NEWTONIAN DYNAMICAL SYSTEMS ADMITTING NORMAL BLOW-UP OF POINTS.

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Abstract. Class of Newtonian dynamical systems admitting normal blow-up of points in Riemannian manifolds is considered. Geometric interpretation for weak normality condition, which arose earlier in the theory of dynamical systems admitting the normal shift of hypersurfaces, is found.

1. Introduction.

Let S be a hypersurface in Riemannian manifold M. One of the ways for deforming S consists in shifting points, which constitute S, along trajectories of some Newtonian dynamical system. Such situation arises in describing the propagation of electromagnetic wave (light) in non-homogeneous media in the limit of geometric optics. Hypersurface S models wave front set (the set of points with constant phase), while trajectories of shift model light beams. Newtonian dynamics of points of Riemannian manifold M in local coordinates $x^1, \dots, x^n$ in M is described by a system of n ordinary differential equations

$$\ddot{x}^k + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma^k_{ij}\, \dot{x}^i\, \dot{x}^j = F^k(x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n), \tag{1.1}$$

where $k = 1, \dots, n$. Here Γ^k_{ij} are components of metric connection for basic metric $\mathbf{g}$ of the manifold M. Quantities F^k are components of force vector $\mathbf{F}$. They determine force field of dynamical system (1.1). In the equations (1.1) they play role of perturbing factor, due to them trajectories of dynamical system (1.1) differ from that of geodesic flow for the metric $\mathbf{g}$.

At each point p of hypersurface S we fix some vector of initial velocity $\mathbf{v}(p)$ and determine trajectories coming out from all points of hypersurface S by setting the following Cauchy problem for the equations (1.1):

$$x^k\Big|_{t=0} = x^k(p), \qquad\qquad \dot{x}^k\Big|_{t=0} = v^k(p). \tag{1.2}$$

Here $v^k(p)$ are the components of vector $\mathbf{v}(p)$ in local coordinates $x^1, \dots, x^n$. Having displaced for time t along trajectories determined by initial data (1.2), points of the hypersurface S constitute another hypersurface S_t. As a result we obtain a family of hypersurfaces and a family of shift maps $f_t\colon S \to S_t$, which are local diffeomorphisms for sufficiently small values of parameter t. All this family of maps is called **a construction of shift** or simply **a shift** of hypersurface S along trajectories of dynamical system (1.1).

Definition 1.1. Shift $f_t\colon S \to S_t$ of hypersurface S along trajectories of Newtonian dynamical system with force field $\mathbf{F}$ is called **a normal shift** if hypersurfaces S_t are orthogonal to the trajectories of shift.

In order to construct the normal shift we should, at least, choose initial velocities $\mathbf{v}(p)$ being perpendicular to initial hypersurface S, i. e. $\mathbf{v}(p) = \nu(p) \cdot \mathbf{n}(p)$, where $\mathbf{n}(p)$ is a unitary normal vector to S at the point p:

$$x^k\Big|_{t=0} = x^k(p), \qquad\qquad \dot{x}^k\Big|_{t=0} = \nu(p) \cdot n^k(p). \tag{1.3}$$

But this is not sufficient. Initial data (1.3) by themselves do not provide orthogonality of S_t and trajectories of shift for $t \neq 0$. We are to use other opportunities due to the choice of hypersurface S, choice of modulus of initial velocity $\nu(p)$ on S, and choice of the force field $\mathbf{F}$ of dynamical system (1.1). If we choose hypersurface S, then, in the case of success, we would have the construction of normal shift realized only on some special hypersurface S (or in some special class of hypersurfaces). In paper [1] we left the choice of S to be arbitrary, and have concentrated efforts to the choice of function $\nu(p)$ in (1.3). Then we found that the proper choice of $\nu(p)$ on an arbitrary hypersurface S is possible only under some definite restrictions for the choice of force field of dynamical system (1.1). This became the origin for the theory of dynamical systems **admitting the normal shift**. It was developed in the series of papers [1–16]. On the base of these papers two theses were prepared: thesis for the degree of **Doctor of Sciences in Russia** [17] and thesis for the degree of **Candidate of Sciences in Russia** [18].

In [1–18] we restricted ourselves to the case of smooth hypersurfaces S_t with no singular points. However, in the process of shifting $f_t\colon S \to S_t$ some singular points can appear (they are called **caustics**). In particular, we can observe the collapse of S_t into a point at some instant of time $t = t_0$ followed by a blow-up of this point into further series of smooth hypersurfaces for $t > t_0$. Without loss of generality we can assume that $t_0 = 0$. Then initial hypersurface $S = \{p_0\}$ consisting of only one point p_0 appears to be singular, and for $t > 0$ we have blow-up of this point p_0 into a series of smooth hypersurfaces S_t.

Definition 1.2. Blow-up $f_t : S \to S_t$ of singular one-point hypersurface $S = \{p_0\}$ along trajectories of Newtonian dynamical system with force field $\mathbf{F}$ is called **a normal blow-up** if for $t > 0$ smooth hypersurfaces S_t are orthogonal to the trajectories of this blow-up.

The idea to consider blow-ups of one-point sets in the framework of normal shift was suggested by A. V. Bolsinov and A. T. Fomenko when author was reporting results of thesis [17] in a seminar at Moscow State University. Partially this idea was realized in [19]. In that paper was shown that Newtonian dynamical systems admitting the normal shift of hypersurfaces are able to implement **normal blow-up** of any point p_0 in Riemannian manifold M. More completely the idea of A. V. Bolsinov and A. T. Fomenko can be realized in special investigation. This is the main goal of present paper.

2. Geometry of normal blow-up.

Let p_0 be some point of Riemannian manifold. Let's consider normal blow-up of this point $f_t : \{p_0\} \to S_t$ along trajectories of Newtonian dynamical system

(1.1). Consider some particular trajectory of shift. For $t = 0$ it passes through the point p_0. Let $\mathbf{v}(0)$ be the velocity vector corresponding to the time instant $t = 0$. If $\mathbf{v}(0) \neq 0$, then this vector can be normalized to unit length:

$$\mathbf{n} = \frac{\mathbf{v}(0)}{|\mathbf{v}(0)|}. \tag{2.1}$$

Unit vectors (2.1) for various trajectories belong to the same tangent space $T_{p_0}(M)$. They can be identified with radius-vectors of points on unit sphere σ in the space $T_{p_0}(M)$.

Definition 2.1. The blow-up $f_t\colon \{p_0\} \to S_t$ of the p_0 along trajectories of Newtonian dynamical system (1.1) is called **regular** if velocity vectors $\mathbf{v}(0)$ at the point p_0 are non-zero for all trajectories of this blow-up and if points corresponding to unit vectors (2.1) fill the whole surface of unit sphere σ in $T_{p_0}(M)$.

In the case of regular blow-up all hypersurfaces S_t possess spherical topology for sufficiently small values of parameter $t \neq 0$. Points q of the unit sphere σ in $T_{p_0}(M)$ can be used to parameterize points of hypersurface S_t. In order to do it we shall write initial data determining regular blow-up of the point p_0 as follows:

$$x^k\Big|_{t=0} = x^k(p_0), \qquad\qquad \dot x^k\Big|_{t=0} = \nu(q)\cdot n^k(q) \tag{2.2}$$

Trajectories of Newtonian dynamical system (1.1) fixed by initial data (2.2) determine a family of maps $f_t\colon \sigma \to S_t$ being diffeomorphisms for sufficiently small values of parameter $t \neq 0$.

Suppose that we have regular blow-up of the point p_0 being normal in the sense of definition 1.2. Let's consider the solution of Cauchy problem (2.2) for the equations (1.1). This is the set of n functions $x^1(q,t), \ldots, x^n(q,t)$. Due to initial data (2.2) we can write Taylor expansions for these functions at the point $t = 0$:

$$x^k(q,t) = x^k(p_0) + \nu(q)\, n^k(q)\cdot t + O(t). \tag{2.3}$$

Denote $\mathbf{v}(q) = \nu(q)\cdot\mathbf{n}(q)$. Vector $\mathbf{v}(q) \in T_{p_0}(M)$ has the meaning of initial velocity for trajectory that corresponds to the point q on unit sphere σ. In terms of components of vector $\mathbf{v}(q)$ the expansions (2.3) can be rewritten as

$$x^k(q,t) = x^k(p_0) + v^k(q)\cdot t + O(t). \tag{2.4}$$

Let $u^1, \ldots, u^{n-1}$ be local coordinates of the point q on unit sphere σ in $T_{p_0}(M)$. Due to local diffeomorphisms of blow-up $f_t\colon \sigma \to S_t$ they can be used as local coordinates on hypersurfaces S_t. Let's represent functions $x^k(q,t)$ and their expansions (2.4) in local coordinates $u^1, \ldots, u^{n-1}$:

$$x^k(u^1,\ldots,u^{n-1},t) = x^k(p_0) + v^k(u^1,\ldots,u^{n-1})\cdot t + O(t). \tag{2.5}$$

Time derivatives of the functions (2.5) determine velocity vector on trajectories of blow-up, their derivatives in $u^1, \ldots, u^{n-1}$ determine tangent vectors to S_t:

$$\mathbf{v}=\sum_{k=1}^{n}\frac{\partial x^k}{\partial t}\cdot\frac{\partial}{\partial x^k},\qquad\qquad \boldsymbol{\tau}_i=\sum_{k=1}^{n}\frac{\partial x^k}{\partial u^i}\cdot\frac{\partial}{\partial x^k}.\tag{2.6}$$

Change in any one of parameters $u^1, \ldots, u^{n-1}$ leads to transfer from one trajectory of blow-up to another. Therefore vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ are called **vectors of variation of trajectories** or simply **vectors of variation**. For the components of the vector $\mathbf{v}(t)$ and for the components of vectors $\boldsymbol{\tau}_i(t)$ from (2.5) we derive

$$v^k(u^1,\ldots,u^{n-1},t)=\frac{\partial x^k}{\partial t}=v^k(u^1,\ldots,u^{n-1})+O(1),\tag{2.7}$$

$$\tau_i^k(u^1,\ldots,u^{n-1},t)=\frac{\partial x^k}{\partial u^i}=\frac{\partial v^k(u^1,\ldots,u^{n-1})}{\partial u^i}\cdot t+O(t).\tag{2.8}$$

The normality condition for the blow-up $f_t: \{p_0\}\to S_t$ implies orthogonality of velocity vector $\mathbf{v}$ to all vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ in (2.6). Let's write this condition as the condition of vanishing of scalar products $(\mathbf{v}\,|\,\boldsymbol{\tau}_i)$:

$$(\mathbf{v}\,|\,\boldsymbol{\tau}_i)=\sum_{k=1}^{n}\sum_{r=1}^{n}g_{kr}(x^1,\ldots,x^n)\,\frac{\partial x^k}{\partial t}\,\frac{\partial x^r}{\partial u^i}=0.\tag{2.9}$$

Substituting the expansions (2.5), (2.7), and (2.8) into the equality (2.9), we determine the asymptotics of left hand side of this equality as $t\to 0$:

$$\sum_{k=1}^{n}\sum_{r=1}^{n}g_{kr}(p_0)\,v^k(u^1,\ldots,u^{n-1})\,\frac{\partial v^k(u^1,\ldots,u^{n-1})}{\partial u^i}\cdot t+O(t)=0.\tag{2.10}$$

Right hand side of (2.10) is identically zero. Therefore from (2.10) we get

$$\sum_{k=1}^{n}\sum_{r=1}^{n}g_{kr}(p_0)\,v^k(u^1,\ldots,u^{n-1})\,\frac{\partial v^k(u^1,\ldots,u^{n-1})}{\partial u^i}=0.\tag{2.11}$$

Here $g_{kr}(p_0)=g_{kr}(x^1(p_0),\ldots,x^n(p_0))$ are the components of metric tensor at the point p_0 referred to local coordinates $x^1, \ldots, x^n$ in M. Looking attentively at the left hand side of (2.11), we see that it is exactly the scalar product of the vector of initial velocity $\mathbf{v}(q)$ and the derivative of this vector with respect to parameter u^i:

$$(\mathbf{v}(q)\,|\,\mathbf{v}'_{u^i}(q))=\frac{1}{2}\,\frac{\partial|\mathbf{v}(q)|^2}{\partial u^i}=\nu(q)\,\frac{\partial\nu(q)}{\partial u^i}=0.\tag{2.12}$$

This equality (2.12) proves the following theorem.

Theorem 2.1. *For the regular blow-up $f_t\colon\{p_0\}\to S_t$ of the point p_0 along the trajectories of Newtonian dynamical system* (1.1) *to be normal it should be determined by initial data* (2.2) *with $\nu(q)$ being constant: $\nu(q)=\mathrm{const}\neq 0$.*

Let's denote by ν_0 the constant appeared in theorem 2.1. Then we can write the initial data (2.2) in the following form:

$$x^k\Big|_{t=0}=x^k(p_0),\qquad\qquad \dot{x}^k\Big|_{t=0}=\nu_0\cdot n^k(q) \tag{2.13}$$

Further we state the following definition, which is central for the theory developed in present paper below.

Definition 2.2. Newtonian dynamical system (1.1) with force field $\mathbf{F}$ on Riemannian manifold M is called a system **admitting normal blow-up of points** if for any point $p_0\in M$ and for arbitrary positive constant $\nu_0>0$ initial data (2.13) determine the normal blow-up $f_t\colon p_0\to S_t$ of the point p_0 along trajectories of this dynamical system.

Definition 2.2 was first formulated in paper [19]. Theorem 2.1 shows that the condition $\nu(q)=\nu_0=\text{const}$ built into definition 2.1 is absolutely inevitable.

Let's substitute $\nu(q)=\nu_0$ into the expansions (2.5), (2.7), and (2.8) and take into account the above notation $\mathbf{v}(q)=\nu(q)\cdot\mathbf{n}(q)$. As a result we get the expansions

$$x^k(u^1,\ldots,u^{n-1},t)=x^k(p_0)+\nu_0\,n^k(u^1,\ldots,u^{n-1})\cdot t+O(t), \tag{2.14}$$

$$v^k(u^1,\ldots,u^{n-1},t)=\nu_0\,n^k(u^1,\ldots,u^{n-1})+O(1), \tag{2.15}$$

$$\tau^k_i(u^1,\ldots,u^{n-1},t)=\nu_0\,\frac{\partial n^k(u^1,\ldots,u^{n-1})}{\partial u^i}\cdot t+O(t), \tag{2.16}$$

which hold due to initial data (2.13) determining blow-up of the point p_0 along trajectories of dynamical system (1.1).

3. Dynamical systems admitting normal blow-up of points.

Definition 2.2 introduces new special class of Newtonian dynamical systems. According to the results of [19], it is not empty (see theorem 12.1 in [19]). In present paper we study this new class of dynamical systems introduced by definition 2.2.

Let $\mathbf{F}$ be the force field of Newtonian dynamical system admitting normal blow-up of points. Then, according to definition 2.2, by choosing an arbitrary point $p_0\in M$ and by fixing an arbitrary constant $\nu_0>0$ one can construct normal blow-up $f_t\colon\{p_0\}\to S_t$. In order to study this blow-up we consider hypersurfaces S_t with spherical topology, determine local coordinates $u^1,\ldots,u^{n-1}$ transferred from unit sphere σ to S_t, and define vectors (2.6) on trajectories of this blow-up. Then we introduce the following scalar products:

$$\varphi_i=(\mathbf{v}\,|\,\boldsymbol{\tau}_i). \tag{3.1}$$

Such scalar products were already considered above in formula (2.9). In thesis [17] they were called **the functions of deviation**. Functions of deviations (3.1) are the measure of deviation of blow-up $f_t\colon\{p_0\}\to S_t$ from normality. In the case of normal blow-up all these functions are identically zero: $\varphi_i=0$.

Vanishing of the functions of deviation φ_i at the initial instant of time $t = 0$ follows from initial conditions (2.13) regardless to the choice of force field $\mathbf{F}$ of Newtonian dynamical system (1.1):

$$\varphi_i\Big|_{t=0} = \lim_{t\to 0}\varphi_i(u^1,\ldots,u^{n-1},t) = 0. \tag{3.2}$$

Indeed, as $t \to 0$ vector of velocity tends to its limit value $\mathbf{v}(0) = \nu_0\cdot\mathbf{n}(q)$, while vector $\boldsymbol{\tau}_i$ tends to zero, this follows from the expansions (2.8) for its components. Hence scalar product $\varphi_i = (\mathbf{v}\,|\,\boldsymbol{\tau}_i)$ tends to zero.

Apart from (3.2), identical vanishing of the functions of deviation in the case of normal blow-up implies vanishing of their time derivatives $\dot\varphi_i$:

$$\dot\varphi_i\Big|_{t=0} = \lim_{t\to 0}\frac{\partial\varphi_i(u^1,\ldots,u^{n-1},t)}{\partial t} = 0. \tag{3.3}$$

In calculating $\dot\varphi_i$ for $t \neq 0$ we can replace differentiation in t by the covariant differentiation with respect to parameter t along the trajectory of blow-up:

$$\dot\varphi_i = \nabla_t\varphi_i = \nabla_t(\mathbf{v}\,|\,\boldsymbol{\tau}_i) = (\nabla_t\mathbf{v}\,|\,\boldsymbol{\tau}_i) + (\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}_i). \tag{3.4}$$

For $\nabla_t\mathbf{v}$ we have $\nabla_t\mathbf{v} = \mathbf{F}$. This follows from the equations of Newtonian dynamics (1.1). Therefore formula (3.4) for $\dot\varphi_i$ now is written as follows:

$$\dot\varphi_i = (\mathbf{F}\,|\,\boldsymbol{\tau}_i) + (\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}_i). \tag{3.5}$$

Vector $\mathbf{F}$ has a finite limit as $t \to 0$, it is determined by initial conditions (2.13): $\mathbf{F} \to \mathbf{F}(p_0, \nu_0\cdot\mathbf{n}(q))$. While vector τ_i tends to zero. Therefore first summand in (3.5) vanishes in the limit as $t \to 0$. Now consider vector $\nabla_t\boldsymbol{\tau}_i$ in the second summand. Let's write components of this vector:

$$\nabla_t\tau^k_i = \frac{\partial\tau^k_i}{\partial t} + \sum_{r=1}^{n}\sum_{s=1}^{n}\Gamma^k_{rs}\,v^r\,\tau^s_i. \tag{3.6}$$

Then let's calculate limit as $t \to 0$ in formula (3.6), using the expansions (2.14), (2.15), and (2.16) for this purpose. As a result we get

$$\lim_{t\to 0}\nabla_t\tau^k_i = \nu_0\,\frac{\partial n^k(u^1,\ldots,u^{n-1})}{\partial u^i}. \tag{3.7}$$

Here $n^k(u^1,\ldots,u^{n-1})$ are components of unitary vector $\mathbf{n}(q)$ being the radius vector of the point q on unit sphere σ in tangent space $T_{p_0}(M)$. Derivatives of the vector $\mathbf{n}(u^1,\ldots,u^{n-1})$ in $u^1,\,\ldots,\,u^{n-1}$ are coordinate tangent vectors to the sphere σ in local coordinates $u^1,\,\ldots,\,u^{n-1}$. Let's denote these vectors by $\mathbf{K}_1,\,\ldots,\,\mathbf{K}_{n-1}$:

$$\mathbf{K}_i(q) = \sum_{k=1}^{n}\frac{\partial\mathbf{n}^k(u^1,\ldots,u^{n-1})}{\partial u^i}\cdot\frac{\partial}{\partial x^k}. \tag{3.8}$$

Then the relationship (3.7) can be rewritten as

$$\nabla_t\boldsymbol{\tau}_i \to \nu_0\cdot\mathbf{K}_i(q)\quad\text{as}\quad t\to 0. \tag{3.9}$$

For vector of velocity $\mathbf{v}$, as was mentioned above, we have the relationship

$$\mathbf{v} \to \nu_0 \cdot \mathbf{n}(q) \quad \text{as} \quad t \to 0. \tag{3.10}$$

From (3.9) and (3.10) we derive vanishing of the second summand in right hand side of (3.5) as $t \to 0$. Indeed, vector $\mathbf{K}_i(q)$ is tangent to unit sphere σ at the point q, while vector $\mathbf{n}(q)$ directed radially. These vectors are perpendicular to each other, their scalar product hence is zero.

Thus, both summands in right hand side of formula (3.5) vanish as $t \to 0$, hence the relationship (3.3) holds. Similar to (3.2), this relationship is fulfilled due to initial data (2.13) regardless to the choice of force field $\mathbf{F}$ of the dynamical system (1.1). Therefore we consider analogous relationship for second order derivatives

$$\ddot{\varphi}_i\Big|_{t=0} = \lim_{t\to 0} \frac{\partial^2 \varphi_i(u^1, \ldots, u^{n-1}, t)}{\partial t^2} = 0, \tag{3.11}$$

which also should be fulfilled in the case of normal blow-up of point p_0. Let's differentiate the equality (3.5) with respect to t. This yields

$$\ddot{\varphi}_i = (\nabla_t \mathbf{F} \,|\, \boldsymbol{\tau}_i) + 2\,(\mathbf{F} \,|\, \nabla_t \boldsymbol{\tau}_i) + (\mathbf{v} \,|\, \nabla_{tt} \boldsymbol{\tau}_i). \tag{3.12}$$

Components of force vector $\mathbf{F}$ depend on double set of arguments: on coordinates $x^1, \ldots, x^n$ of the point on trajectory and on components $v^1, \ldots, v^n$ of velocity vector of this point. This means that vector $\mathbf{F}$ depend on the point of tangent bundle TM. Such vectors are not embraced by the ordinary concept of vector field on a manifold. Therefore in paper [6] the concept of **extended vector field** was introduced. The concept of **extended tensor field** is its natural generalization.

Definition 2.3. Extended tensor field $\mathbf{X}$ of the type (r, s) on the manifold M is a tensor-valued function that to each point $q = (p, \mathbf{v})$ of tangent bundle TM puts into correspondence some tensor from tensor space $T^r_s(p, M)$ at the point p on M.

Smooth extended tensor fields constitute an algebra over the ring of smooth functions on tangent bundle. It was called **the extended algebra of tensor fields** on M. In the case of Riemannian manifold one can naturally define two covariant differentiations ∇ and $\tilde{\nabla}$ in extended algebra of tensor fields on it. First was called **spatial gradient**, second was called **velocity gradient**. Covariant derivative $\nabla_t \mathbf{F}$ of the force vector in formula (3.12) can be expressed through corresponding gradients of extended vector field $\mathbf{F}$. For the components of the vector $\nabla_t \mathbf{F}$ in formula (3.12) we have the following expression:

$$\nabla_t F^k = \sum_{s=1}^{n} \nabla_s F^k \, v^s + \sum_{s=1}^{n} \tilde{\nabla}_s F^k \, F^s. \tag{3.13}$$

We shall not comment formula (3.13), and we shall not describe in details all things related with extended algebra of tensor fields (see Chapters II, III, and IV in thesis [17]). Technique of using extended tensor fields is assumed to be known to reader.

Second covariant derivative $\nabla_{tt} \boldsymbol{\tau}_i$ in formula (3.12) is expressed through $\boldsymbol{\tau}_i$ and $\nabla_t \boldsymbol{\tau}_i$. The matter is that components of any vector of variation of trajectories

$\boldsymbol{\tau}$ in case of Newtonian dynamical systems satisfy the system of linear ordinary differential equations of the second order:

$$\begin{aligned}\nabla_{tt}\tau^k=&-\sum_{m=1}^{n}\sum_{s=1}^{n}\sum_{r=1}^{n}R^k_{msr}\,\tau^s\,v^r\,v^m+\\&+\sum_{s=1}^{n}\nabla_t\tau^s\,\tilde\nabla_sF^k+\sum_{s=1}^{n}\tau^s\,\nabla_sF^k.\end{aligned}\tag{3.14}$$

Taking into account (3.13) and (3.14), we can bring formula (3.12) to the form

$$\begin{aligned}\ddot\varphi_i=&\sum_{r=1}^{n}\left(2\,F_r+\sum_{s=1}^{n}v^s\,\tilde\nabla_rF_s\right)\nabla_t\tau^r_i\,+\\&+\sum_{r=1}^{n}\left(\sum_{s=1}^{n}v^s\,(\nabla_sF_r+\nabla_rF_s)+\sum_{s=1}^{n}F^s\,\tilde\nabla_sF_r\right)\tau^r_i.\end{aligned}\tag{3.15}$$

Here and everywhere below, aside with contravariant components of vectors, we use their covariant components obtained by lowering index by means of metric:

$$v_i=\sum_{j=1}^{n}g_{ij}\,v^j,\qquad\qquad F_i=\sum_{j=1}^{n}g_{ij}\,F^j.$$

Quantities F_k, $\tilde\nabla_kF_s$, ∇_sF_k in formula (3.15) are the components of smooth extended tensor fields. They all have finite limits as $t\to 0$. Limits are determined by substituting local coordinates of the point p_0 and components of the vector $\mathbf{v}(0)=\nu_0\cdot\mathbf{n}(q)$ for their arguments. Therefore in order to calculate limit of the derivative $\ddot\varphi_i$ it is sufficient to use the relationship (3.9) and remember that $\boldsymbol{\tau}_i\to 0$ as $t\to 0$ (the latter is due to the expansions (2.16)):

$$\lim_{t\to 0}\ddot\varphi_i=\sum_{r=1}^{n}\nu_0\,\left(2\,F_r+\sum_{s=1}^{n}v^s\,\tilde\nabla_rF_s\right)\,K^r_i.\tag{3.16}$$

Substituting (3.16) into (3.11), we obtain the following relationship:

$$\sum_{r=1}^{n}\left(2\,F_r+\sum_{s=1}^{n}v^s\,\tilde\nabla_rF_s\right)K^r_i=0.\tag{3.17}$$

Here K^r_i are the components of the vector $\mathbf{K}_i$ from (3.8). Note that left hand side of (3.17) is linear with respect to components of the vector $\mathbf{K}_i$, while vectors $\mathbf{K}_1,\,\dots\,,\,\mathbf{K}_{n-1}$ form a base in the hyperplane perpendicular to the vector $\mathbf{n}(q)$. Vector $\mathbf{n}(q)$, in turn, is collinear to the velocity vector $\mathbf{v}(0)=\nu_0\cdot\mathbf{n}(q)$. Therefore if we introduce the operator P of orthogonal projection to the hyperplane perpendicular to velocity vector $\mathbf{v}$ and if we denote by P^r_i its components, we can replace (3.17) by an equivalent relationship

$$\sum_{r=1}^{n}\left(2\,F_r+\sum_{s=1}^{n}v^s\,\tilde\nabla_rF_s\right)P^r_i=0.\tag{3.18}$$

Orthogonal projectors $\mathbf{P}$ form an extended tensor field of the type $(1,1)$. Components of this field can be written in explicit form:

$$P^r_i = \delta^r_i - N^r\,N_i. \tag{3.19}$$

Here δ^r_i is Kronecker delta symbol, while N^r and N_i are contravariant and covariant components of extended vector field $\mathbf{N}$ formed by unitary vectors collinear to the vector of velocity $\mathbf{v}$:

$$v = |\mathbf{v}|, \qquad\qquad \mathbf{N} = \frac{\mathbf{v}}{v}. \tag{3.20}$$

What is the meaning of the derived relationships (3.18)? The matter is that the relationships (3.11), in contrast to (3.2) and (3.3), cannot be fulfilled only due to initial conditions (2.13). They are equivalent to the relationships (3.18) that should be fulfilled at the point p_0 for all vectors $\mathbf{v}$ such that $|\mathbf{v}| = \nu_0$. If dynamical system (1.1) belongs to the class of systems admitting normal blow-up of points, as we assumed above in the beginning of this section, then the relationships (3.18) for its force field $\mathbf{F}$ are fulfilled at all points of tangent bundle TM, where $|\mathbf{v}| \neq 0$. In this case they are partial differential equations with respect to the components of force vector $\mathbf{F}$.

Further we continue to study the relationships like (3.11). Next in the series of relationships (3.2), (3.3), and (3.11) is the vanishing condition for third derivatives of the functions of deviation $\varphi_1,\,\ldots\,,\,\varphi_{n-1}$:

$$\dddot{\varphi}_i\Bigr|_{t=0} = \lim_{t\to 0}\frac{\partial^3\varphi_i(u^1,\ldots,u^{n-1},t)}{\partial t^3} = 0. \tag{3.21}$$

In order to calculate third derivative $\dddot{\varphi}_i$ we differentiate the equality (3.15) with respect to t. The equality (3.15) has the following structure:

$$\ddot{\varphi}_i = \sum_{r=1}^{n}\alpha_r\,\nabla_t\tau^r_i + \sum_{r=1}^{n}\beta_r\,\tau^r_i. \tag{3.22}$$

Here α_r and β_r are components of extended covector fields. Therefore

$$\begin{aligned}\dddot{\varphi}_i &= \sum_{r=1}^{n}\alpha_r\,\nabla_{tt}\tau^r_i + \sum_{r=1}^{n}(\nabla_t\alpha_r + \beta_r)\,\nabla_t\tau^r_i + \sum_{r=1}^{n}\nabla_t\beta_r\,\tau^r_i = \\ &= \sum_{r=1}^{n}\alpha_r\,\nabla_{tt}\tau^r_i + \sum_{r=1}^{n}\left(\sum_{s=1}^{n}\nabla_s\alpha_r\,v^s + \sum_{s=1}^{n}\tilde{\nabla}_s\alpha_r\,F^s\right)\nabla_t\tau^r_i + \\ &\quad + \sum_{r=1}^{n}\beta_r\,\nabla_t\tau^r_i + \sum_{r=1}^{n}\left(\sum_{s=1}^{n}\nabla_s\beta_r\,v^s + \sum_{s=1}^{n}\tilde{\nabla}_s\beta_r\,F^s\right)\tau^r_i.\end{aligned}$$

In order to calculate $\nabla_t\alpha_r$ and $\nabla_t\beta_r$ above we used formulas similar to (3.13). Further we take into account that $\tau^r_i(0) = 0$ (this follows from (2.16)). Then

$$\dddot{\varphi}_i(0) = \sum_{r=1}^{n}\alpha_r\,\nabla_{tt}\tau^r_i(0) + \sum_{r=1}^{n}\left(\beta_r + \sum_{s=1}^{n}\nabla_s\alpha_r\,v^s + \sum_{s=1}^{n}\tilde{\nabla}_s\alpha_r\,F^s\right)\nabla_t\tau^r_i(0).$$

Second covariant derivative $\nabla_{tt}\tau^r_i$ can be determined from the equation (3.14). In the limit as $t\to 0$ this equation yields

$$\nabla_{tt}\tau^r_i(0)=\sum_{s=1}^{n}\tilde\nabla_sF^r\,\nabla_t\tau^s_i(0). \tag{3.23}$$

Let's substitute (3.23) into the above expression for $\dddot\varphi_i(0)$. As a result we get

$$\dddot\varphi_i(0)=\sum_{r=1}^{n}\left(\beta_r+\sum_{s=1}^{n}\tilde\nabla_rF^s\,\alpha_s+\sum_{s=1}^{n}\nabla_s\alpha_r\,v^s+\sum_{s=1}^{n}\tilde\nabla_s\alpha_r\,F^s\right)\nabla_t\tau^r_i(0).$$

The value of $\nabla_t\tau^r_i$ for $t=0$ is determined from (3.9). Therefore the condition of vanishing of third derivatives (3.21) leads to the following relationship:

$$\sum_{r=1}^{n}\left(\beta_r+\sum_{s=1}^{n}\tilde\nabla_rF^s\,\alpha_s+\sum_{s=1}^{n}\nabla_s\alpha_r\,v^s+\sum_{s=1}^{n}\tilde\nabla_s\alpha_r\,F^s\right)K^r_i=0.$$

Components of vectors $\mathbf{K}_1,\,\ldots\,,\mathbf{K}_{n-1}$ in the relationship just obtained can be replaced by components of orthogonal projector $\mathbf{P}$. Arguments for doing this are the same as in replacing the relationship (3.17) by (3.18):

$$\sum_{r=1}^{n}\left(\beta_r+\sum_{s=1}^{n}\tilde\nabla_rF^s\,\alpha_s+\sum_{s=1}^{n}\nabla_s\alpha_r\,v^s+\sum_{s=1}^{n}\tilde\nabla_s\alpha_r\,F^s\right)P^r_i=0. \tag{3.24}$$

Now we are to substitute explicit expressions for α_r and β_r into the relationship (3.24). They should be taken in comparing formulas (3.15) and (3.22) for $\ddot\varphi$:

$$\begin{aligned}\alpha_r&=2\,F_r+\sum_{s=1}^{n}v^s\,\tilde\nabla_rF_s,\\ \beta_r&=\sum_{s=1}^{n}v^s\,(\nabla_sF_r+\nabla_rF_s)+\sum_{s=1}^{n}F^s\,\tilde\nabla_sF_r.\end{aligned} \tag{3.25}$$

But before doing this substitution, note that previously obtained equations (3.18) for the components of force vector $\mathbf{F}$ can be written as

$$\sum_{r=1}^{n}\alpha_r\,P^r_i=0. \tag{3.26}$$

Let's apply the differentiations ∇ and $\tilde\nabla$ to (3.26) and let's contract the resulting equalities with components of vectors $\mathbf{v}$ and $\mathbf{F}$ respectively. This yields

$$\begin{aligned}&\sum_{r=1}^{n}\sum_{s=1}^{n}v^s\,\nabla_s\alpha_r\,P^r_i=0,\\ &\sum_{r=1}^{n}\sum_{s=1}^{n}F^s\,\tilde\nabla_s\alpha_r\,P^r_i=\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{\alpha_s\,N^s\,P^r_i\,F_r}{|\mathbf{v}|}.\end{aligned} \tag{3.27}$$

In deriving (3.27) we took into account (3.26) and we used the relationships

$$\nabla_s P^r_i = 0, \qquad \tilde{\nabla}_s P^r_i = -\frac{1}{|\mathbf{v}|}\left(N_i\,P^r_s + \sum_{j=1}^{n} g_{ij}\,P^j_s\,N^r\right). \tag{3.28}$$

The relationships (3.28) can be proved by direct calculations on the base of formulas (3.19) and (3.20) (see § 5 in Chapter V of thesis [17]).

Let's use the relationships (3.27) in order to simplify the equations (3.24). Due to the first of these relationships the third summand in (3.24) vanishes. Second relationship (3.27) enables us to transform fourth summand in (3.24). As a result of both these transformations we obtain

$$\sum_{r=1}^{n}\left(\beta_r + \sum_{s=1}^{n}\tilde{\nabla}_r F^s\,\alpha_s + \sum_{s=1}^{n}\frac{\alpha_s\,N^s\,F_r}{|\mathbf{v}|}\right)P^r_i = 0. \tag{3.29}$$

Now let's substitute α_r and β_r taken from (3.25) into the equations (3.29). Then

$$\begin{aligned}
&\sum_{r=1}^{n}\sum_{s=1}^{n} v^s(\nabla_s F_r + \nabla_r F_s)\,P^r_i + \sum_{r=1}^{n}\sum_{s=1}^{n} F^s\tilde{\nabla}_s F_r\,P^r_i\,+\\
&+\sum_{r=1}^{n}\sum_{s=1}^{n} 2\,F_s\,\tilde{\nabla}_r F^s\,P^r_i + \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n} v^q\,\tilde{\nabla}_s F_q\,\tilde{\nabla}_r F^s\,P^r_i\,+\\
&+\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{2\,F_r\,N^s\,F_s}{|\mathbf{v}|}\,P^r_i + \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n} F_r\,N^s\,N^q\,\tilde{\nabla}_s\,F_q\,P^r_i = 0.
\end{aligned} \tag{3.30}$$

Further transformation of the obtained equations we begin with the fourth summand in (3.30). Due to (3.19) we have $\delta^r_i = P^r_i + N^r\,N_i$. Therefore

$$\begin{aligned}
&\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n} v^q\,\tilde{\nabla}_s F_q\,\tilde{\nabla}_r F^s\,P^r_i = \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{j=1}^{n} v^q\,\tilde{\nabla}_s F_q\,P^s_j\,\tilde{\nabla}_r F^j\,P^r_i\,+\\
&+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{j=1}^{n} v^q\,\tilde{\nabla}_s F_q\,N^s\,N_j\,\tilde{\nabla}_r F^j\,P^r_i = -\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{n} 2\,F_s\,P^s_j\,\tilde{\nabla}_r F^j\,P^r_i\,+\\
&+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{j=1}^{n} N^q\,\tilde{\nabla}_s F_q\,N^s\,v^j\,\tilde{\nabla}_r F_j\,P^r_i = -\sum_{r=1}^{n}\sum_{s=1}^{n} 2\,F_s\,\tilde{\nabla}_r F^s\,P^r_i\,+\\
&+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{j=1}^{n} 2\,F_j\,N^j\,N_s\,\tilde{\nabla}_r F^s\,P^r_i - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n} 2\,N^q\,N^s\,\tilde{\nabla}_s F_q\,F_r\,P^r_i.
\end{aligned}$$

Apart from the relationship $\delta^r_i = P^r_i + N^r\,N_i$ following from (3.19), here we used the relationship (3.26) written as

$$\sum_{r=1}^{n}\sum_{s=1}^{n} |\mathbf{v}|\,N^s\,\tilde{\nabla}_r F_s\,P^r_i = -\sum_{r=1}^{n} 2\,F_r\,P^r_i.$$

Let's apply this relationship once more for to transform second summand in the above expression. As a result we have

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n} v^q\,\tilde\nabla_s F_q\,\tilde\nabla_r F^s\,P^r_i = -\sum_{r=1}^{n}\sum_{s=1}^{n} 2\,F_s\,\tilde\nabla_r F^s\,P^r_i\,-$$
$$-\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{4\,F_s\,N^s\,F_r}{|\mathbf{v}|}\,P^r_i-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n} 2\,N^q\,N^s\,\tilde\nabla_s F_q\,F_r\,P^r_i.$$

In substituting this expression for the fourth summand into (3.30) we find that third summand cancels, while fifth and sixth summands change their signs. The equations (3.30) in whole now look like

$$\begin{aligned}&\sum_{r=1}^{n}\sum_{s=1}^{n} v^s(\nabla_s F_r+\nabla_r F_s)\,P^r_i+\sum_{r=1}^{n}\sum_{s=1}^{n} F^s\tilde\nabla_s F_r\,P^r_i\,-\\ &-\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{2\,F_r\,N^s\,F_s}{|\mathbf{v}|}\,P^r_i-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n} F_r\,N^s\,N^q\,\tilde\nabla_s\,F_q\,P^r_i=0.\end{aligned}\tag{3.31}$$

And finally, let's do some slight (purely cosmetic) transformations in the equations (3.18) and (3.31). Then write them combining into a system:

$$\begin{cases}\displaystyle\sum_{i=1}^{n}\left(v^{-1}\,F_i+\sum_{j=1}^{n}\tilde\nabla_i\left(N^j\,F_j\right)\right)P^i_k=0,\\[2ex] \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_i F_j+\nabla_j F_i-2\,v^{-2}\,F_i\,F_j\right)N^j\,P^i_k\,+\\[2ex] \displaystyle+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde\nabla_j F_i}{v}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde\nabla_j F_r}{v}\,F_i\right)P^i_k=0.\end{cases}\tag{3.32}$$

Let's state the result following from the above calculations in form of a theorem.

Theorem 3.1. *For Newtonian dynamical system* (1.1) *in Riemannian manifold M to admit normal blow-up of points its force field should satisfy the equations* (3.32) *at all points $q=(p,\mathbf{v})$ of tangent bundle TM, where $\mathbf{v}\neq 0$.*

4. Weak normality condition.

Above we considered vanishing conditions for the functions of deviation φ_i and for their derivatives $\dot\varphi_i$, $\ddot\varphi_i$, and $\dddot\varphi_i$ at the initial instant of time $t=0$. And we have found that first two conditions $\varphi_i(0)=0$ and $\dot\varphi_i(0)=0$ are fulfilled only due to initial data (2.13) determining the blow-up of point. They make no restriction for the choice of force field $\mathbf{F}$ of dynamical system. Considering next two conditions $\ddot\varphi_i(0)=0$ and $\dddot\varphi_i(0)=0$, we derived the restrictions for $\mathbf{F}$ in form of the equations (3.32) for the components of force vector.

Further we could step by step consider the vanishing conditions for the derivatives of the functions of deviation of higher order getting more and more equations for $\mathbf{F}$

in each step. However, as we shall see soon, it is not necessary. The matter is that the equations (3.32) are exactly the same as **weak normality equations**, which arose in considering Newtonian dynamical systems admitting the normal shift of hypersurfaces. For the case $M = \mathbb{R}^2$ they were first derived in [1], then in [3] they were generalized for the case $M = \mathbb{R}^n$. In form of (3.32) corresponding to the case of arbitrary Riemannian manifold these equations were derived in [6] (see also Chapter V in thesis [17]). Weak normality equations are equivalent to the following condition **of weak normality**.

Definition 4.1. Newtonian dynamical system (1.1) in Riemannian manifold M satisfies **weak normality condition** if for each its trajectory there exists some ordinary differential equation

$$\ddot{\varphi} = \mathcal{A}(t)\,\dot{\varphi} + \mathcal{B}(t)\,\varphi \tag{4.1}$$

such that any function of deviation $\varphi = (\mathbf{v}\,|\,\boldsymbol{\tau})$ corresponding to the arbitrary choice of the vector of variation $\boldsymbol{\tau}$ on that trajectory is the solution of this equation.

Words "any function of deviation" and "arbitrary choice of the vector of variation" in this definition should be commented. Suppose that $p = p(t)$ is some trajectory of Newtonian dynamical system (1.1). Let's include it into some (arbitrary) one-parametric family of trajectories $p = p(u,t)$, so that for $u = 0$ we would have $p(0,t) = p(t)$. Parameter u can be introduced, for instance, by making dependent on u the initial data in Cauchy problem that fixes our trajectory $p(t)$. In local coordinates the family of trajectories $p = p(u,t)$ is given by the functions

$$\begin{aligned} &x^1 = x^1(u,t),\\ &.\ .\ .\ .\ .\ .\ .\ .\ .\ .\\ &x^n = x^n(u,t). \end{aligned} \tag{4.2}$$

Derivatives of the functions (4.2) with respect to parameter u determine vector $\boldsymbol{\tau}(t)$ on the trajectory $p(t)$:

$$\boldsymbol{\tau} = \sum_{k=1}^{n} \frac{\partial x^k}{\partial u}\biggr|_{u=0} \cdot \frac{\partial}{\partial x^k} \tag{4.3}$$

(compare with formula (2.6)). This vector is called **the vector of variation of trajectories**. This vector $\boldsymbol{\tau}(t)$ (constructed as described above) is implied in definition 4.1. Its scalar product with the vector of velocity $\mathbf{v}$ is **a function of deviation** corresponding to it: $\varphi = (\mathbf{v}\,|\,\boldsymbol{\tau})$.

It is easy to show that components of any vector of variation $\boldsymbol{\tau}(t)$ constructed as above satisfy the differential equations (3.14) (see paper [6] or Chapter V in thesis [17]). And conversely, any vector $\boldsymbol{\tau}(t)$ with components satisfying the equations (3.14) can be obtained by formula (4.3) in the above construction. Therefore words "arbitrary choice of the vector of variation" in definition 4.1 can be understood as the choice of an arbitrary solution of the system of linear homogeneous ordinary differential equations (3.14). Function of deviation

$$\varphi(t) = (\mathbf{v}\,|\,\boldsymbol{\tau}) = \sum_{k=1}^{n} v_k\,\tau^k$$

corresponding to this choice of $\boldsymbol{\tau}$ in the case of general position satisfies linear homogeneous ordinary differential equation of the order $2\,n$ (see theorem 6.1 in Chapter V of thesis [17]). In special cases (for special choice of force field $\mathbf{F}$) the order of this equation can be lower. Definition 4.1 separates the case, when order of the equation for φ is 2. For this case in [6] the following proposition was proved (see also theorem 6.2 in Chapter V of thesis [17]).

Theorem 4.1. *Newtonian dynamical system in Riemannian manifold of the dimension $n \geqslant 2$ satisfies weak normality condition if and only if its force field satisfies the system of differential equations* (3.32) *at all points $q = (p, \mathbf{v})$ of tangent bundle TM, where $\mathbf{v} \neq 0$.*

Note that vectors of variation $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ arising in blow-up of points are naturally embedded into the above construction (4.3). Therefore if force field $\mathbf{F}$ of dynamical system (1.1) satisfies the equations (3.32), then corresponding functions of deviation $\varphi_1, \ldots, \varphi_{n-1}$ satisfy the differential equations of the form (4.1). In this case the conditions

$$\varphi_i\Bigr|_{t=0} = 0, \qquad\qquad \dot\varphi_i\Bigr|_{t=0} = 0 \tag{4.4}$$

from (3.2) and (3.3) provide identical vanishing of the functions of deviation. The conditions (4.4) by themselves, as we noted above, are provided only by initial data (2.13). Therefore we can strengthen the theorem 3.1 formulating it as follows.

Theorem 4.2. *Newtonian dynamical system* (1.1) *in Riemannian manifold M admits normal blow-up of points if and only if its force field satisfies the equations* (3.32) *at all points $q = (p, \mathbf{v})$ of tangent bundle TM, where $\mathbf{v} \neq 0$.*

The restriction in dimension $n \geqslant 2$ from theorem 4.1 is inessential. We do not formulate it explicitly, since in the dimension $n = 1$ the concept of normal blow-up of points has no meaning.

5. Concluding remarks.

Theorem 4.2 is a central result of present paper. It reduces the study of Newtonian dynamical systems admitting normal blow-up of points to the analysis of the system of partial differential equations (3.32) for their force fields. Moreover it provides geometric interpretation for weak normality condition, reducing this rather abstract condition to visually obvious condition that dynamical system is able to perform normal blow-up of points.

Note that in the dimension $n = 2$ (this case was studied in details in thesis [18]) the condition that Newtonian dynamical system admits the normal shift of hypersurfaces is also reduced to the system of the equations (3.32). Therefore we have the following proposition.

Proposition 5.1. *Class of Newtonian dynamical systems admitting the normal shift of hypersurfaces and class of Newtonian dynamical systems admitting normal blow-up of points for $n = 2$ do coincide.*

For the dimension $n \geqslant 3$ in the theory of Newtonian dynamical systems admitting the normal shift of hypersurfaces, apart from weak normality equations (3.32), we have so called **additional normality equations**. They have the following form:

$$\begin{cases} \sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n} N^m\,\frac{F_i\,\tilde\nabla_m F_j}{v}-\nabla_i F_j\right)=\\ \qquad=\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n} N^m\,\frac{F_j\,\tilde\nabla_m F_i}{v}-\nabla_j F_i\right),\\ \sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\,\tilde\nabla_j F^i\,P^\varepsilon_i=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}\frac{P^j_m\,\tilde\nabla_j F^i\,P^m_i}{n-1}\,P^\varepsilon_\sigma. \end{cases} \tag{5.1}$$

Analysis of complete system of normality equations combined by (3.32) and (5.1) was undertook in [16]. However, in paper [16] an error was committed. Therefore part of results of [16] are not valid. This error was corrected in Chapter VII of thesis [17] (see also paper [19]). As a result an explicit formula for general solution of complete system of normality equations (3.32) and (5.1) was derived.

Currently the analysis of separate system of weak normality equations (3.32) is urgent. In particular, would be worth to know whether something like proposition 5.1 is valid in the dimension $n \geqslant 3$. Theorem 4.2 reduces this problem to the study of the equations (3.32)

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[1]Papers [1–16] are arranged here in the order they were written. However, the order of publication not always coincides with the order of writing.

[2]For the convenience of reader we give direct reference to archive file. This is the following URL address: **http://xxx.lanl.gov/eprint/math.DG/0002202**.