nabla}_r A\, N^q\, \tilde{\nabla}_q \tilde{\nabla}_s A\right) = \\ = \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\, \nabla_r b_s + \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\, b_r\, b'_s. \end{aligned} \tag{6.7}$$

Similarly we calculate right hand side of the equation (5.3):

$$\begin{aligned} \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon \left(\nabla_s \tilde{\nabla}_r A + \sum_{q=1}^{n} \tilde{\nabla}_s A\, N^q\, \tilde{\nabla}_q \tilde{\nabla}_r A\right) = \\ = \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\, \nabla_s b_r + \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\, b_s\, b'_r. \end{aligned} \tag{6.8}$$

On the base of (6.7) and (6.8) we conclude that the equation (5.3) is reduced to

$$\sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon \left(\nabla_r b_s + b_r\, b'_s - \nabla_s b_r - b_s\, b'_r\right) = 0. \tag{6.9}$$

For the further analysis of the obtained equations (6.9) one should use the peculiarity of covectorial field $\mathbf{b}$ from extended algebra of tensor fields on M. Components of this field $b_1, \ldots, b_n$ depend only on modulus of velocity vector, but they do not depend on its direction. By calculating derivatives $\nabla_r b_s$ and $\nabla_s b_r$ in (6.9) we apply the following theorem.

Theorem 6.1. *Let $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(x^1,\ldots,x^n,v)$ be components of fiberwise spherically symmetric tensor field $\mathbf{X}$ from extended algebra $\mathbf{T}(M)$. Then components of spatial gradient $\nabla\mathbf{X}$ for this field are given by formula*

$$
\begin{aligned}
\nabla_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} =&\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m} + \sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s} -\\
&-\sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.
\end{aligned}
\tag{6.10}
$$

Proof. Formula (6.10) is obtained as a result of reduction from formula (4.5). For the components of fiberwise spherically symmetric tensor field $\mathbf{X}$ (as in the statement of theorem) natural arguments are $x^1, \ldots, x^n$, and v, where

$$
v = |\mathbf{v}| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}(x^1,\ldots,x^n)\,v^i\,v^j}\,.
\tag{6.11}
$$

While partial derivatives in formula (4.5) are assumed to be respective to the variables $x^1, \ldots, x^n, v^1, \ldots, v^n$. Recalculation of these derivatives to natural variables for spherically symmetric field consists in the following substitutions:

$$
\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^b}\quad\text{by}\quad\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v}\cdot\frac{\partial v}{\partial v^b},
\tag{6.12}
$$

$$
\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m}\quad\text{by}\quad\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m}+\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v}\cdot\frac{\partial\,v}{\partial x^m}.
\tag{6.13}
$$

Derivatives $\partial\,v/\partial v^b$ and $\partial\,v/\partial x^m$ are calculated due to (6.11). Upon finding explicit expressions for these derivatives and upon making substitutions (6.12) and (6.13) in formula (4.5), we get two extra summands:

$$
\sum_{a=1}^{n}\sum_{b=1}^{n}\frac{X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v}\,\frac{1}{2}\,\frac{\partial g_{ab}}{\partial x^m}\,\frac{v^a\,v^b}{v}\quad\text{and}\quad-\sum_{a=1}^{n}\sum_{b=1}^{n}v^a\,\Gamma^b_{ma}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v}\,N_b.
$$

If we take into account the explicit formula for components of metric connection

$$
\Gamma^b_{ma} = \frac{1}{2}\sum_{q=1}^{n} g^{bq}\left(\frac{\partial g_{qa}}{\partial x^m}+\frac{\partial g_{mq}}{\partial x^a}-\frac{\partial g_{ma}}{\partial x^q}\right)
\tag{6.14}
$$

(see [19–22]), then we easily see that above two summands cancel each other. As a result formula (4.5) transforms into the form (6.10). $\square$

Corollary. *Components $\omega_{rs} = \nabla_r b_s + b_r\, b'_s - \nabla_s b_r - b_s\, b'_r$ of skew-symmetric extended tensor field $\boldsymbol{\omega}$ in the equations (6.9) depend only on modulus of velocity vector $|\mathbf{v}|$, but they do not depend on the direction of the vector $\mathbf{v}$.*

This fact immediately follows from the formula (6.10). It allows us to make further simplifications in the equations (6.9). Let $\mathbf{c}$ and $\mathbf{d}$ be arbitrary two vectors from tangent space $T_p(M)$. In multidimensional case $n \geqslant 3$ we can rotate velocity vector $\mathbf{v}$, keeping its modulus unchanged, and can direct it so that it will be perpendicular to vectors $\mathbf{c}$ and $\mathbf{d}$ simultaneously. Then

$$\sum_{\sigma=1}^{n} P^r_\sigma\, c^\sigma = c^r, \qquad\qquad \sum_{\varepsilon=1}^{n} P^s_\varepsilon\, d^\varepsilon = d^s.$$

Therefore the equations (6.9) are transformed as follows:

$$\sum_{s=1}^{n}\sum_{r=1}^{n} c^r\, d^s \left(\nabla_r b_s + b_r\, b'_s - \nabla_s b_r - b_s\, b'_r\right) = 0. \tag{6.15}$$

Since $\mathbf{c}$ and $\mathbf{d}$ are arbitrary two vectors, we can further simplify the obtained equations (6.15), bringing them to the form

$$\nabla_r b_s + b_r\, b'_s = \nabla_s b_r + b_s\, b'_r. \tag{6.16}$$

Next step consists in reducing the equations (5.2). In order to do it we substitute (5.5) into (5.2). From (6.1) we derive

$$\sum_{s=1}^{n} \nabla_s A\, P^s_k = \sum_{s=1}^{n} \nabla_s a\, P^s_k + \sum_{s=1}^{n}\sum_{r=1}^{n} \nabla_s b_r\, v^r\, P^s_k. \tag{6.17}$$

Then from (6.3) and (6.4) we obtain the following two relationships:

$$\sum_{r=1}^{n} P^{qr}\, \tilde{\nabla}_q A = \sum_{s=1}^{n} b_s\, P^{sr}, \tag{6.18}$$

$$|\mathbf{v}| \sum_{r=1}^{n}\sum_{s=1}^{n} P^{qr}\, \tilde{\nabla}_r \tilde{\nabla}_s A\, P^s_k = \left(a' + \sum_{r=1}^{n} b'_r\, v^r\right) P^q_k. \tag{6.19}$$

In (6.18) we have free index r, and in (6.19) we have free index q. Let's multiply these two equalities (6.18) and (6.19) and do contract with respect to indices r and q upon multiplying the resulting equality by g_{rq}. This yields

$$|\mathbf{v}| \sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} P^{qr}\, \tilde{\nabla}_q A\, \tilde{\nabla}_r \tilde{\nabla}_s A\, P^s_k = \left(a' + \sum_{r=1}^{n} b'_r\, v^r\right) \sum_{s=1}^{n} b_s\, P^s_k.$$

One more relationship is obtained from (6.4) upon multiplying by $N^r\, A\, P^s_q$ and upon contracting with respect to r and s:

$$-\sum_{r=1}^{n}\sum_{s=1}^{n} N^r\, A\, \tilde{\nabla}_r \tilde{\nabla}_s A\, P^s_k = -\left(a + \sum_{r=1}^{n} b_r\, v^r\right) \sum_{s=1}^{n} b'_s\, P^s_k.$$

Let's multiply (6.2) by N^r and P^s_k, then contract it with respect to r and s:

$$-|\mathbf{v}|\sum_{r=1}^{n}\sum_{s=1}^{n}N^r\,\nabla_r\tilde{\nabla}_sA\,P^s_k=-\sum_{r=1}^{n}\sum_{s=1}^{n}v^r\,\nabla_rb_s\,P^s_k.$$

Now, in order to write the result of substituting (5.5) into the equations (5.2), we have to add (6.17) and three above equalities:

$$\sum_{s=1}^{n}\left(\nabla_sa+b_s\,a'-a\,b'_s+\sum_{r=1}^{n}v^r\bigl(\nabla_sb_r+b_s\,b'_r-\nabla_rb_s-b_r\,b'_s\bigr)\right)P^s_k=0.$$

Let's take into account (6.16), this leads to vanishing the whole expression under summation with respect to r. As a result we obtain the following equation:

$$\sum_{s=1}^{n}(\nabla_sa+b_s\,a'-a\,b'_s)\,P^s_k=0. \tag{6.20}$$

Equations (6.20) are analogous to the equations (6.9), the operation of contraction with components of projector $\mathbf{P}$ can be omitted:

$$\nabla_sa+b_s\,a'=a\,b'_s. \tag{6.21}$$

Arguments used in deriving the equations (6.21) are similar to those used in deriving (6.16) from (6.9).

Theorem 6.2. *Force field* $\mathbf{F}$ *given by scalar ansatz* (5.1) *corresponds to some Newtonian dynamical system admitting the normal shift on the Riemannian manifold* M *if and only if scalar field* A *in ansatz* (5.1) *is defined by formula* (5.5)*, while extended fields* a *and* $\mathbf{b}$ *in* (5.5) *are fiberwise spherically symmetric and satisfying the equations* (6.16) *and* (6.21).

Note that the equations (6.16) and (6.21) coincide with reduced normality equations (5.7) and (5.6) we were to derive.

7. Analysis of reduced equations.

With the aim of further study of the equations (5.6) and (5.7) let's express covariant derivatives in them through partial derivatives. In order to do it we use formula (6.10) and take into account symmetry of connection components:

$$\left(\frac{\partial}{\partial x^s}+b_s\,\frac{\partial}{\partial v}\right)a=\left(a\,\frac{\partial}{\partial v}\right)b_s, \tag{7.1}$$

$$\left(\frac{\partial}{\partial x^s}+b_s\,\frac{\partial}{\partial v}\right)b_r=\left(\frac{\partial}{\partial x^r}+b_r\,\frac{\partial}{\partial v}\right)b_s. \tag{7.2}$$

The equations (7.2) form closed system of equations with respect to components $b_1,\,\ldots\,,b_n$ of covector field $\mathbf{b}$. We can study them separately. Let's consider the differential operators in these equations:

$$\mathbf{L}_i=\frac{\partial}{\partial x^i}+b_i\,\frac{\partial}{\partial v},\ \text{ where }\ i=1,\,\ldots\,,n. \tag{7.3}$$

Now by means of direct calculations we can check that the equations (7.2) are exactly the conditions of permutability of operators (7.3):

$$[\mathbf{L}_s,\,\mathbf{L}_r]=0. \tag{7.4}$$

Let $\mathbb{R}^+=(0,+\infty)$ be positive semiaxis on real axis $\mathbb{R}$. Operators (4.3) have natural interpretation as vector fields on the direct product of manifolds $M\times\mathbb{R}^+$. Let's complement $\mathbf{L}_1,\,\dots\,,\,\mathbf{L}_n$ by one more vector field $\mathbf{L}_{n+1}$, which possibly is not commutating with fields $\mathbf{L}_1,\,\dots\,,\,\mathbf{L}_n$, but which should complete $\mathbf{L}_1,\,\dots\,,\,\mathbf{L}_n$ up to a moving frame on the manifold $M\times\mathbb{R}^+$. Each field $\mathbf{L}_i$ has its own local one-parametric group of local diffeomorphisms (see [19]) with parameter y_i:

$$\varphi_i(y^i):\;M\times\mathbb{R}^+\to M\times\mathbb{R}^+. \tag{7.5}$$

Let's fix some point $p_0\in M\times\mathbb{R}^+$ and let's consider composition of such diffeomorphisms applied to the point p_0:

$$p(y^1,\dots,y^n,w)=\varphi_1(y^1)\circ\;\dots\;\circ\varphi_n(y^n)\circ\varphi_{n+1}(w)(p_0). \tag{7.6}$$

In left hand side of the equality (7.6) we have the point p parameterized by real numbers $y^1,\,\dots\,,\,y^n,\,w$. This is equivalent to defining local coordinates on $M\times\mathbb{R}^+$ in some neighborhood of the point p_0. Permutability of vector fields (7.4) implies permutability of first n maps (7.5) in the composition (7.6). For the vector fields $\mathbf{L}_1,\,\dots\,,\,\mathbf{L}_n$ this fact yields the following expressions:

$$\begin{aligned}
&\mathbf{L}_1=\frac{\partial}{\partial y^1}=\sum_{i=1}^n\frac{\partial x^i}{\partial y^1}\,\frac{\partial}{\partial x^i}+\frac{\partial v}{\partial y^1}\,\frac{\partial}{\partial v},\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\\
&\mathbf{L}_n=\frac{\partial}{\partial y^n}=\sum_{i=1}^n\frac{\partial x^i}{\partial y^n}\,\frac{\partial}{\partial x^i}+\frac{\partial v}{\partial y^n}\,\frac{\partial}{\partial v}.
\end{aligned} \tag{7.7}$$

Let's compare formulas (7.7) and (7.3) for vector fields $\mathbf{L}_1,\,\dots\,,\,\mathbf{L}_n$. This yields

$$\frac{\partial x^i}{\partial y^k}=\delta^i_k=\begin{cases}1 & \text{for}\;\;i=k,\\ 0 & \text{for}\;\;i\neq k.\end{cases} \tag{7.8}$$

From the same comparison for the functions b_k in variables $y^1,\,\dots\,,\,y^n,\,w$ we get

$$b_k=\frac{\partial v}{\partial y^k}. \tag{7.9}$$

The relationships (7.8) show that newly constructed local coordinates $y^1,\,\dots\,,\,y^n,\,w$ on $M\times\mathbb{R}^+$ and initial local coordinates $x^1,\,\dots\,,\,x^n,\,v$ on this manifold are related by means of only one function $V(y^1,\dots,y^n,w)$:

$$\begin{cases}x^1=y^1,\,\dots\,,\,x^n=y^n,\\ v=V(y^1,\dots,y^n,w).\end{cases} \tag{7.10}$$

Inverse relation is also given by the only one function $W(x^1,\ldots,x^n,v)$:

$$\begin{cases} y^1=x^1,\ \ldots,\ y^n=x^n,\\ w=W(x^1,\ldots,x^n,v). \end{cases} \tag{7.11}$$

Functions $V(y^1,\ldots,y^n,w)$ and $W(x^1,\ldots,x^n,v)$ in (7.10) and in (7.11) are bound by the obvious relationships that express the fact that the changes of variables (7.10) and (7.11) are inverse to each other:

$$\begin{aligned} &V(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))=v,\\ &W(x^1,\ldots,x^n,V(x^1,\ldots,x^n,w))=w. \end{aligned} \tag{7.12}$$

Let's use the following natural notations for partial derivatives of the first order:

$$\begin{aligned} &V_i(x^1,\ldots,x^n,w)=\frac{\partial V(x^1,\ldots,x^n,w)}{\partial x^i},\\ &V_w(x^1,\ldots,x^n,w)=\frac{\partial V(x^1,\ldots,x^n,w)}{\partial w}. \end{aligned} \tag{7.13}$$

Analogous notations will be used for partial derivatives of the second order:

$$\begin{aligned} &V_{ij}(x^1,\ldots,x^n,w)=\frac{\partial^2 V(x^1,\ldots,x^n,w)}{\partial x^i\,\partial x^j},\\ &V_{iw}(x^1,\ldots,x^n,w)=\frac{\partial^2 V(x^1,\ldots,x^n,w)}{\partial x^i\,\partial w}. \end{aligned} \tag{7.14}$$

Now we can rewrite formula (7.9) in initial local coordinates $x^1,\ \ldots,\ x^n,\ v$:

$$b_k=V_k(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v)). \tag{7.15}$$

Theorem 7.1. *Functions $b_1,\ \ldots,\ b_n$ satisfy nonlinear system of partial differential equations* (7.2) *if and only if they are determined by some function $V(x^1,\ldots,x^n,w)$ with non-zero derivative $\partial V/\partial w$ according to the formula* (7.15).

Proof. In theorem 7.1 we have two propositions. Direct proposition is already proved: each solution of the system of equations (7.2) is given by (7.15). Conversely, suppose that some function $V(x^1,\ldots,x^n,w)$ with non-zero derivative $\partial V/\partial w$ is chosen. From $\partial V/\partial w\neq 0$, relying on the theorem on implicit functions (see [51], [52]), we derive the existence of the function $W(x^1,\ldots,x^n,v)$ such that it is bound with $V(x^1,\ldots,x^n,w)$ by the relationships (7.12). Differentiating these relationships and taking into account the notations (7.13), we derive

$$\frac{\partial W}{\partial v}=\frac{1}{V_w(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))} \tag{7.16}$$

$$\frac{\partial W}{\partial x^s}=-\frac{V_s(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))}{V_w(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))} \tag{7.17}$$

Let's substitute V and W into (7.15) and calculate functions $b_1, \ldots, b_n$. Then by means of direct calculations we check that the functions obtained satisfy differential equations (7.2). Indeed, here we have

$$\begin{aligned}\frac{\partial b_r}{\partial x^s} &= V_{rs}(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v)) +\\ &\qquad + V_{rw}(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))\,\frac{\partial W}{\partial x^s},\end{aligned}\tag{7.18}$$

$$\begin{aligned}b_s\,\frac{\partial b_r}{\partial v} &= V_s(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))\,\times\\ &\qquad \times\, V_{rw}(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))\,\frac{\partial W}{\partial v}.\end{aligned}\tag{7.19}$$

Let's add the equalities (7.18) and (7.19) and let's take into account formulas (7.16) and (7.17) for derivatives. This yields the equality

$$\left(\frac{\partial}{\partial x^s}+b_s\,\frac{\partial}{\partial v}\right)b_r = V_{rs}(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v)).\tag{7.20}$$

Due to first formula (7.14) we can transpose indices r and s in right hand side of (7.20), i. e. $V_{rs} = V_{sr}$. This provides the equations (7.2) for the functions (7.15) we have constructed above. $\square$

Having constructed general solution for the equations (7.2), now let's study the equations (7.1). Let's implement the change of variables (7.10) and transfer to variables $y^1, \ldots, y^n, w$. In left hand side of (7.1) we have the same differential operator $\mathbf{L}_s$ as in (7.2). In variables $y^1, \ldots, y^n, w$ this operator is written as $\mathbf{L}_s = \partial/\partial y^s$ (see relationships (7.7)). Let's transform the operator in right hand side of (7.1) to the variables $y^1, \ldots, y^n, w$:

$$\frac{\partial}{\partial v} = \sum_{i=1}^{n}\frac{\partial y^i}{\partial v}\,\frac{\partial}{\partial y^i}+\frac{\partial w}{\partial v}\,\frac{\partial}{\partial w} = \frac{\partial W}{\partial v}\,\frac{\partial}{\partial w}.$$

For further transformation of the above expression for operator $\partial/\partial v$ we use formula (7.16). As a result we get the following relationship:

$$\frac{\partial}{\partial v} = \frac{1}{V_w(y^1,\ldots,y^n,w)}\,\frac{\partial}{\partial w}.$$

Now in variables $y^1, \ldots, y^n, w$ the equations (7.1) are written as

$$\frac{\partial a}{\partial y^s} = \frac{V_{sw}}{V_w}\,a.\tag{7.21}$$

Here we used the above notations (7.13) and (7.14). The equations (7.21) can be easily solved if we rewrite them as follows:

$$\frac{\partial}{\partial y^s}\left(\frac{a}{V_w}\right) = 0.\tag{7.22}$$

General solution of the equations (7.22) contains an arbitrary function of one variable $h(w)$. It is given by the following formula:

$$a=h(w)\ V_w(y^1,\ldots,y^n,w). \tag{7.23}$$

Upon coming back to initial variables $x^1,\ \ldots,\ x^n,\ v$ from (7.23) we obtain

$$\begin{aligned}a=h(W(x^1,\ldots,x^n,v))\,\times\\ \times\ V_w(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v)).\end{aligned} \tag{7.24}$$

With the aim of additional verification we substitute the above expressions (7.24) and (7.15) into the equations (7.1). Let's do the appropriate calculations:

$$\begin{aligned}&\frac{\partial a}{\partial x^s}=\bigl(h'(W)\,V_w+h(W)\,V_{ww}\bigr)\,\frac{\partial W}{\partial x^s}+h(W)\,V_{sw},\\ &b_s\,\frac{\partial a}{\partial v}=V_s\,\bigl(h'(W)\,V_w+h(W)\,V_{ww}\bigr)\,\frac{\partial W}{\partial v}.\end{aligned}$$

We add two above equalities and take into account formulas (7.16) and (7.17) for partial derivatives $\partial W/\partial x^s$ and $\partial W/\partial v$:

$$\left(\frac{\partial}{\partial x^s}+b_s\,\frac{\partial}{\partial v}\right)a=h(W)\,V_{sw}. \tag{7.25}$$

Similar calculations for the right hand side of the equations (7.1) yield

$$\left(a\,\frac{\partial}{\partial v}\right)b_s=h(W)\,V_w\,V_{sw}\,\frac{\partial W}{\partial v}. \tag{7.26}$$

Comparing (7.25) with (7.26) and taking into account formulas (7.16) for the derivative $\partial W/\partial v$ completes the proof of the following theorem.

Theorem 7.2. *Functions $b_1,\ \ldots,\ b_n$ and a satisfy nonlinear differential equations* (7.1) *and* (7.2) *if and only if they are determined by formulas* (7.15) *and* (7.24).

8. General formula for force field.

Analyzing reduced normality equations above, we have found their general solution. Now we are able to write **formula** for the force field of arbitrary Newtonian dynamical system **admitting the normal shift** on Riemannian manifold of the **dimension** $n\geqslant 3$. Let's substitute (5.5) into the formula for the components of force field $\mathbf{F}$. Thereby we take into account that fields a and $\mathbf{b}$ determining scalar parameter A are fiberwise spherically symmetric:

$$F_k=a\,N_k+|\mathbf{v}|\sum_{i=1}^{n}b_i\,\bigl(2\,N^i\,N_k-\delta^i_k\bigr). \tag{8.1}$$

Formula (8.1) completely determines the dependence of force field on velocity vector $\mathbf{v}=v\,\mathbf{N}$. Taking into account (7.15) and (7.24) we get

$$\begin{aligned}F_k=h(W)\ V_w(x^1,\ldots,x^n,W)\ N_k\,+\\ +\ |\mathbf{v}|\sum_{i=1}^{n}V_i(x^1,\ldots,x^n,W)\ \bigl(2\,N^i\,N_k-\delta^i_k\bigr).\end{aligned} \tag{8.2}$$

Here $h(W)$ is an arbitrary function of one variable; through V_i and V_w we denoted derivatives (7.13), where functions $V(x^1, \dots, x^n, W)$ and $W(x^1, \dots, x^n, v)$ are bound with each other by the relationships (7.12).

Formula (8.2) for the force field of dynamical system admitting the normal shift contains the arbitrariness determined by one function of $(n+1)$ variables $V(x^1, \dots, x^n, w)$. Arbitrariness determined by the function $h(W)$ can be eliminated by means of gauge transformation that changes a but doesn't change $\mathbf{b}$:

$$\begin{aligned} &V(x^1, \dots, x^n, w) \longrightarrow V(x^1, \dots, x^n, \rho^{-1}(w)),\\ &W(x^1, \dots, x^n, v) \longrightarrow \rho(W(x^1, \dots, x^n, v)),\\ &h(w) \longrightarrow h(\rho^{-1}(w))\ \rho'(\rho^{-1}(w)). \end{aligned} \tag{8.3}$$

Hence transformation (8.3) doesn't change components of force field $\mathbf{F}$, though it changes parameters V, W, and h in (8.2). If $h(w) \neq 0$, we can choose function $\rho(w)$ such that $h(w)\,\rho'(w) = 1$. Upon doing gauge transformation (8.3) in this case we obtain the following formula for force field $\mathbf{F}$:

$$\begin{aligned} F_k = {} & V_w(x^1, \dots, x^n, W)\ N_k +\\ & + |\mathbf{v}| \sum_{i=1}^{n} V_i(x^1, \dots, x^n, W)\ \bigl(2\,N^i\,N_k - \delta^i_k\bigr). \end{aligned} \tag{8.4}$$

Formula (8.4) is almost as universal as formula (8.2). The only exception is the case $h = 0$, which is not embraced by formula (8.4).

9. Effectivization of general formula.

Formulas (8.2) and (8.4) determine components of the force field of Newtonian dynamical system admitting the normal shift on Riemannian manifold M. However, both these formulas have common fault. They are ineffective since we are to use function $W(x^1, \dots, x^n, v)$ determined implicitly by the equations (7.12). With the aim to get more effective formula we use the relationships (7.16) and (7.17). Let's rewrite these relationships as follows:

$$V_w(x^1, \dots, x^n, W(x^1, \dots, x^n, v)) = \frac{1}{\partial W/\partial v}, \tag{9.1}$$

$$V_k(x^1, \dots, x^n, W(x^1, \dots, x^n, v)) = -\frac{\partial W/\partial x^k}{\partial W/\partial v}. \tag{9.2}$$

Substituting (9.2) into the formula (7.15), for the components of covector $\mathbf{b}$ we get

$$b_k = -\frac{\partial W(x^1, \dots, x^n, v)/\partial x^k}{\partial W(x^1, \dots, x^n, v)/\partial v}. \tag{9.3}$$

Here $W(x^1, \dots, x^n, v)$ can be understood as absolutely arbitrary function provided the derivative in denominator of the fraction in (9.3) is non-zero. The same function, upon substituting (9.1) into (7.24), determines the field a:

$$a = \frac{h(W(x^1, \dots, x^n, v))}{\partial W(x^1, \dots, x^n, v)/\partial v}. \tag{9.4}$$

Let's substitute the expressions (9.3) and (9.4) into the formula (8.1) for $\mathbf{F}$:

$$
\begin{aligned}
F_k=&\frac{h(W(x^1,\ldots,x^n,v))}{\partial W(x^1,\ldots,x^n,v)/\partial v}\ N_k+\\
&-\ |\mathbf{v}|\sum_{i=1}^{n}\frac{\partial W(x^1,\ldots,x^n,v)/\partial x^i}{\partial W(x^1,\ldots,x^n,v)/\partial v}\ \bigl(2\,N^i\,N_k-\delta^i_k\bigr).
\end{aligned}
\tag{9.5}
$$

Let's rewrite formula (9.5) in terms of covariant derivatives. We formulate the result in form of the theorem.

Theorem 9.1. *Newtonian dynamical system on Riemannian manifold M of the dimension $n\geqslant 3$ admits the normal shift if and only if its force field $\mathbf{F}$ has the components determined by formula*

$$
F_k=\frac{h(W)\,N_k}{W_v}-|\mathbf{v}|\sum_{i=1}^{n}\frac{\nabla_iW}{W_v}\,\bigl(2\,N^i\,N_k-\delta^i_k\bigr),
\tag{9.6}
$$

where W is fiberwise spherically symmetric scalar field from extended algebra of tensor fields on M with non-zero derivative

$$
W_v=\frac{\partial W}{\partial v}=\sum_{i=1}^{n}N^i\ \tilde\nabla_iW\neq 0,
$$

and $h=h(W)$ is an arbitrary function of one variable.

10. Kinematics of normal shift.

Having explicit formula (6.6) for the force field of the dynamical admitting the normal shift, we are able to describe in details the process of normal shift of a given hypersurface S along trajectories of this dynamical system. According to the definition 3.2 one chooses some point p_0 on S, then on some part $S'=O_S(p_0)$ of hypersurface S one should define the function ν that determines the modulus of initial velocity on the trajectories of shift. At the point p_0 this function ν should be normalized by the condition

$$
\nu(p_0)=\nu_0,
\tag{10.1}
$$

where ν_0 is some nonzero number (see condition (3.3) above). Let's choose local coordinates $x^1,\ldots,x^n$ on M in some neighborhood of the point p_0 and local coordinates $u^1,\ldots,u^{n-1}$ on S in a neighborhood of the same point. The choice of local coordinates $u^1,\ldots,u^{n-1}$ determines coordinate tangent vectors $\boldsymbol{\tau}_1,\ldots,\boldsymbol{\tau}_{n-1}$ to the hypersurface S. Let τ^i_k be components of vector $\boldsymbol{\tau}_k$ in coordinates $x^1,\ldots,x^n$ on M. If hypersurface S is defined parametrically

$$
\begin{aligned}
&x^1=x^1(u^1,\ldots,u^{n-1}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
&x^n=x^n(u^1,\ldots,u^{n-1}),
\end{aligned}
\tag{10.2}
$$

then components of coordinate tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ are determined by derivatives of the functions $x^i(u^1, \ldots, u^{n-1})$ in (10.2):

$$\tau^i_k = \frac{\partial x^i}{\partial u^k}. \tag{10.3}$$

Function $\nu = \nu(p) = \nu(u^1, \ldots, u^{n-1})$ determines the initial velocity on the trajectories of shift starting from S (see formula (2.3) above):

$$\mathbf{v}(t)\Bigr|_{t=0} = \nu(p) \cdot \mathbf{n}(p). \tag{10.4}$$

Normal vector $\mathbf{n}(p)$ is determined up to a sign: $\mathbf{n}(p) \to \pm\mathbf{n}(p)$. Therefore without loss of generality we can assume that constant ν_0 in (10.1) is positive. Then function $\nu(p)$ in (10.4) is also positive. This means that

$$|\mathbf{v}(t)|\Bigr|_{t=0} = \nu(p). \tag{10.5}$$

The condition (10.4) provides normality of shift at initial instant of time $t = 0$. How to provide normality condition for other instants of time $t \neq 0$? For this purpose we consider the solution of Cauchy problem (2.1) for the system of differential equations (1.2), which describes Newtonian dynamical system with force field $\mathbf{F}$. This solution is given by the functions (2.2). Let's write them as

$$\begin{aligned} &x^1 = x^1(u^1, \ldots, u^{n-1}, t),\\ &\dots\dots\dots\dots\dots\dots\\ &x^n = x^n(u^1, \ldots, u^{n-1}, t). \end{aligned} \tag{10.6}$$

Functions (10.6) describe the shift $f_t \colon S \to S_t$. In sufficiently small neighborhood $S' = O_S(p_0)$ of the point p_0 and for sufficiently small values of t the map f_t is a diffeomorphism: $f_t : S' \to S'_t$. Due to this diffeomorphism we can carry local coordinates $u^1, \ldots, u^{n-1}$ from S to S_t. Then for any fixed t the functions (10.6) can be treated as parametric equations of hypersurface S_t similar to the equations (10.2). Let's calculate derivatives (10.3) for the functions (10.6). Doing this, we define vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ tangent to all hypersurfaces S_t:

$$\boldsymbol{\tau}_k = \boldsymbol{\tau}_k(u^1, \ldots, u^{n-1}, t). \tag{10.7}$$

Denote by φ_i the scalar product of the vector (10.7) and the vector of velocity:

$$\varphi_k = (\boldsymbol{\tau}_k \,|\, \mathbf{v}). \tag{10.8}$$

In thesis [17] the functions $\varphi_1, \ldots, \varphi_{n-1}$ determined by formula (10.8) were called **functions of deviation**. Such functions play an important role in deriving the normality equations (3.1) and (3.2), since the condition of normality for the shift $f_t \colon S \to S_t$ consists in identical vanishing of all functions (10.8):

$$\varphi_k = \varphi_k(u^1, \ldots, u^{n-1}, t) = 0. \tag{10.9}$$

From identical vanishing of the functions of deviation (10.9) it follows that their time derivatives at the initial instant of time $t = 0$ are zero:

$$\dot{\varphi}_k\Big|_{t=0} = 0. \tag{10.10}$$

Moreover from the relationships (10.9) it follows that φ_k for $t = 0$ are zero as well:

$$\varphi_k\Big|_{t=0} = 0. \tag{10.11}$$

In papers [6] and [7] it was shown that for Newtonian dynamical systems admitting the normal shift (in the sense of definition 3.2 and theorem 3.1) the conditions (10.10) and (10.11) are not only necessary, but also sufficient for identical vanishing of all functions of deviation (10.8).

The conditions (10.11) are trivial consequences of (10.4). They give no information on how to choose the function $\nu(p)$ on S. Therefore let's consider the conditions (10.10). Let's calculate $\dot{\varphi}$ by differentiating (10.8):

$$\dot{\varphi}_k = \nabla_t\varphi_k = (\nabla_t\boldsymbol{\tau}_k\,|\,\mathbf{v}) + (\boldsymbol{\tau}_k\,|\,\mathbf{F}). \tag{10.12}$$

In deriving (10.12) we took into account the equations of dynamics (1.2) written as $\nabla_t\mathbf{v} = \mathbf{F}$. Now let's calculate the derivative $\nabla_t\boldsymbol{\tau}_k$. It's the vector with components

$$\nabla_t\tau^i_k = \frac{\partial\tau^i_k}{\partial t} + \sum^n_{r=1}\sum^n_{s=1}\Gamma^i_{rs}\,v^r\,\tau^s_k. \tag{10.13}$$

If we take into account formulas (10.3) determining τ^i_k, then from (10.13) we obtain

$$\nabla_t\boldsymbol{\tau}^i_k = \frac{\partial v^i}{\partial u^k} + \sum^n_{r=1}\sum^n_{s=1}\Gamma^i_{rs}\,v^r\,\frac{\partial x^s}{\partial u^k} = \nabla_{u^k}v^i. \tag{10.14}$$

In vectorial form (10.14) is written as $\nabla_t\boldsymbol{\tau}_k = \nabla_{u^k}\mathbf{v}$. That is $\nabla_t\boldsymbol{\tau}_k$ coincides with covariant derivative of vector function $\mathbf{v}(u^1, \dots, u^{n-1}, t)$ with respect to parameter u^k along k-th coordinate line on hypersurface S_t. Let's substitute the obtained expression for $\nabla_t\boldsymbol{\tau}_k$ into the formula (10.12). This yields

$$\dot{\varphi}_k = (\nabla_{u^k}\mathbf{v}\,|\,\mathbf{v}) + (\boldsymbol{\tau}_k\,|\,\mathbf{F}). \tag{10.15}$$

Further, we take into account the obvious relationship $\nabla_{u^k}(\mathbf{v}\,|\,\mathbf{v}) = 2\,(\nabla_{u^k}\mathbf{v}\,|\,\mathbf{v})$ and formula (10.5), which determines modulus of velocity vector for $t = 0$. Then we can bring formula (10.15) for the derivative $\dot{\varphi}_k$ to the following form:

$$\dot{\varphi}_k\Big|_{t=0} = \nu\,\frac{\partial\nu}{\partial u^k} + (\boldsymbol{\tau}_k\,|\,\mathbf{F}). \tag{10.16}$$

Now, due to (10.16), the relationships (10.10) for derivatives $\dot{\varphi}_k$ are written as partial differential equations for the function $\nu = \nu(u^1, \dots, u^{n-1})$ on S:

$$\frac{\partial \nu}{\partial u^k} = -\nu^{-1}\,(\mathbf{F}\,|\,\boldsymbol{\tau}_k). \tag{10.17}$$

Let's substitute force field (9.6) into the equations (10.17) and take into account the fact that vector $\mathbf{N}$ for $t = 0$ coincides with unitary normal vector $\mathbf{n}(p)$ on S. Upon rather simple calculations this yields

$$\frac{\partial \nu}{\partial u^k} = -\sum_{i=1}^{n} \frac{\nabla_i W}{W_v}\,\tau^i_k. \tag{10.18}$$

Let's multiply the equation (10.18) by W_v and transfer the sum from left to right hand side of this equation. Moreover, let's write explicitly all derivatives:

$$\frac{\partial W}{\partial v}\,\frac{\partial \nu}{\partial u^k} + \sum_{i=1}^{n} \frac{\partial W}{\partial x^i}\,\frac{\partial x^i}{\partial u^k} = 0. \tag{10.19}$$

It's not difficult to see that left hand side of (10.19) is the derivative of the function $W(x^1, \dots, x^n, v)$ with respect to u^k upon substituting the functions (10.2) for $x^1, \dots, x^n$ and the function $\nu(u^1, \dots, u^{n-1})$ for v. Therefore the equations (10.19) are easily integrated in form of functional equation

$$W(x^1(p), \dots, x^n(p), \nu(p)) = W_0 = \text{const}, \tag{10.20}$$

which determines the function $\nu = \nu(p) = \nu(u^1, \dots, u^{n-1})$ in implicit form. The value of constant W_0 in (10.20) is fixed by normalizing condition (10.1):

$$W_0 = W(x^1(p_0), \dots, x^n(p_0), \nu_0).$$

Theorem 10.1. *In order to construct the normal shift of hypersurface S, given in parametric form by functions $x^1(p), \dots, x^n(p)$ from* (10.2)*, along trajectories of Newtonian dynamical system with force field* (9.6) *one should determine the function $\nu(p)$ in* (10.4) *by means of the equation* (10.20).

Having constructed the normal shift $f_t\colon S \to S_t$ along trajectories of dynamical system with force field (9.6), we obtain the family of hypersurfaces S_t. By changing the initial instant for counting the time $t \to t + t_0$ we can treat each hypersurface of this family as initial hypersurface. Therefore on each of them the following equality similar to (10.20) is fulfilled:

$$W(p, |\mathbf{v}|) = W_0(t) = \text{const}\,. \tag{10.21}$$

Note that the values of constants $W_0(t)$ in (10.21) can be different on different hypersurfaces S_t. Let's calculate the dynamics of $W_0(t)$ in t. First find the time dynamics of the modulus of velocity vector on the trajectories of shift:

$$\frac{d\,|\mathbf{v}|}{dt} = \nabla_t v = \frac{(\mathbf{v}\,|\,\nabla_t \mathbf{v})}{v} = (\mathbf{N}\,|\,\mathbf{F}) = \sum_{k=1}^{n} N^k\,F_k. \tag{10.22}$$

Then substitute (9.6) into (10.22). As a result of this substitution we obtain

$$\frac{dv}{dt}=\frac{h(W)}{W_v}-\sum_{i=1}^{n}\frac{\nabla_i W}{W_v}\,\frac{dx^i}{dt}. \tag{10.23}$$

Now let's multiply (10.23) by W_v and transfer the sum from left to right hand side of this equation. Moreover, let's write explicitly all derivatives:

$$\sum_{i=1}^{n}\frac{\partial W}{\partial x^i}\,\frac{dx^i}{dt}+\frac{\partial W}{\partial v}\,\frac{dv}{dt}=h(W). \tag{10.24}$$

In left hand side of (10.24) we see the time derivative of the scalar field on the trajectories of normal shift. Hence (10.24) is the required equation that determines time dynamics of constants $W_0(t)$ in (10.21). Let's write this equation as follows:

$$\frac{dW_0}{dt}=h(W_0). \tag{10.25}$$

If function $h(w)$ in (9.6) is zero, then due to (10.25) the field W in (10.21) not only is constant on each separate hypersurface S_t, but has equal values on all such hypersurfaces. For $h(w)\neq 0$ differential equation (10.25) is easily integrated. So, knowing the value of W on S, we can find its value on any one of hypersurfaces S_t.

11. Coordinates associated with Newtonian normal shift.

It is known that the construction of geodesic normal shift of hypersurface S in Riemannian manifold M provides some special choice of local coordinates in a neighborhood of S. They are called **semigeodesic coordinates** (see [20] or [53]). Newtonian normal shift of hypersurface S also can provide some special choice of local coordinates in M. Let $f_t\colon S\to S_t$ be the normal shift of S along trajectories of Newtonian dynamical system with force field (9.6), function $\nu(p)$ for which is fixed by normalizing condition (10.1) at the point $p_0\in S$ (without loss of generality we can assume that $\nu_0>0$). Then some neighborhood of the point p_0 foliates into the union of not intersecting parts of hypersurfaces S_t. Choosing local coordinates $u^1,\,\dots\,,\,u^{n-1}$ on S in a neighborhood of p_0, we can carry them from S to S_t by means of shift diffeomorphism $f_t\colon S\to S_t$. Therefore the set of n quantities $u^1,\,\dots\,,\,u^{n-1}$, and t can be considered as local coordinates in M in some neighborhood of the point p_0. Such coordinates are called **associated with normal shift** $f_t\colon S\to S_t$. If we denote associated coordinates by $x^1,\,\dots\,,\,x^n$, then the functions $x^i(u^1,\dots,u^{n-1},t)$ in (10.6) become extremely simple:

$$\begin{aligned}
&x^1(u^1,\dots,u^{n-1},t)=u^1,\\
&\dots\dots\dots\dots\dots\dots\\
&x^{n-1}(u^1,\dots,u^{n-1},t)=u^{n-1},\\
&x^n(u^1,\dots,u^{n-1},t)=t.
\end{aligned} \tag{11.1}$$

Differentiating functions (11.1) according to (10.3), we obtain the components of vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ tangent to the hypersurfaces S_t in associated coordinates:

$$\tau^i_k = \delta^i_k = \begin{cases} 1 & \text{for } \ i = k, \\ 0 & \text{for } \ i \neq k. \end{cases}$$

Differentiating functions (11.1) in t, we find components of velocity vector $\mathbf{v}$:

$$v^i = \delta^i_n = \begin{cases} 1 & \text{for } \ i = n, \\ 0 & \text{for } \ i \neq n. \end{cases} \tag{11.2}$$

Denote by ν the modulus of velocity vector on trajectories of shift:

$$\nu = \nu(u^1, \ldots, u^{n-1}, t) = |\mathbf{v}|. \tag{11.3}$$

Doing this, we extend the domain of function ν from (10.5), which is initially defined only for $t = 0$ on initial hypersurface S. From (11.2) we obtain the relationship

$$\nu = |\mathbf{v}| = \sqrt{g_{nn}(u^1, \ldots, u^{n-1}, t)}. \tag{11.4}$$

On the other hand, the modulus of velocity vector can be calculated from the functional equation (10.21). One can solve this equation in explicit form by using the function $V(x^1, \ldots, x^n, w)$ from (7.10):

$$|\mathbf{v}| = V(u^1, \ldots, u^{n-1}, t, W_0(t)). \tag{11.5}$$

Comparing (11.4) and (11.5), we get the following formula for diagonal component g_{nn} of metric tensor in local coordinates associated with normal shift $f_t\colon S \to S_t$:

$$g_{nn} = V(x^1, \ldots, x^n, W_0(x^n))^2. \tag{11.6}$$

Following non-diagonal components of g_{ij} are zero due to normality of shift:

$$g_{nk} = 0 \ \text{ for all } \ k = 1, \ldots, n-1. \tag{11.7}$$

Function of one variable $W_0(t)$ from (11.5) and (11.6) is determined as the solution of ordinary differential equation (10.25) fixed by initial condition

$$W_0(t)\Big|_{t=0} = W(p_0, \nu_0). \tag{11.8}$$

Initial condition (11.8) follows from (10.21), from the relationship (10.5), and from normalizing condition (10.1).

The relationships (11.6) and (11.7) mean that matrix formed by components of metric tensor in local coordinates $x^1, \ldots, x^n$ associated with Newtonian normal shift $f_t\colon S \to S_t$ has blockwise-diagonal structure:

$$g = \left\| \begin{array}{cc} \boxed{\quad G \quad} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \begin{matrix} 0 & \ldots & 0 \end{matrix} & g_{nn} \end{array} \right\|. \tag{11.9}$$

Trajectories of normal shift $f_t\colon S\to S_t$ correspond to the variation of parameter t in (11.1) by fixed values of parameters $u^1,\,\ldots\,,\,u^{n-1}$. Let's write the equation of Newtonian dynamics of points $\nabla_t\mathbf{v}=\mathbf{F}$ for such trajectories. In associated local coordinates this vectorial equation reduces to the series of scalar equations. Taking into account the relationship (11.2), we obtain

$$\Gamma^k_{nn}=F^k,\qquad k=1,\,\ldots\,,\,n. \tag{11.10}$$

Let's lower the upper index k in (11.10). Thereby we take into account blockwise-diagonal structure of matrix of metric tensor (11.9) and explicit formula (6.14) for components of metric connection. This yields

$$\frac{\partial g_{nn}}{\partial x^k}=-2\,F_k\ \text{ for }\ k<n, \tag{11.11}$$

$$\frac{\partial g_{nn}}{\partial x^n}=2\,F_n\quad\text{ for }\ k=n. \tag{11.12}$$

The quantity g_{nn} in left hand sides of (11.11) and (11.12) is determined by formula (11.6). Covariant components of force vector are determined by formula (9.6), or equivalent formula (8.2). By substituting (11.6) and (8.2) into (11.11) and into (11.12) we take into account that components of velocity vector on trajectories of shift are determined by formulas (11.2), while its modulus is determined by formula (11.3). Therefore for components of unitary vector $\mathbf{N}$ we have

$$N^k=\frac{\delta^k_n}{|\mathbf{v}|},\qquad\qquad N_k=|\mathbf{v}|\,\delta^n_k.$$

If we remember all circumstances listed above, then by substituting (11.6) and (8.2) into the equalities (11.11) and (11.12) we find that these equalities turn to identities. So we get no restrictions for the choice of functions $h(w)$ and $V(x^1,\ldots,x^n,w)$. This is not surprising, since all restrictions due to normality of shift $f_t:S\to S_t$ are already handled by normality equations (3.1) and (3.2), and by explicit formula (8.2) that follows from these equations. As for the normalizing condition (3.3) for ν, it is provided by the relationship (11.6), by the equation (10.25) for the function $W_0(t)$, and by initial condition (11.8).

The above result gives the answer to one of the questions by A. V. Bolsinov and A. T. Fomenko. It is formulated as follows: *to what extent the normal shift of some particular hypersurface $f_t:S\to S_t$ characterizes the structure of force field of dynamical system admitting the normal shift*? The answer is: **yes, it characterizes, but partially; it doesn't determine F completely**.

Indeed, if normal shift $f_t:S\to S_t$ is already constructed, then constructing associated coordinates $x^1,\,\ldots\,,\,x^n$ in a neighborhood of S reduces to the choice of local coordinates $u^1,\,\ldots\,,\,u^{n-1}$ on initial hypersurface S. Diagonal component g_{nn} of metric tensor in these coordinates determines the function of n variables

$$V(x^1,\ldots,x^n,W_0(x^n))=\sqrt{g_{nn}(x^1,\ldots,x^n)} \tag{11.13}$$

(see the relationship (11.6) above). But by function (11.3) one cannot reconstruct the function of $n+1$ variables $V(x^1, \ldots, x^n, w)$, which is contained in the formula (8.2) for components of force field $\mathbf{F}$.

If we suppose that $h(w) = 0$ in formula (8.2), then from (10.25) it follows that $W_0(x^n) = W_0 = \text{const}$. In this case formula (11.13) simplifies to

$$V(x^1, \ldots, x^n, W_0) = \sqrt{g_{nn}(x^1, \ldots, x^n)}. \tag{11.14}$$

But for the fixed normal shift $f_t \colon S \to S_t$ the constant W_0 is strictly fixed. Therefore by (11.14) we cannot reconstruct the function $V(x^1, \ldots, x^n, w)$ in whole.

Now suppose that $h(w) \neq 0$. Let's consider gauge transformations (8.3) that do not change force field (8.2). Transformations (8.3) are supplemented by the rule for transforming $W_0(t)$. It looks like

$$W_0(t) \longrightarrow \rho(W_0(t)). \tag{11.15}$$

Gauge transformations (8.3) supplemented by the additional rule (11.15) preserve not only the force field $\mathbf{F}$ of dynamical system, but the maps of normal shift f_t as well. Therefore they do not change the choice of associated local coordinates and the function g_{nn} in right hand side of (11.13). Invariance of the left hand side of (11.13) with respect to these transformations is easily checked by direct calculations. At the expense of the gauge transformations (8.3) the case when $h(w) \neq 0$ can be reduced to the case $h(w) = 1$ (see comment preceding formula (8.4)). For $h(w) = 1$ by means of integrating (10.25) we get $W_0(t) = W_0 + t$, where $W_0 = \text{const}$. This reduces (11.13) to the following form:

$$V(x^1, \ldots, x^n, W_0 + x^n) = \sqrt{g_{nn}(x^1, \ldots, x^n)}. \tag{11.16}$$

Constant W_0 in (11.6) is strictly fixed for the fixed normal shift $f_t : S \to S_t$. Therefore by (11.16) one cannot reconstruct the function (11.16) in whole.

Consider a simple example. Let $M = \mathbb{R}^n$ be euclidean space with standard metric, and let S be hyperplane given by the equation $x^n = 0$. Then the relationships (11.1) determine parametrization of S and define the normal shift of this hypersurface $f_t \colon S \to S_t$, being the parallel displacement of S along n-th coordinate axis. Thereby $g_{nn} = 1$.

1. The above shift can be implemented by Newtonian dynamical system with identically zero force field $\mathbf{F}(\mathbf{r}, \mathbf{v}) = 0$. This corresponds to the choice $h(w) = 0$ and $V(x^1, \ldots, x^n, w) = w$ in formula (8.2), and to the choice $W_0 = 1$ in formula (11.14) respectively.

2. Function $V(x^1, \ldots, x^n, w)$ can be changed, keeping $h(w) = 0$ and $W_0 = 1$ meanwhile. Let's set $V(x^1, \ldots, x^n, w) = w + (1 - w) \cdot \varphi(x^1, \ldots, x^n)$. Then the above normal shift of hyperplane S will be implemented by Newtonian dynamical system, force field of which is non-zero:

$$\mathbf{F}(\mathbf{r}, \mathbf{v}) = \frac{1 - |\mathbf{v}|}{1 - \varphi} \cdot \frac{2\,(\mathbf{v} \,|\, \nabla\varphi) \cdot \mathbf{v} - |\mathbf{v}|^2 \cdot \nabla\varphi}{|\mathbf{v}|}.$$

Here $\varphi = \varphi(\mathbf{r})$ is an arbitrary function of coordinates $x^1, \ldots, x^n$, for which is natural to assume, that its values are distinct from 1.

3. Taking $h(w) = 1$, we can choose $W(x^1, \ldots, x^n) = w - x^n$ in formula (8.2), and $W_0 = 1$ in formula (11.16) respectively. For the force field $\mathbf{F}$ this yields

$$\mathbf{F}(\mathbf{r}, \mathbf{v}) = \frac{\mathbf{v}}{|\mathbf{v}|} - \frac{2\,(\mathbf{v}\,|\,\mathbf{M}) \cdot \mathbf{v} - |\mathbf{v}|^2 \cdot \mathbf{M}}{|\mathbf{v}|}.$$

The example, which we have just examined, confirms our conclusion that knowing the normal shift $f_t\colon S \to S_t$ of some particular hypersurface is not sufficient for to determine the force field of dynamical system implementing this shift, even if we know that this system belong to the class of systems admitting the normal shift.

12. Blowing up the points. Generalization of theory as suggested by A. V. Bolsinov and A. T. Fomenko.

Before now, studying normal shift $f_t\colon S \to S_t$, we restricted ourselves to the case of smooth hypersurfaces and took parameter t small enough for hypersurfaces S_t to be non-singular as well. However, one case with singularity appears to be interesting now. This is the case when hypersurface S_t collapses into a point at a time for some $t = t_0$. By reverting the direction of time we can speak about blowing up the point. Moreover, without loss of generality we can assume that $t_0 = 0$. In this case we have singular initial hypersurface $S = \{p_0\}$ consisting of only one point p_0, and a fan-shaped pencil of trajectories coming out from this point (see figure 12.1). Velocity vectors $\mathbf{v} = \mathbf{v}(0)$ on these trajectories corresponding to the time instant $t = 0$ belong to the tangent space $T_{p_0}(M)$. They determine a hypersurface s in the fiber of tangent bundle TM over the point $p_0 \in M$. It can be understood as "limiting variety" for hypersurfaces S_t in "infinitesimal scale":

$$s = \lim_{t \to 0} \frac{S_t}{t}. \tag{12.1}$$

If initial values of velocity vectors on all trajectories at the point p_0 are non-zero, then s have topology of $(n-1)$-dimensional sphere. The same topology is inherited by all hypersurfaces S_t for sufficiently small values of parameter t. Let σ be the unit sphere in the fiber of tangent bundle over the point p_0, let q be a point of this sphere, and let $\mathbf{n}(q)$ be radius-vector of the point q in $T_{p_0}(M)$. Then radius-vectors of the points on the hypersurface s are given by formula $\mathbf{v}(q) = \nu(q) \cdot \mathbf{n}(q)$, where $\nu = \nu(q)$ is some positive function on unit sphere σ, while trajectories coming out from the point p_0 are determined by initial data

$$x^k\Big|_{t=0} = x^k(p_0), \qquad\qquad \dot{x}^k\Big|_{t=0} = \nu(q) \cdot n^k(q) \tag{12.2}$$

for the equations of Newtonian dynamics (1.2). Components of force vector in (1.2) are determined by formulas (8.2). Due to normality of shift $f_t\colon S \to S_t$ all results obtained above in sections 10 and 11 remain valid for non-singular hypersurfaces

S_t with $t = 0$. Formula (11.5) from section 11 now is written as follows:

$$|\mathbf{v}| = V(x^1(t,q), \dots, x^n(t,q), W_0(t)). \tag{12.3}$$

We can return to initial form of formula (11.5) if we denote by $u^1, \dots, u^{n-1}$ local coordinates of the point q on unit sphere σ.

Modulus of initial velocity $|\mathbf{v}| = \nu(q)$, which is contained in formula (12.2), can be found by passing to the limit $t \to 0$ in formula (12.3):

$$\nu(q) = V(x^1(p_0), \dots, x^n(p_0), W_0). \tag{12.4}$$

Here $x^1(p_0), \dots, x^n(p_0)$ are coordinates of the point p_0, and W_0 is initial value of function $W_0(t)$ for $t = 0$. The function $W_0(t)$ itself is determined as solution of ordinary differential equation (10.25). Note that right hand side of (12.4) doesn't depend on q, i. e. function $\nu(q)$ is constant:

$$\nu(q) = V(p_0, W_0) = \nu_0 = \mathrm{const}\,. \tag{12.5}$$

Constant W_0 can be expressed through ν_0 if we take into account (7.12):

$$W_0 = W(p_0, \nu_0) = \mathrm{const}\,. \tag{12.6}$$

On account of the relationships (12.5) and (12.6) we can rewrite (12.2) as follows:

$$x^k\,\Bigr|_{t=0} = x^k(p_0), \qquad\qquad \dot x^k\,\Bigr|_{t=0} = \nu_0 \cdot n^k(q). \tag{12.6}$$

Thus, we can state a theorem that follows from the results of sections 10 and 11 by passing to the limit (12.1).

Theorem 12.1. *Suppose that on Riemannian manifold M some Newtonian dynamical system admitting the normal shift is defined, i. e. we have the system with force field $\mathbf{F}$ given by formula* (8.2)*. Then for any point $p_0 \in M$ and for any positive constant $\nu_0 > 0$ initial data* (12.6) *determine the normal blow-up $f_t : p_0 \to S_t$ of the point p_0 along trajectories of this dynamical system.*

Consideration of normal blow-ups for separate point of the manifold M gives the opportunity for further development of the theory of dynamical systems admitting the normal shift. We can formulate the following definition similar to definition 3.1.

Definition 12.1. Newtonian dynamical system (1.2) with force field $\mathbf{F}$ on Riemannian manifold M is called a system **admitting the normal blow-ups of points** if for any point $p_0 \in M$ and for arbitrary positive constant $\nu_0 > 0$ initial data (12.6) determine the normal blow-up $f_t : p_0 \to S_t$ of the point p_0 along trajectories of this dynamical system.

The idea of constructing new theory on the base of definition 12.1 was suggested by A. V. Bolsinov and A. T. Fomenko in February of 2000 in the seminar at Moscow State University during the discussion on the results of thesis [17]. Theorem 12.1 shows that dynamical systems with force field (8.2) are included into the framework of new theory. But, possibly, one can find some new dynamical systems, which aren't embraced by formula (8.2). We can compare this situation with that of the theory of distributions, where narrowing class of test functions extends the class of distributions. Here, narrowing class of initial hypersurfaces in the construction of normal shift $f_t : S \to S_t$ to singular one-point sets, we have a good chance to extend class of dynamical systems that can implement such shift. Is it really so? The answer to this question can be given only as a result of constructing new theory. But this falls out of the limits of this paper.

13. On the problems of metrizability.

Problem of metrizability arose on initial stage of developing theory of dynamical systems admitting the normal shift by testing theory for non-triviality. It was noted (see papers [5] and [7]) that if metric $\tilde{\mathbf{g}}$ is conformally equivalent to the basic metric $\mathbf{g}$ of Riemannian manifold M, i. e. if we have

$$\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}, \tag{13.1}$$

then geodesic flow of metric $\tilde{\mathbf{g}}$ is a dynamical system admitting the normal shift with respect to metric $\mathbf{g}$. Its force field is given by formula

$$F_k = -|\mathbf{v}|^2\,\nabla_k f + 2\sum_{s=1}^{n} \nabla_s f\,v^s\,v_k. \tag{13.2}$$

Here ∇f is a gradient of scalar field f determining conformal factor e^{-2f} in (13.1).

Definition 13.1. Newtonian dynamical system on Riemannian manifold is called **metrizable system** if it inherits trajectories of the system with force field (13.2).

Trajectory inheriting and trajectory equivalence for two dynamical systems are understood in the sense of the following definitions.

Definition 13.2. Suppose that on the Riemannian manifold M two Newtonian dynamical systems are defined with force fields $\mathbf{F}$ and $\tilde{\mathbf{F}}$ respectively . Say that second system **inherits trajectories** of the first system if any trajectory of the second system as a line (up to a regular reparametrization) coincides with some trajectory of the first system.

Definition 13.3. Two Newtonian dynamical systems on the Riemannian manifold M are called **trajectory equivalent** if they inherit trajectories of each other.

Note that definitions 13.2 and 13.3 are somewhat different from corresponding definitions used in papers [54–60]. Our definitions are more specialized and adopted to the case of Newtonian dynamical systems with common configuration space.

Metrizable dynamical systems are trivial regarding to their use in the construction of normal shift. Normal shift along trajectories of such systems, in essential, is reduced to geodesic normal shift. Therefore in papers [5] and [9] we considered the problem of describing all metrizable Newtonian dynamical systems admitting the normal shift, and in paper [15] we constructed examples of non-metrizable ones. Main result of papers [5] and [9] is formulated in the following theorem.

Theorem 13.1. *Newtonian dynamical system admitting the normal shift on the Riemannian manifold M with metric $\mathbf{g}$ is metrizable by means of conformally equivalent metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$ if and only if its force field is given by formula*

$$F_k = -|\mathbf{v}|^2\,\nabla_k f + 2\sum_{s=1}^{n} \nabla_s f\, v^s\, v_k + \frac{H(v\,e^{-f})\,e^f}{|\mathbf{v}|}\,v_k,$$

where $H = H(v)$ is some arbitrary function of one variable.

Theorem 13.1 solved the problem of describing dynamical systems admitting the normal shift and being metrizable by means of conformally equivalent metric. However, the requirement of conformal equivalence of metrics $\tilde{\mathbf{g}}$ and $\mathbf{g}$ in this theorem is **a priori**. One can exclude this requirement. Then geodesic flow of metric $\tilde{\mathbf{g}}$ will correspond to the dynamical system with less special force field

$$F^k = \sum_{i=1}^{n}\sum_{j=1}^{n}(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})\,v^i\,v^j \tag{13.3}$$

in the metric $\mathbf{g}$. One can step further, i. e. one can avoid metric $\tilde{\mathbf{g}}$ at all assuming $\tilde{\Gamma}^k_{ij}$ in (13.3) to be components of some symmetric affine connection in M.

Definition 13.4. Newtonian dynamical system on Riemannian manifold M is called **metrizable by geodesic flow** of affine connection $\tilde{\Gamma}$ in M if it inherits trajectories of this geodesic flow.

This is the very treatment of the concept of metrizability that was considered in paper [11], which is not published unfortunately. In paper [11] the following theorem was proved.

Theorem 13.2. *Newtonian dynamical system admitting the normal shift on the Riemannian manifold M of the dimension $n \geqslant 3$ is metrizable by geodesic flow of affine connection $\tilde{\Gamma}$ if and only if this connection is (at least locally) a metric connection for some metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$, which is conformally equivalent to basic metric $\mathbf{g}$ of the manifold M.*

In other words, if there exists some Newtonian dynamical system which inherits trajectories of geodesic flow of affine connection $\tilde{\Gamma}$, and which is admitting the normal shift, then connection $\tilde{\Gamma}$ is necessarily a metric connection for some metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$. And conversely, if $\tilde{\Gamma}$ is defined by metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$, then one can find some dynamical system admitting the normal shift and inheriting trajectories of geodesic flow of $\tilde{\Gamma}$. Though the converse proposition of the theorem is obvious, since geodesic flow of metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$ admits the normal shift in metric $\mathbf{g}$.

Theorem 13.2 shows that a priori assumption on conformal equivalence of metrics $\tilde{\mathbf{g}}$ and $\mathbf{g}$ used in papers [5] and [9] at the first approach to the problem of metrizability doesn't cause the loss of generality. Proof of the theorem 13.2 is based on the following fact, which was proved in [5].

Theorem 13.3. *Suppose that force field $\mathbf{F}$ of the first Newtonian dynamical system on Riemannian manifold M is a homogeneous function of degree 2 with respect to components of velocity vector $\mathbf{v}$ in the fibers of tangent bundle TM. Then second*

Newtonian dynamical system inherits trajectories of the first system if and only if its force field $\tilde{\mathbf{F}}$ is given by the following formula:

$$\tilde{\mathbf{F}}(p,\mathbf{v}) = \mathbf{F}(p,\mathbf{v}) + \frac{H(p,\mathbf{v})}{|\mathbf{v}|}\cdot\mathbf{v}.$$

Components of force field (13.3) are quadratic functions with respect to the components of velocity vector. Hence the for force field of dynamical system inheriting trajectories of geodesic flow of affine connection $\tilde{\Gamma}$ we have the formula

$$F^k = \sum_{i=1}^{n}\sum_{j=1}^{n}(\Gamma^k_{ij} - \tilde{\Gamma}^k_{ij})\,v^i\,v^j + H\,N^k. \tag{13.4}$$

Further proof of theorem 13.2 in unpublished paper [11] consisted in substituting (13.4) into the normality equations (3.1) and (3.2). Here we shall give more simple proof of this theorem based on comparison of formulas (13.4) and (8.2). Let's denote $M^k_{ij} = \tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}$. The quantities M^k_{ij} are the components of some (not extended) tensor field $\mathbf{M}$ on the manifold M. It is called the field of **deformation** or the field of **variation** for connection Γ. Now formula (13.4) is written as follows:

$$F^k = -\sum_{i=1}^{n}\sum_{j=1}^{n} M^k_{ij}\,v^i\,v^j + H\,N^k. \tag{13.5}$$

Proof of the theorem 13.2. Let's contract both sides of formula (13.5) with components of orthogonal projector $\mathbf{P}$ from (4.3). This immediately excludes the term containing scalar function $H = H(x^1,\ldots,x^n,v^1,\ldots,v^n)$:

$$\sum_{k=1}^{n} P^q_k\,F^k = -\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} P^q_k\,M^k_{ij}\,v^i\,v^j. \tag{13.6}$$

Similar contracting in formula (8.2) cancels the entry of function $h(W)$:

$$\sum_{k=1}^{n} P^q_k\,F^k = -|\mathbf{v}|^2\sum_{k=1}^{n} P^q_k\,U^k(x^1,\ldots,x^n,W). \tag{13.7}$$

Here in (13.7) by $U^k = U^k(x^1,\ldots,x^n,|\mathbf{v}|)$ we denote the following quantities:

$$U^k = \sum_{i=1}^{n}\frac{g^{ki}\,V_i(x^1,\ldots,x^n,W(x^1,\ldots,x^n,|\mathbf{v}|))}{|\mathbf{v}|}. \tag{13.8}$$

From (3.10) we see that U^k are components of extended vector field $\mathbf{U}$, which is fiberwise spherically symmetric (see definition 5.1 above). Let's compare the relationships (13.7) and (13.6). Their left hand sides coincide. Hence we can equate right hand sides of these two relationships:

$$\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} P^q_k\,M^k_{ij}\,v^i\,v^j = |\mathbf{v}|^2\sum_{k=1}^{n} P^q_k\,U^k. \tag{13.9}$$

This relationship is remarkable, since the dependence on $\mathbf{v}$ in it is almost explicit. Indeed, the quantities M^k_{ij} do not depend on $\mathbf{v}$, while quantities U^k depend only on modulus of velocity vector. Let's fix the point $p \in M$. This means that we fix local coordinates $x^1, \ldots, x^n$. Then rewrite the relationship (13.9) in vectorial form:

$$\mathbf{P}(\mathbf{M}(\mathbf{v},\mathbf{v}) - |\mathbf{v}|^2 \cdot \mathbf{U}) = 0. \tag{13.10}$$

Here $\mathbf{M}(\mathbf{v},\mathbf{v})$ is vector valued quadratic form determined by tensor $\mathbf{M}$ when twice contracting it with vector $\mathbf{v}$. Note that according to (13.10) the operator of projection $\mathbf{P}$, when applied to the expression $\mathbf{M}(\mathbf{v},\mathbf{v}) - |\mathbf{v}|^2 \cdot \mathbf{U}$, yields zero. Therefore the equality (13.10) can be rewritten as follows:

$$\mathbf{M}(\mathbf{v},\mathbf{v}) - |\mathbf{v}|^2 \cdot \mathbf{U}(|\mathbf{v}|) = \lambda(\mathbf{v}) \cdot \mathbf{v}. \tag{13.11}$$

Vector $\mathbf{U} = \mathbf{U}(|\mathbf{v}|)$ in (13.11) depend only on modulus of velocity vector, while scalar $\lambda = \lambda(\mathbf{v})$ can contain full scale dependence on $\mathbf{v}$. Let's study this dependence. Consider vectors $\mathbf{U}(|\mathbf{v}|)$ and $\mathbf{U}(|\alpha \cdot \mathbf{v}|)$, where α is a number. Remember that in theorem 13.2 we deal with multidimensional case $n \geqslant 3$. In the space of the dimension $n \geqslant 3$ vector $\mathbf{v}$ ran be turned so that it doesn't belong to the linear span of vectors $\mathbf{U}(|\mathbf{v}|)$ and $\mathbf{U}(|\alpha \cdot \mathbf{v}|)$, while its modulus $|\mathbf{v}|$ being preserved unchanged. Let's substitute $\alpha \cdot \mathbf{v}$ for vector $\mathbf{v}$ into the equality (13.11):

$$\alpha^2 \cdot \mathbf{M}(\mathbf{v},\mathbf{v}) - \alpha^2\,|\mathbf{v}|^2 \cdot \mathbf{U}(|\alpha \cdot \mathbf{v}|) = \alpha\,\lambda(\alpha \cdot \mathbf{v}) \cdot \mathbf{v}. \tag{13.12}$$

Then multiply both sides of (13.11) by α^2 and subtract the obtained equality from (13.12). As a result we get the following relationship:

$$\alpha\,|\mathbf{v}|^2 \cdot (\mathbf{U}(|\alpha \cdot \mathbf{v}|) - \mathbf{U}(|\mathbf{v}|)) + (\lambda(\alpha \cdot \mathbf{v}) - \alpha\,\lambda(\mathbf{v})) \cdot \mathbf{v} = 0. \tag{13.13}$$

For the case when vector $\mathbf{v}$ doesn't belong to linear span of vectors $\mathbf{U}(|\mathbf{v}|)$ and $\mathbf{U}(|\alpha \cdot \mathbf{v}|)$ from the equality (13.13) we derive

$$\mathbf{U}(|\alpha \cdot \mathbf{v}|) = \mathbf{U}(|\mathbf{v}|), \tag{13.14}$$

$$\lambda(\alpha \cdot \mathbf{v}) = \alpha\,\lambda(\mathbf{v}). \tag{13.15}$$

Though the equality (13.14) holds for the case when $\mathbf{v}$ belongs to linear span of $\mathbf{U}(|\mathbf{v}|)$ and $\mathbf{U}(|\alpha \cdot \mathbf{v}|)$ as well, since $\mathbf{U}$ depend on $|\mathbf{v}|$, but not on the direction of $\mathbf{v}$. Substituting (3.14) back to (3.13), we prove (3.15) for all $\mathbf{v} \neq 0$. For $\mathbf{v} = 0$ the value of $\lambda(\mathbf{v})$ is not determined by formula (13.11). Therefore we can extend the function $\lambda(\mathbf{v})$ by taking $\lambda(0) = 0$. This cancels the restriction $\mathbf{v} \neq 0$ in applying formula (13.15).

Due to (13.14) vector $\mathbf{U}$ do not depend on $\mathbf{v}$ at all. Therefore left hand side of (13.11) is quadratic function in $\mathbf{v}$. The equality (13.11) can be rewritten as

$$\mathbf{K}(\mathbf{v}) = \mathbf{K}(\mathbf{v},\mathbf{v}) = \lambda(\mathbf{v}) \cdot \mathbf{v}. \tag{13.16}$$

Any quadratic in $\mathbf{v}$ function satisfies the following identity, which can be checked by direct calculations: $\mathbf{K}(\mathbf{v}_1+\mathbf{v}_2)+\mathbf{K}(\mathbf{v}_1-\mathbf{v}_2) = 2\cdot\mathbf{K}(\mathbf{v}_1)+2\cdot\mathbf{K}(\mathbf{v}_2)$. Substituting (13.16) into this identity, we get the equality for $\lambda(\mathbf{v})$:

$$\lambda(\mathbf{v}_1 + \mathbf{v}_2)\cdot(\mathbf{v}_1 + \mathbf{v}_2) + \lambda(\mathbf{v}_1 - \mathbf{v}_2)\cdot(\mathbf{v}_1 - \mathbf{v}_2) = 2\,\lambda(\mathbf{v}_1)\cdot\mathbf{v}_1 + 2\,\lambda(\mathbf{v}_2)\cdot\mathbf{v}_2.$$

If vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, then the above vectorial equality leads to the pair of scalar equalities:

$$\begin{aligned}\lambda(\mathbf{v}_1 + \mathbf{v}_2) + \lambda(\mathbf{v}_1 - \mathbf{v}_2) = 2\,\lambda(\mathbf{v}_1),\\ \lambda(\mathbf{v}_1 + \mathbf{v}_2) - \lambda(\mathbf{v}_1 - \mathbf{v}_2) = 2\,\lambda(\mathbf{v}_2).\end{aligned}$$

Let's add them and divide the result by 2. Then we get the relationship

$$\lambda(\mathbf{v}_1 + \mathbf{v}_2) = \lambda(\mathbf{v}_1) + \lambda(\mathbf{v}_2). \tag{13.17}$$

If vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, then (13.17) follows from (13.15). The relationships (13.15) and (13.17) mean that $\lambda(\mathbf{v})$ is a linear function in $\mathbf{v}$.

Thus $\lambda(\mathbf{v}) = (\mathbf{v}\,|\,\mathbf{\Lambda})$, where $\mathbf{\Lambda}$ is some (not extended) vector field on M. When applied to quadratic form $\mathbf{M}(\mathbf{v},\mathbf{v})$ in (13.11), this yields

$$\mathbf{M}(\mathbf{v},\mathbf{v}) = |\mathbf{v}|^2\cdot\mathbf{U} + (\mathbf{v}\,|\,\mathbf{\Lambda})\cdot\mathbf{v}. \tag{13.18}$$

Above we have proved that $\mathbf{U}$ doesn't depend on $\mathbf{v}$. Therefore $\mathbf{U}$ is also some (not extended) vector field on M. Now let's return to (13.8) and rewrite this equality in terms of covariant components of the vector $\mathbf{U}$:

$$U_i(x^1,\ldots,x^n)\,|\mathbf{v}| = V_i(x^1,\ldots,x^n,W(x^1,\ldots,x^n,|\mathbf{v}|)).$$

The quantity $|\mathbf{v}|$ plays the role of independent variable in this equality. Let's substitute $V(x^1,\ldots,x^n,w)$ for $|\mathbf{v}|$ and take into account the relationships (7.12):

$$U_i(x^1,\ldots,x^n)\,V(x^1,\ldots,x^n,w) = V_i(x^1,\ldots,x^n,w).$$

Now remember the relationships (7.13). They show that $U_i(x^1,\ldots,x^n)$ is a logarithmic derivative of the function $V(x^1,\ldots,x^n,w)$ with respect to the variable x^i. This logarithmic derivative doesn't depend on w:

$$U_i(x^1,\ldots,x^n) = \frac{\partial\ln(V(x^1,\ldots,x^n,w))}{\partial x^i}. \tag{13.19}$$

From (13.19) it follows that there exist (at least locally) two functions, a function $f = f(x^1,\ldots,x^n)$ and a function $\rho = \rho(w)$ such that

$$\begin{aligned}&U_i = \nabla_i f = \frac{\partial f(x^1,\ldots,x^n)}{\partial x^i},\\ &V = \exp(f(x^1,\ldots,x^n))\,\rho(w).\end{aligned} \tag{13.20}$$

At the expense of gauge transformation (8.3) function $V = V(x^1, \ldots, x^n, w)$ in (13.20) can be brought to the following form:

$$V = \exp(f(x^1, \ldots, x^n))\, w. \tag{13.21}$$

Let's substitute (13.21) into the formula (8.2) for the components of force field $\mathbf{F}$:

$$F_k = h(|\mathbf{v}|\, e^{-f})\, e^f\, N_k + \sum_{i=1}^{n} |\mathbf{v}|^2\, U_i\, \bigl(2\, N^i\, N_k - \delta^i_k\bigr). \tag{13.22}$$

Now let's compare formula (13.22) with formula (13.5) and take into account the above relationship (13.18) for tensor field $\mathbf{M}$. This yields

$$\frac{H - h(|\mathbf{v}|\, e^{-f})\, e^f}{|\mathbf{v}|^2} = (2\, \mathbf{U} + \boldsymbol{\Lambda}\, |\, \mathbf{N}). \tag{13.23}$$

Here $\mathbf{N}$ is unitary vector directed along the vector of velocity. Note that left hand side of (13.23) depend only on modulus of velocity vector $\mathbf{v}$, while right hand side depend on the direction of this vector. Therefore from (13.23) we get

$$H = h(|\mathbf{v}|\, e^{-f})\, e^f, \qquad\qquad \boldsymbol{\Lambda} = -2\, \mathbf{U}. \tag{13.24}$$

The relationships (13.20) and (13.24) completely determine the components of tensor field $\mathbf{M}$. For components of connection $\tilde{\Gamma}$ we have

$$\tilde{\Gamma}^k_{ij} = \Gamma^k_{ij} - \nabla_i f\, \delta^k_j - \nabla_j f\, \delta^k_i + \sum_{q=1}^{n} g^{kq}\, \nabla_q f\, g_{ij}. \tag{13.25}$$

Substituting (13.25) into (13.3), we come to the force field (13.2). Force field (13.2) corresponds to geodesic flow of metric connection for metric $\tilde{\mathbf{g}} = e^{-2f}\, \mathbf{g}$. Thus, theorem 13.2 is completely proved. $\square$

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