ON RATIONAL EXTENSION OF HEISENBERG ALGEBRA.

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Abstract. Construction of rational extension for Heisenberg algebra with one pair of generators is discussed.

1. Introduction.

Heisenberg algebra with one pair of generators arises in describing one-dimensional dynamics of spinless particle in non-relativistic quantum mechanics. Stationary states of such particle are described by scalar functions $\Psi(x)$ from the space $L_2(\mathbb{R})$:

$$\int\limits_{-\infty}^{+\infty}|\Psi(x)|^2\,dx<\infty.$$

Physical parameters of a particle in quantum mechanics are represented by corresponding operators that can be applied to wave-functions Ψ. Thus coordinate of particle is represented by the operator of multiplication by x, while momentum of particle is represented by differential operator

$$p=-i\,\hbar\,\frac{d}{dx}.$$

Operators x and p are bound by commutational relationship $[x,\,p]=-i\,\hbar$. Denote by y the operator of differentiation with respect to x. This operator differs from the operator of momentum only by constant factor. For operators x and y we have the following commutational relationship:

$$x\,y-y\,x=[x,\,y]=-1. \tag{1.1}$$

Transferring from quantum mechanics to purely algebraic situation, we consider associative (but not commutative) algebra with unity $H=H(x,y)$ defined by two generators x and y, which are bound by commutational relationship (1.1). It is called **Heisenberg algebra**. In such treatment y is no longer an operator of differentiation with respect to x, but it is simply a symbol for denoting one of the generators, playing the same role as a symbol x denoting another generator.

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As a linear space, algebra $H(x,y)$ is generated by its unit element and by monomials $x^{\alpha_1}\cdot y^{\beta_1}\cdot\ldots\cdot x^{\alpha_n}\cdot y^{\beta_n}$ with entire non-negative powers. Due to the commutational relationship (1.1) such monomial can be transformed to the linear combination of monomials of the form $x^{\alpha}\cdot y^{\beta}$. Therefore, as a linear space, algebra $H(x,y)$ can be identified with the set of polynomials of two variables $\mathbb{C}[x,y]$. However, the operation of multiplication in $H(x,y)$ differs from multiplication of polynomials in $\mathbb{C}[x,y]$. Let's study this operation in more details. First let's calculate commutator $[x,\,y^q]$, where power q is some entire non-negative number. For $q\geqslant 2$ we get

$$[x,\,y^q]=[x,\,y\cdot\ldots\cdot y]=[x,\,y]\cdot y^{q-1}+y\cdot[x,\,y]\cdot y^{q-2}+\ldots+y^{q-1}\cdot[x,\,y]=-q\,y^{q-1}.$$

Denote $f(y)=y^q$, then $q\,y^{q-1}=f'(y)$. Now the above formula is rewritten as

$$[x,\,f(y)]=-f'(y). \tag{1.2}$$

Taking for f the polynomial of two variables $f(x,y)$, one can generalize formula (1.2). Namely we have the following two formulas:

$$[x,\,f]=-f'_y, \qquad\qquad [y,\,f]=f'_x. \tag{1.3}$$

Here, when substituting generators of Heisenberg algebra x and y into polynomial $f(x,y)$, they are assumed to be ordered in a following way:

$$f(x,y)=\sum_{p\,q}f_{pq}\;x^p\cdot y^q. \tag{1.4}$$

Derivation of formulas (1.3) is quite similar to derivation of previous formulas (1.2).

In formulas (1.3) we see that transposition of the element $f=f(x,y)$ with generators x and y in Heisenberg algebra leads to the differentiation of corresponding polynomial $f(x,y)$. Formulas (1.3) admit further generalization:

$$\begin{aligned} f\cdot x^k&=\sum_{\alpha=0}^{k}C^{\alpha}_k\;x^{\alpha}\cdot D^{k-\alpha}_y f,\\ y^k\cdot f&=\sum_{\alpha=0}^{k}C^{\alpha}_k\;D^{k-\alpha}_x f\cdot y^{\alpha}. \end{aligned} \tag{1.5}$$

Here D_x and D_y are operators of differentiation in x and y respectively, while C^{α}_k in formulas (1.5) are binomial coefficients:

$$C^{\alpha}_k=\frac{k!}{\alpha!\;(k-\alpha)!}. \tag{1.6}$$

Formulas (1.5) are proved by induction in k. For $k=0$ they are obvious. For $k=1$ they reduces to (1.3). Inductive step from k to $k+1$ is provided by the following well-known identity for binomial coefficients:

$$C^{\alpha}_{k+1}=\begin{cases} C^{\alpha-1}_k+C^{\alpha}_k & \text{for}\ \ 0<\alpha<k+1,\\ C^{\alpha-1}_k & \text{for}\ \ \alpha=k+1,\\ C^{\alpha}_k & \text{for}\ \ \alpha=0. \end{cases}$$

Let's substitute $f = y^{k+q}$ and $f = x^{k+q}$, where $q \geqslant 0$, into (1.5). This yields

$$y^{k+q} \cdot x^k = \sum_{\alpha=0}^{k} \frac{k!\ (k+q)!}{\alpha!\ (k-\alpha)!\ (q+\alpha)!}\ x^\alpha \cdot y^{q+\alpha}, \tag{1.7}$$

$$y^k \cdot x^{k+q} = \sum_{\alpha=0}^{k} \frac{k!\ (k+q)!}{\alpha!\ (k-\alpha)!\ (q+\alpha)!}\ x^{\alpha+q} \cdot y^\alpha. \tag{1.8}$$

Formulas (1.7) and (1.8) can be united into one formula if we write them as follows:

$$y^q \cdot x^r = \sum_{\substack{0 \leqslant \alpha \leqslant q \\ 0 \leqslant \alpha \leqslant r}} C^\alpha_{qr}\ x^{q-\alpha} \cdot y^{r-\alpha}. \tag{1.9}$$

Coefficients C^α_{qr} in this relationship (1.9) are determined by formula

$$C^\alpha_{qr} = \frac{q!\ r!}{(q-\alpha)!\ \alpha!\ (r-\alpha)!}. \tag{1.10}$$

Formula (1.10) for C^α_{qr} is similar to formula (1.6) for binomial coefficients. This is why we have chosen symbols C^α_{qr} for coefficients in (1.9).

2. Some properties of Heisenberg algebra.

Consider two polynomials of the form (1.4). Suppose that these are f and g:

$$f = \sum_{p=0}^{n}\sum_{q=0}^{n} f_{pq}\ x^p \cdot y^q, \qquad\qquad g = \sum_{r=0}^{n}\sum_{s=0}^{n} g_{rs}\ x^r \cdot y^s. \tag{2.1}$$

Let's calculate the product of polynomials (2.1) in Heisenberg algebra:

$$f \cdot g = \sum_{p=0}^{n}\sum_{q=0}^{n} f_{pq}\ x^p \cdot y^q \cdot g(x,y) = \sum_{p=0}^{n}\sum_{q=0}^{n}\sum_{\alpha=0}^{q} f_{pq}\ C^\alpha_q\ x^p \cdot D^\alpha_x g(x,y) \cdot y^{q-\alpha}.$$

Here we used second relationship (1.5) and took into account the following symmetry of binomial coefficients: $C^\alpha_q = C^{q-\alpha}_q$. Further let's note that

$$\frac{D^\alpha_y(y^q)}{\alpha!} = \begin{cases} C^\alpha_q\ y^{q-\alpha} & \text{for}\ \ \alpha \leqslant q, \\[2ex] \ \ 0 & \text{for}\ \ \alpha > q. \end{cases}$$

Therefore if we construct by f and g a new polynomial

$$h(x,y) = \sum_{\alpha=0}^{n+1} \frac{D^\alpha_y f\ D^\alpha_x g}{\alpha!},$$

and if, upon collecting similar terms in it, we arrange variables x and y in a natural order as in (1.4), then for the product of polynomials $f \cdot g$ we can write $f \cdot g = h$:

$$f \cdot g = \sum_{\alpha=0}^{n+1} \frac{D^\alpha_y f\ D^\alpha_x g}{\alpha!}. \tag{2.2}$$

For us it's important that formula (2.2) reduces multiplication of polynomials in Heisenberg algebra to the ordinary multiplication of polynomials in the ring $\mathbb{C}[x, y]$ and to the differentiation of these polynomials. If we do not restrict the degree of polynomials f and g by particular number n, we can rewrite formula (2.2) as

$$f \cdot g = \sum_{\alpha=0}^{\infty} \frac{D_y^\alpha f \; D_x^\alpha g}{\alpha!}. \tag{2.3}$$

Theorem 2.1. *Heisenberg algebra with one pair of generators $H(x, y)$ has no divisors of zero.*

Proof. Recall that two nonzero elements f and g are called divisors of zero if their product is zero: $f \cdot g = 0$ (see [1] or [2]). Suppose that such two elements in $H(x, y)$ do exist. They should be represented by two polynomials

$$f = \sum_{q=0}^{m} f_q(x) \cdot y^q, \qquad\qquad g = \sum_{q=0}^{n} g_q(x) \cdot y^q, \tag{2.4}$$

where $f_m(x) \neq 0$ and $g_n(x) \neq 0$. Substituting polynomials (2.4) into the formula (2.3), for the product of these polynomials we get

$$h = \sum_{q=0}^{m+n} h_q(x) \cdot y^q. \tag{2.5}$$

For the leading term in polynomial (2.5) we have $h_{m+n}(x) = f_m(x)\; g_n(x)$. Therefore from $f_m(x) \neq 0$ and $g_n(x) \neq 0$ we derive $h_{m+n}(x) \neq 0$. Hence $h \neq 0$. This contradicts to the assumption that $f \cdot g = 0$. Theorem is proved. $\square$

If the equalities $g_1 \cdot f = 1$ and $f \cdot g_2 = 1$ are fulfilled, then element g_1 is called **left inverse element** for f, and g_2 is called **right inverse element** for f. It's easy to show that if element f in associative algebra has both left and right inverse elements, then these two inverse elements do coincide (see [1] or [2]):

$$g_1 = g_1 \cdot (f \cdot g_2) = (g_1 \cdot f) \cdot g_2 = g_2.$$

In associative algebra without divisors of zero the existence of left inverse element g_1 for f implies the existence of right inverse element $g_2 = g_1$. Indeed, from $1 = g_1 \cdot f$ it follows that $f = f \cdot (g_1 \cdot f) = (f \cdot g_1) \cdot f$. Then we have $(f \cdot g_1 - 1) \cdot f = 0$. The element $f \neq 0$ cannot be a divisor of zero, therefore $f \cdot g_1 - 1 = 0$. This yields the relationship $f \cdot g_1 = 1$.

And conversely, the existence of right inverse element g_2 for f implies the existence of left inverse element $g_1 = g_2$. This fact is proved similarly. From $1 = f \cdot g_2$ it follows that $f = (f \cdot g_2) \cdot f = f \cdot (g_2 \cdot f)$. Further we have $f \cdot (1 - g_2 \cdot f) = 0$. This yields the required relationship $g_2 \cdot f = 1$.

If element g is both left and right inverse for the element f, then g is called **bilateral inverse element** for f or simply **inverse element**. It is denoted as $g = f^{-1}$. Element f possessing bilateral inverse element f^{-1} is called **invertible**.

Theorem 2.2. *Element f of Heisenberg algebra $H(x,y)$ is invertible if and only if it is in the field of scalars.*

Proof. Suppose that f is invertible and $f \notin \mathbb{C}$. Then it is represented by a polynomial $f(x,y)$ which is not constant. Suppose, for instance, that $f(x,y)$ has an actual entry of the variable y. Then

$$f = \sum_{q=0}^{m} f_q(x) \cdot y^q, \tag{2.6}$$

where $m \neq 0$ and $f_m(x) \neq 0$. Let $g = f^{-1}$. We write polynomial g as

$$g = \sum_{q=0}^{n} g_q(x) \cdot y^q, \tag{2.7}$$

where $g_n(x) \neq 0$. Substituting polynomials (2.6) and (2.7) into formula (2.3), for the product of these polynomials $h = f \cdot g$ we get formula (2.5) with $n + m \neq 0$ and $h_{n+m}(x) = f_m(x)\ g_n(x)$. Hence $h_{n+m}(x) \neq 0$. This result contradicts to the equality $h = f \cdot g = f \cdot f^{-1} = 1$.

Now we have to consider the case, when $f(x,y)$ doesn't contain actual entries of the variable y. Then $f = f(x) \notin \mathbb{C}$. In this case the product of polynomials $f \cdot g$ in Heisenberg algebra coincides with their product in the ring $\mathbb{C}[x,y]$:

$$f \cdot g = f\ g = 1.$$

But non-constant polynomial $f(x)$ isn't invertible element of the ring $\mathbb{C}[x,y]$. The contradictions obtained prove that $f = \text{const} \in \mathbb{C}$. $\square$

Definition 2.1. Two nonzero elements f and g in noncommutative algebra are called **left comeasurable** if one can find nonzero elements a and b such that $a \cdot f = b \cdot g$.

Definition 2.2. Two nonzero elements f and g in noncommutative algebra are called **right comeasurable** if one can find nonzero elements a and b such that $f \cdot a = g \cdot b$.

In commutative algebra the concept of comeasurability is trivial, since any two elements in commutative algebra are comeasurable. In this aspect noncommutative Heisenberg algebra is similar to commutative ones. Namely we have the theorem.

Theorem 2.3. *Arbitrary two nonzero elements f and g in Heisenberg algebra $H(x,y)$ are left comeasurable.*

Proof. Elements of Heisenberg algebra are represented by polynomials. Their product is determined by formula (2.3). Suppose that elements f and g are given by polynomials (2.6) and (2.7), where m and n are degrees of these polynomials with respect to the variable y. For elements a and b in definition 2.1 (if they exist) from the equality $a \cdot f = b \cdot g$ we derive

$$m + \deg_y(a) = n + \deg_y(b).$$

Therefore we shall construct the elements a and b in form of polynomials

$$a=\sum_{k=0}^{n}a_k(x)\cdot y^k,\qquad\qquad b=\sum_{k=0}^{m}b_k(x)\cdot y^k. \tag{2.8}$$

Applying formula (2.3), we transform the equality $a\cdot f=b\cdot g$, which should be satisfied by polynomials (2.6), (2.7), and (2.8):

$$\sum_{k=0}^{n}\frac{D^k_y a\ D^k_x f}{k!}=\sum_{k=0}^{m}\frac{D^k_y b\ D^k_x g}{k!}. \tag{2.9}$$

Now (2.9) is an ordinary polynomial equality in the ring $\mathbb{C}[x,y]$. Let's do further transformations in this equality:

$$\sum_{k=0}^{n}\sum_{q=k}^{n}\sum_{s=0}^{m}C^k_q\ a_q(x)\ D^k_x f_s(x)\ y^{q-k+s}=\sum_{k=0}^{m}\sum_{q=k}^{m}\sum_{s=0}^{n}C^k_q\ b_q(x)\ D^k_x g_s(x)\ y^{q-k+s}.$$

First we change order of summation and replace index k by an index $p=q-k$:

$$\sum_{s=0}^{m}\sum_{q=0}^{n}\sum_{p=0}^{q}C^p_q\ a_q(x)\ D^{q-p}_x f_s(x)\ y^{p+s}=\sum_{s=0}^{n}\sum_{q=0}^{m}\sum_{p=0}^{q}C^p_q\ b_q(x)\ D^{q-p}_x g_s(x)\ y^{p+s}.$$

Then we do another change of the order of summation and replace s by $r=p+s$:

$$\begin{aligned}&\sum_{r=0}^{m+n}\left(\sum_{p=\max(0,r-m)}^{\min(n,r)}\left(\sum_{q=p}^{n}C^p_q\ a_q(x)\ D^{q-p}_x f_{r-p}(x)\right)\right)y^r=\\&=\sum_{r=0}^{m+n}\left(\sum_{p=\max(0,r-n)}^{\min(m,r)}\left(\sum_{q=p}^{m}C^p_q\ b_q(x)\ D^{q-p}_x g_{r-p}(x)\right)\right)y^r.\end{aligned} \tag{2.10}$$

Here (2.10) is an equality of two polynomials of the order $m+n$ with respect to variable y. In their leading terms with $r=n+m$ we find

$$a_n(x)\,f_m(x)=b_m(x)\,g_n(x). \tag{2.11}$$

The equation (2.11) can be easily satisfied if we define a_n and b_m by formulas

$$a_n(x)=g_n(x)\,\varphi(x),\qquad\qquad b_m(x)=f_m(x)\,\varphi(x). \tag{2.12}$$

We shall determine polynomial $\varphi(x)$ later. Now let's consider again the polynomial equality (2.10). In leading order $r=n+m$ it is fulfilled due to (2.12). Equating coefficients of y^r for other powers, we get $n+m$ equalities which can be treated as the equations with respect to $n+m$ polynomials $a_0(x)$, ... , $a_{m-1}(x)$

and $b_0(x), \ldots, b_{n-1}(x)$. Note that (2.10) doesn't contain the derivatives of polynomials $a_0(x), \ldots, a_{m-1}(x)$ and $b_0(x), \ldots, b_{n-1}(x)$. Therefore we have the non-homogeneous system of linear algebraic equations with respect to these polynomials. It can be written in matrix form as follows

$$\begin{Vmatrix} M_1^1 & \ldots & M_n^1 & M_{n+1}^1 & \ldots & M_{n+m}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_1^n & \ldots & M_n^n & M_{n+1}^n & \ldots & M_{n+m}^n \\ M_1^{n+1} & \ldots & M_n^{n+1} & M_{n+1}^{n+1} & \ldots & M_{n+m}^{n+1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_1^{n+m} & \ldots & M_n^{n+m} & M_{n+1}^{n+m} & \ldots & M_{n+m}^{n+m} \end{Vmatrix} \begin{Vmatrix} a_{n-1} \\ \vdots \\ a_0 \\ b_{m-1} \\ \vdots \\ b_0 \end{Vmatrix} = \begin{Vmatrix} A_{n-1} \\ \vdots \\ A_0 \\ B_{m-1} \\ \vdots \\ B_0 \end{Vmatrix}. \tag{2.13}$$

Elements of matrix M in (2.13) are polynomials with respect to the variable x, they are determined by coefficients $f_q(x)$ and $g_q(x)$ in polynomials (2.6) and (2.7). Quantities $A_0, \ldots, A_{m-1}$ and $B_0, \ldots, B_{n-1}$ in right hand side of (2.13) are also polynomials in x. They are determined by coefficients $f_q(x)$ and $g_q(x)$ in polynomials (2.6) and (2.7), and by our choice of leading coefficients $a_n(x)$ and $b_m(x)$ in (2.8). In particular, if we determine $a_n(x)$ and $b_m(x)$ by formula (2.12), then polynomials $A_0, \ldots, A_{m-1}$ and $B_0, \ldots, B_{n-1}$ has common factor $\varphi(x)$.

First let's study the case, when $\mu(x) = \det M \neq 0$. The quantity $\mu(x)$ can be treated as **noncommutative analog of resultant** in Heisenberg algebra for two polynomials $f(x, y)$ and $g(x, y)$ respective to the variable y:

$$\mu(x) = \mathrm{HRes}_y(f, g).$$

For the case $\mu(x) \neq 0$ we choose $\varphi(x) = \mu(x)$ in (2.12). In this case system of equations (2.13) has unique solution given by polynomials in x. So the required polynomials (2.8) are constructed.

Now let's consider degenerate case $\mu(x) = \det M = 0$. In this case we choose $\varphi(x) = 0$ in (2.12). This choice makes zero the right hand sides of the equations (2.13). These equations become homogeneous. Homogeneous system of linear algebraic equations with degenerate square matrix always has at least one nontrivial solution. We can find it applying Gauss's method. In general case this solution $a_0(x), \ldots, a_{m-1}(x)$, $b_0(x), \ldots, b_{n-1}(x)$ will be represented by rational functions. However, the solution of homogeneous system of linear equations is always determined only up to an arbitrary common factor. We can choose this factor so that $a_0(x), \ldots, a_{m-1}(x)$, $b_0(x), \ldots, b_{n-1}(x)$ will become polynomials. Thus, the required polynomials (2.8) do exist in degenerate case $\mathrm{HRes}_y(f, g) = \det M = 0$ as well. Theorem is proved. $\square$

3. Rational extension of Heisenberg algebra.

It is known that rational functions are determined as ratio of two polynomials. In abstract algebraic situation for commutative rings and algebras this is generalized in form of the construction of the field of fractions (see [1] and [2]). In the case of Heisenberg algebra we need noncommutative analog of this construction.

Consider a set $\mathcal{M}$ consisting of ordered pairs of elements from Heisenberg algebra. We shall denote such pairs as $b^{-1} \circ a$, where a and b are two elements of $H(x, y)$. Into $\mathcal{M}$ we include only those pairs $b^{-1} \circ a$ for which $b \neq 0$. Two pairs $b^{-1} \circ a$ and $d^{-1} \circ c$ are called **equivalent** if there are two elements u and v in $H(x, y)$ such that

$$u \cdot b = v \cdot d, \qquad\qquad u \cdot a = v \cdot c. \tag{3.1}$$

Properties of reflexivity and symmetry for the equivalence relation just introduced are obvious. Let's check the property of transitivity. Suppose that $b^{-1} \circ a \sim d^{-1} \circ c$ and $d^{-1} \circ c \sim f^{-1} \circ e$. Then there are some elements u, v, w and z in $H(x, y)$ such that the following relationships are fulfilled:

$$\begin{aligned} &u \cdot b = v \cdot d, &\qquad &u \cdot a = v \cdot c,\\ &w \cdot d = z \cdot f, &\qquad &w \cdot c = z \cdot e, \end{aligned}$$

Elements v and w are nonzero. This follows from $b \neq 0$, $d \neq 0$, and $f \neq 0$ due to the theorem 2.1. Let's apply theorem 2.3 to v and w. It asserts that there are two elements p and q in $H(x, y)$ such that $p \cdot v = q \cdot w$. Therefore we can write

$$\begin{aligned} &(p \cdot u) \cdot b = p \cdot v \cdot d = q \cdot w \cdot d = (q \cdot z) \cdot f,\\ &(p \cdot u) \cdot a = p \cdot v \cdot c = q \cdot w \cdot c = (q \cdot z) \cdot e. \end{aligned}$$

From the above relationships we get $b^{-1} \circ a \sim f^{-1} \circ e$. This means that the property transitivity is present.

Thus the set $\mathcal{M}$ consisting of pairs $b^{-1} \circ a$, which further will be called fractions, breaks into classes of equivalence. Denote by $\mathcal{H}(x, y)$ the set of such classes. The class of equivalence for the fraction $b^{-1} \circ a$ might be denoted by $\mathrm{Cl}(b^{-1} \circ a)$. However, for sake of simplicity the sign of class is usually omitted. The sign of equivalence $\sim$ also is often replaced by the sign of equality, emphasizing that equivalent fractions are treated as unseparable objects.

Lemma 3.1. *If $b^{-1} \circ a \sim d^{-1} \circ c$, then from the equality of denominators follows the equality of numerators $a = c$.*

Indeed, equivalence $b^{-1} \circ a \sim d^{-1} \circ c$ means that there exist two elements u and v such that the relationships (3.1) are fulfilled. Due to $b = d \neq 0$ first of them can be brought to the form $(u - v) \cdot b = 0$. According to the theorem 2.1, Heisenberg algebra has no divisors of zero. Therefore $u = v \neq 0$. Now second relationship (3.1) yields $u(a - c) = 0$. Hence $a = c$.

Let's define algebraic operations in factor set $\mathcal{H}(x, y) = \mathcal{M}/\sim$. Let's begin with the operation of summation. Suppose that we have two fractions. Their denominators are nonzero. We apply theorem 2.3 to them and find elements u and v from $H(x, y)$ such that the following equality is fulfilled:

$$u \cdot b = v \cdot d = f \neq 0. \tag{3.2}$$

Now we can bring fractions $b^{-1} \circ a$ and $d^{-1} \circ c$ to the common denominator:

$$b^{-1} \circ a = (u \cdot b)^{-1} \circ (u \cdot a), \qquad\qquad d^{-1} \circ c = (v \cdot d)^{-1} \circ (v \cdot c).$$

Now we define the sum of two fractions $b^{-1} \circ a$ and $d^{-1} \circ c$ as

$$b^{-1} \circ a + d^{-1} \circ c = f^{-1} \circ (u \cdot a + v \cdot c), \tag{3.3}$$

where u, v, and denominator f are determined by the relationship (3.2). One should check correctness of determining sum of fractions by means of formula (3.3).

Lemma 3.2. *If fraction $b^{-1} \circ a$ is equivalent to $\tilde{b}^{-1} \circ \tilde{a}$, and fraction $d^{-1} \circ c$ is equivalent to $\tilde{d}^{-1} \circ \tilde{c}$, then sum of fractions $b^{-1} \circ a + d^{-1} \circ c$ constructed by formula* (3.3) *is equivalent to the sum $\tilde{b}^{-1} \circ \tilde{a} + \tilde{d}^{-1} \circ \tilde{c}$ constructed by the same formula.*

Idea of proof consist in bringing fractions $f^{-1} \circ (u \cdot a + v \cdot c)$ and $\tilde{f}^{-1} \circ (\tilde{u} \cdot \tilde{a} + \tilde{v} \cdot \tilde{c})$, arising due to formula (3.3), to the common denominator by applying theorem 2.3. Thereby fractions $b^{-1} \circ a$, $d^{-1} \circ c$, $\tilde{b}^{-1} \circ \tilde{a}$, $\tilde{d}^{-1} \circ \tilde{c}$ are also brought to the same denominator. Further steps of proof consist in applying lemma 3.1 and in elementary calculations.

Note that in the set of fractions $\mathcal{H}(x, y) = \mathcal{M}/\sim$ there is a zero element. It is represented by fractions with numerator $a = 0$ and arbitrary denominator $b \neq 0$.

Now let's determine **multiplication** in the set of fractions $\mathcal{H}(x, y)$. Suppose that we have two fractions $b^{-1} \circ a$ and $d^{-1} \circ c$. If $a = 0$, then for the product of these fractions we set by definition

$$(b^{-1} \circ a) \cdot (d^{-1} \circ c) = 0. \tag{3.4}$$

Suppose $a \neq 0$. Then $d \neq 0$ and $d \cdot a \neq 0$. Let's apply theorem 2.3 to this pair of nonzero elements. As a result we find elements u and v such that

$$u \cdot d = v \cdot (d \cdot a). \tag{3.5}$$

From the equality (3.5) it follows that

$$b^{-1} \circ a \sim (v \cdot d \cdot b)^{-1} \circ (v \cdot d \cdot a) \sim (v \cdot d \cdot b)^{-1} \circ (u \cdot d).$$

These relationships could be a motivation for determining the product of fractions $b^{-1} \circ a$ and $d^{-1} \circ c$ by means of formula

$$(b^{-1} \circ a) \cdot (d^{-1} \circ c) = (v \cdot d \cdot b)^{-1} \circ (u \cdot c), \tag{3.6}$$

where elements u and v are determined according to the formula (3.5). One should check correctness of defining the operation of multiplication by formula (3.6). For the formula (3.4), which defines the product of fractions in the case $a = 0$, the checking procedure is trivial.

Lemma 3.3. *If fraction $b^{-1} \circ a$ is equivalent to the fraction $\tilde{b}^{-1} \circ \tilde{a}$, and fraction $d^{-1} \circ c$ is equivalent to the fraction $\tilde{d}^{-1} \circ \tilde{c}$, then the product $(b^{-1} \circ a) \cdot (d^{-1} \circ c)$ constructed by formula* (3.6) *is equivalent to the product $(\tilde{b}^{-1} \circ \tilde{a}) \cdot (\tilde{d}^{-1} \circ \tilde{c})$ constructed by the same formula.*

Proof. The use of formula (3.6) for multiplying fractions listed in the statement of lemma 3.3 assumes that we find nonzero elements u, v, $\tilde{u}$, $\tilde{v}$ such that

$$u \cdot d = v \cdot d \cdot a, \qquad \tilde{u} \cdot \tilde{d} = \tilde{v} \cdot \tilde{d} \cdot \tilde{a}. \tag{3.7}$$

Their existence is granted by theorem 2.3. Under these conditions we have

$$\begin{aligned} z &= (b^{-1} \circ a) \cdot (d^{-1} \circ c) = (v \cdot d \cdot b)^{-1} \circ (u \cdot c), \\ \tilde{z} &= (\tilde{b}^{-1} \circ \tilde{a}) \cdot (\tilde{d}^{-1} \circ \tilde{c}) = (\tilde{v} \cdot \tilde{d} \cdot \tilde{b})^{-1} \circ (\tilde{u} \cdot \tilde{c}). \end{aligned} \tag{3.8}$$

Let's prove the equivalence of fractions z and $\tilde{z}$ in (3.8). In order to prove this fact we shall bring z and $\tilde{z}$ to common denominator. However, we shall do it in two steps. First we apply theorem 2.3 and find elements p and $\tilde{p}$ such that

$$p \cdot v \cdot d = \tilde{p} \cdot \tilde{v} \cdot \tilde{d} = w. \tag{3.9}$$

Multiplying numerators and denominators of fractions z and $\tilde{z}$ by p and $\tilde{p}$ respectively, they can be transformed as follows:

$$z \sim (w \cdot b)^{-1} \circ (p \cdot u \cdot c), \qquad \tilde{z} \sim (w \cdot \tilde{b})^{-1} \circ (\tilde{p} \cdot \tilde{u} \cdot \tilde{c}). \tag{3.10}$$

Moreover, we multiply the equalities (3.7) by p and $\tilde{p}$ respectively and take into account the above relationship (3.9):

$$p \cdot u \cdot d = w \cdot a, \qquad \tilde{p} \cdot \tilde{u} \cdot \tilde{d} = w \cdot \tilde{a}. \tag{3.11}$$

In the second step, applying theorem 2.3 once more, we find q and $\tilde{q}$ such that

$$q \cdot w \cdot b = \tilde{q} \cdot w \cdot \tilde{b} = m. \tag{3.12}$$

This allows us to bring fractions (3.10) to common denominator:

$$z \sim m^{-1} \circ (q \cdot p \cdot u \cdot c), \qquad \tilde{z} \sim m^{-1} \circ (\tilde{q} \cdot \tilde{p} \cdot \tilde{u} \cdot \tilde{c}). \tag{3.13}$$

Now let's note that the equality (3.12) can be used for bringing fractions $b^{-1} \circ a$ and $\tilde{b}^{-1} \circ \tilde{a}$ to common denominator m:

$$b^{-1} \circ a \sim m^{-1} \circ (q \cdot w \cdot a), \qquad \tilde{b}^{-1} \circ \tilde{a} \sim m^{-1} \circ (\tilde{q} \cdot w \cdot \tilde{a}).$$

We know that fractions $b^{-1} \circ a$ and $\tilde{b}^{-1} \circ \tilde{a}$ are equivalent. Using lemma 3.1, we get

$$q \cdot w \cdot a = \tilde{q} \cdot w \cdot \tilde{a}. \tag{3.14}$$

Let's multiply the equalities (3.11) from left hand side by q and $\tilde{q}$ respectively. Then take into account the equality (3.14). This yields the relationship

$$q \cdot p \cdot u \cdot d = \tilde{q} \cdot \tilde{p} \cdot \tilde{u} \cdot \tilde{d} = h. \tag{3.15}$$

The relationship (3.15) can be used for bringing fractions $d^{-1} \circ c$ and $\tilde{d}^{-1} \circ \tilde{c}$ to common denominator h:

$$d^{-1} \circ c \sim h^{-1} \circ (q \cdot p \cdot u \cdot c), \qquad \tilde{d}^{-1} \circ \tilde{c} \sim h^{-1} \circ (\tilde{q} \cdot \tilde{p} \cdot \tilde{u} \cdot \tilde{c}).$$

Let's recall that $d^{-1} \circ c \sim \tilde{d}^{-1} \circ \tilde{c}$. Applying lemma 3.1, we now get

$$q \cdot p \cdot u \cdot c = \tilde{q} \cdot \tilde{p} \cdot \tilde{u} \cdot \tilde{c}. \tag{3.16}$$

Comparing (3.16) and (3.13) we conclude that fractions z and $\tilde{z}$ are equivalent. This is the very result we were to obtain. $\square$

Conclusions. Having defined the operations of addition and multiplication in factor set $\mathcal{M}/\!\sim\, = \mathcal{H}(x, y)$, we turn it into noncommutative ring. Moreover, we can keep the operation of multiplication by scalars $\alpha \in \mathbb{C}$:

$$\alpha\ (b^{-1} \circ a) = b^{-1} \circ (\alpha\, a).$$

Therefore $\mathcal{H}(x, y)$ is an algebra. It is natural to call it the **rational extension** of Heisenberg algebra $H(x, y)$. Algebra $H(x, y)$ is embedded into $\mathcal{H}(x, y)$ in form of fractions with unitary denominator: $a \mapsto (1)^{-1} \circ a$.

Note that any nonzero element in $\mathcal{H}(x, y)$ is invertible: nonzero fraction $b^{-1} \circ a$ has an inverse fraction $a^{-1} \circ b$. Therefore $\mathcal{H}(x, y)$ is an **algebra with division**. It is known (see. [3]) that finite-dimensional associative algebras with division over the field of reals $\mathbb{R}$ are exhausted by two examples: these are complex numbers $\mathbb{C}$ and algebra of quaternions $\mathbb{K}$. Algebra $\mathcal{H}(x, y)$ is an example of an algebra with division which is, though infinite-dimensional, but rather effectively constructed.

Recently I. Z. Golubchik said me that the above construction of algebra $\mathcal{H}(x, y)$ is a special case of more general construction applicable to so called "skew-polynomial algebras" and to abstract associative algebras satisfying some rather natural conditions (see [4] and [5] for more details). Nevertheless, I think that constructive algorithm for finding the "comeasuring factors" and the concept of "noncommutative resultant" suggested in the proof of theorem 2.3 could be worth for actual calculations in extended Heisenberg algebra $\mathcal{H}(x, y)$ and in developing computer programs for such calculations.

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