ON THE SOLUTIONS OF WEAK NORMALITY EQUATIONS IN MULTIDIMENSIONAL CASE.

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Abstract. The system of weak normality equations constitutes a part in the complete system of normality equations. Solutions of each of these two systems of equations are associated with some definite classes of Newtonian dynamical systems in Riemannian manifolds. In this paper for the case of simplest flat Riemannian manifold $M=\mathbb{R}^n$ with $n\geqslant 3$ we show that there exist solutions of weak normality equations that do not solve complete system of normality equations in whole. Hence associated classes of Newtonian dynamical systems do not coincide with each other.

1. Introduction.

Let M be a Riemannian manifold of the dimension n, and let S be a hypersurface in M. One of the ways for deforming S consists in moving points of S along trajectories of some Newtonian dynamical system. In local coordinates $x^1, \ldots, x^n$ in M such system is given by n ordinary differential equations

$$\ddot{x}^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j=F^k(x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n), \tag{1.1}$$

where $k=1,\ldots,n$. Here $\Gamma^k_{ij}=\Gamma^k_{ij}(x^1,\ldots,x^n)$ are components of metric connection for basic metric $\mathbf{g}$ of the manifold M, and F^k are components of force vector $\mathbf{F}$. They determine force field of dynamical system (1.1).

In order to obtain a shift of hypersurface S we set up the following Cauchy problem for the system of ordinary differential equations (1.1):

$$x^k\,\Big|_{t=0}=x^k(p), \qquad\qquad \dot{x}^k\,\Big|_{t=0}=\nu(p)\cdot n^k(p). \tag{1.2}$$

Here p is a point of hypersurface S, while $n^k(p)$ are components of unitary normal vector to S. The function $\nu(p)$ is interpreted as modulus of initial velocity $|\mathbf{v}|$; we assume it to be a smooth function of p. For the fixed point p initial data (1.2) determine some trajectory of dynamical system (1.1) coming out from the point p and being perpendicular to S at this point. Let's map the point p to the point $p(t)$ on such trajectory. If such correspondence can be extended to the whole surface S, then we have a shifting map $f_t\colon S\to S_t$. In general case the theorem on smooth dependence of the solution of ODE upon initial data (see [1] or [2]) warranties only the possibility to extend this map to some neighborhood of the point p in S. Let S' be such neighborhood. When t is sufficiently close to zero, shifting maps $f_t\colon S\to S_t$ are diffeomorphisms, their images are smooth hypersurfaces. In whole, diffeomorphisms $f_t : S' \to S'_t$, which are defined locally, are glued into a one-parametric family of local[1] diffeomorphisms $f_t \colon S \to S_t$.

Definition 1.1. One-parametric set of local diffeomorphisms $f_t : S \to S_t$ determined by the equations (1.1) and by initial data (1.2) is called **a shift** of S along trajectories of dynamical system (1.1). Such shift is called **a normal shift** if all hypersurfaces S_t produced by this shift are perpendicular to its trajectories.

Let p_0 be some point of hypersurface S, and let ν_0 be some nonzero number. We normalize the function $\nu(p)$ in (1.2) by the condition:

$$\nu(p_0) = \nu_0. \tag{1.3}$$

Definition 1.2. Newtonian dynamical system (1.1) with force field $\mathbf{F}$ is called a system **admitting the normal shift in strong sense**[2] if for any hypersurface S in M, for any point $p_0 \in S$, and for any real number $\nu_0 \neq 0$ there exists a neighborhood S' of the point S, and there exists smooth nonzero function $\nu(p)$ in S' normalized by the condition (1.3) and such that the shift $f_t \colon S' \to S'_t$ determined by this function is a normal shift in the sense of definition 1.1.

Definitions 1.1 and 1.2 underlie in the base of the theory of dynamical systems admitting the normal shift. This theory was developed in papers [3–18]; results of these papers were used in preparing theses [19] and [20]. In paper [8] we have derived the following equations, which were called **weak normality equations**:

$$\begin{cases} \displaystyle\sum_{i=1}^{n}\left(v^{-1}\,F_i+\sum_{j=1}^{n}\tilde\nabla_i\left(N^j\,F_j\right)\right)P^i_k=0,\\[2ex] \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j+\nabla_jF_i-2\,v^{-2}\,F_i\,F_j\right)N^j\,P^i_k\,+\\[2ex] \displaystyle+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde\nabla_jF_i}{v}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde\nabla_jF_r}{v}\,F_i\right)P^i_k=0. \end{cases} \tag{1.4}$$

In paper [9] **additional normality equations** were derived:

$$\begin{cases} \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_i\,\tilde\nabla_mF_j}{v}-\nabla_iF_j\right)=\\[2ex] \displaystyle\qquad=\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_j\,\tilde\nabla_mF_i}{v}-\nabla_jF_i\right),\\[2ex] \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}P^j_\sigma\,\tilde\nabla_jF^i\,P^\varepsilon_i=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}\frac{P^j_m\,\tilde\nabla_jF^i\,P^m_i}{n-1}\,P^\varepsilon_\sigma. \end{cases} \tag{1.5}$$

[1]Locality here means that domain of the map f_t depends on t. For sufficiently large t it can be empty at all.

[2]Earlier we used the definition without normalizing condition (1.3) for the function $\nu(p)$. Such definition was called the normality condition. The definition 1.2 strengthens this condition making it more strict with respect to the choice of force field $\mathbf{F}$ of the dynamical system (1.1). It is called the strong normality condition.

Leaving the equations (1.4) and (1.5) with no comments for a while, now we shall formulate two theorems binding these equations with definitions 1.1 and 1.2.

Theorem 1.1. *Newtonian dynamical system in two-dimensional Riemannian manifold M satisfies strong normality condition if and only if its force field $\mathbf{F}$ satisfies weak normality equations* (1.4) *for* $v = |\mathbf{v}| \neq 0$.

Theorem 1.2. *Newtonian dynamical system in Riemannian manifold M of the dimension $n \geqslant 3$ satisfies strong normality condition if and only if its force field $\mathbf{F}$ satisfies the normality equations* (1.4) *and* (1.5) *for* $v = |\mathbf{v}| \neq 0$.

Theorems 1.1 and 1.2 show that two and multidimensional cases are substantially different. In multidimensional case complete system of the equations (1.4) and (1.5) is strongly overdetermined, so that in reducing it we find that it is integrable in explicit form. General solution for the equations (1.4) and (1.5) is determined by two arbitrary functions $h = h(w)$ and $W = W(x^1, \ldots, x^n, v)$:

$$F_k(p, \mathbf{v}) = \frac{h(W)\,N_k}{W_v} - v\sum_{i=1}^{n}\frac{\nabla_i W}{W_v}\left(2\,N^i\,N_k - \delta^i_k\right). \tag{1.6}$$

Formula (1.6) was obtained in thesis [19]. Here $\nabla_i W$ is the partial derivative of the function W in i-th coordinate of the point p, while W_v is partial derivative of W in the variable v, which is is interpreted as modulus of velocity vector: $v = |\mathbf{v}|$.

In two-dimensional case we have only the equations (1.4). Here they were reduced to one scalar partial differential equation of the second order, then various special solutions of this equation were constructed (see thesis [20]). Most of these solutions correspond to the force fields that can be obtained by means of formula (1.6) when taking $n = 2$ in it. These force fields were called fields of **multidimensional type**. However, one of the most important results of thesis [20] is that there essentially two-dimensional solution of the equations (1.4) was constructed. This solution is not given by formula (1.6). Here we also construct the solution of the equations (1.4), which is not expressed by formula (1.6), but in multidimensional case.

Formula (1.6) obtained in thesis [19] describes all Newtonian dynamical systems admitting the normal shift in Riemannian manifolds of the dimension $n \geqslant 3$. However, when thesis [19] has been already written, new statement of the problem of normal shift was found. It leads to the equations (1.4) in pure form (without additional equations (1.5)). Indeed, in papers [3–18] and in theses [19] and [20] we considered only smooth hypersurfaces S and we choosed sufficiently small values of t for the sift $f_t\colon S \to S_t$ to result in smooth hypersurfaces S_t only. If we eliminate this restriction for t, then in the process of shifting we sometimes can observe singular points on hypersurface S_t (they are called **caustics**). In particular, under the definite sircumstances hypersurface S_t can contract into a point at a time for some $t = t_0$. This process is called **collapse**. Immediately after the callapse for $t > t_0$ we shall observe a blow-up of the point into a series of expanding hypersurfaces S_t. The idea to consider the blow-ups of points by means of Newtonian dynamical systems were suggested to me by A. V. Bolsinov and A. T. Fomenko when I was reporting results of thesis [19] in the seminar at Moscow State University in February of 2000. This idea was realized in paper [21] and in paper [22].

Let p_0 be some point of Riemannian manifold M. Let's consider the set of all unitary vectors in tangent space $T_{p_0}(M)$. They can be interpreted as radius-vectors of the points of unit sphere σ in $T_{p_0}(M)$. Let $q \in \sigma$ and let $\mathbf{n}(q)$ be the radius-vector of the point q on unit sphere. Let's fix some constant number $\nu_0 \neq 0$ and set up the following Cauchy problem for the equations of Newtonian dynamical system (1.1):

$$x^k\Big|_{t=0} = x^k(p_0), \qquad\qquad \dot{x}^k\Big|_{t=0} = \nu_0 \cdot n^k(q) \tag{1.7}$$

For the fixed q initial data (1.2) determine some trajectory of dynamical system (1.1) coming out fromn the point p_0. To the point $q \in \sigma$ we put into correspondence the point $p(t)$ on such trajectory. Theorem on smooth dependence of the solution of ODE upon initial data (see [1] or [2]) says that we can extend this map to some neighborhood of the point q in σ. If σ' is such neighborhood of the point q, then we have one-parametric family of diffeomorphisms $f_t\colon \sigma' \to S'_t$. Due to compactness of unit sphere σ we can glue local maps into one map $f_t\colon \sigma \to S_t$. Here diffeomorphisms $f_t\colon \sigma \to S_t$ are determined globally on the whole sphere σ, though parameter t can be restricted by some interval $(-\varepsilon, +\varepsilon)$ on real axis as before.

Definition 1.3. One parametric family of diffeomorphisms $f_t\colon \sigma \to S_t$ given by the equations (1.1) and initial data (1.7) is called **a blow-up** of the point p_0 along trajectories of dynamical system (1.1). It is called **a normal blow-up** if all hypersurfaces S_t arising in this blow-up are perpendicular to its trajectories.

Definition 1.4. Newtonian dynamical system (1.1) with force field $\mathbf{F}$ in Riemannian manifold M is called **admitting normal blow-up of points** if for any point $p_0 \in M$, and for any positive constant ν_0 initial data (1.7) determine normal blow-up of this point along trajectories of dynamical system (1.1).

Definitions 1.3 and 1.4 were first formulated in paper [21]. They introduced new object: a class of newtonian dynamical systems admitting the normal blow-up of points in Riemannian manifolds. In paper [21] it was shown (see theorem 12.1) that this new class of systems comprises the class of dynamical systems admitting the normal shift of hypersurfaces, which was previously considered. More exact description of new class of dynamical systems is given by the following theorem proved in paper [22].

Theorem 1.3. *Newtonian dynamical system* (1.1) *on Riemannian manifold M admits normal blow-up of points if and only if its force field* $\mathbf{F}$ *satisfies weak normality equations* (1.4) *for* $|\mathbf{v}| \neq 0$.

From theorems 1.1 and 1.3 we see that class of dynamical systems admitting the normal shift of hypersurfaces in two dimensional case $n = 2$ coincides with the class of systems admitting normal blow-up of points. This was proved in [22]. But for the multidimensional case $n \geqslant 3$ the question on **coinciding** or **not coinciding** of these two classes remained open. I. A. Taimanov was strongly interested in this question during my report in the seminar of Yu. G. Reshetnyak at MI SB RAS (Mathematical Institute of Siberian Brunch of Russian Academy of Sciences) in

October of 2000. The main goal of present paper is to give answer to this question and, thus, eliminate one more obstacle for defending[1] thesis (19).

2. Normality equations and extended tensor fields.

Let's consider the force field $\mathbf{F}$ in the equations of Newtonian dynamics (1.1). Left hand side of these equations are components of acceleration vector $\nabla_t \mathbf{v}$ (covariant derivative of velocity vector with respect to parameter τ along the trajectory). Therefore F^k are components of tangent vector to M. However, they depend on double set of arguments: on coordinates $x^1, \ldots, x^n$ of the point $p \in M$ and on the components of tangent vector $\mathbf{v} \in T_p(M)$. Pair $q = (p, \mathbf{v})$ is a point of tangent bundle TM, so that $p = \pi(q)$. Thus, considering the equations of the form (1.1), we come to the concept of **extended vector field**. Its generalization is a concept of **extended tensor field**.

Definition 2.1. Function $\mathbf{X}$ that maps each point $q = (p, \mathbf{v})$ of tangent bundle TM to some tensor of the type (r, s) from the space $T^r_s(p, M)$ at the point $p = \pi(q)$ is called **extended tensor field** of the type (r, s) on the manifold M.

The concept of extended tensor field stems from thesis [23] of Finsler, which gave rise to Finslerian geometry. Another approach to constructing extended tensor fields consists in rising them from M to TM. Here they constitute some special subset in the set of traditional tensor fields on TM. Such tensor fields were considered in the book [24], they were called **semibasic tensor fields**. I am grateful to N. S. Dairbekov from IM SB RAS, who noted that theories of extended and semibasic tensor fields are isomorphic to each other.

Below we shall use theory of extended tensor fields, which is described in details in Chapters II–IV of thesis [19]. It is based on the definition 2.1.

For the beginning let's consider some particular examples of extended tensor fields on the Riemannian manifold M.

1. Let's take the point $q = (p, \mathbf{v})$ of tangent bundle TM and let's map it to the vector $\mathbf{v}$ belonging to tangent space $T_p(M)$. This yields an extended vector field, which is called **the field of velocity**.

2. Let's map the point $q = (p, \mathbf{v})$ of TM to the number $v = |\mathbf{v}|$. This yields an extended scalar field, which is called **the field of modulus of velocity vector**.

3. Extended field of unitary vectors $\mathbf{N}$ is determined as the ratio of two previous fields: $\mathbf{N} = \mathbf{v}/v$.

4. Extended field of operators $\mathbf{P}$ is formed by operators of orthogonal projection onto the hyperplanes perpendicular to velocity vector $\mathbf{v}$. Its components can be written explicitly: $P^i_j = \delta^i_j - N^i\, N_j$.

Components of all above fields are present in normality equations (1.4) and (1.5). Moreover, in these equations we see the operators of covariant differentiation ∇ and $\tilde{\nabla}$; they called **spatial** and **velocity** gradients. The simplest way to define them is to use explicit formulas in coordinates:

$$\tilde{\nabla}_m X^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial X^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial v^m}, \tag{2.1}$$

[1] The matter is that in Russia one can pretend for the degree of Doctor of Sciences only upon writing thesis and passing so called Defense Procedure in Specialized Council.

$$\begin{aligned}\nabla_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} &= \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m} - \sum_{a=1}^{n}\sum_{b=1}^{n} v^a\,\Gamma^b_{ma}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^b} + \\ &+ \sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned} \tag{2.2}$$

3. Scalar ansatz.

In order to simplify the normality equations (1.4) and (1.5) in paper [18] the scalar ansatz was suggested. Note that it is applicable either in two-dimensional case $n = 2$, and in multidimensional case $n \geqslant 3$ as well:

$$F_k = A\,N_k - |\mathbf{v}|\sum_{i=1}^{n} P^i_k\,\tilde{\nabla}_i A. \tag{3.1}$$

Scalar ansatz (3.1) follows from first normality equation in the system (1.4). Substituting (3.1) back into this equation, we turn it into identity.

Formula (3.1) expresses force vector $\mathbf{F}$ through one extended scalar field A. This scalar field A can be expressed back through $\mathbf{F}$:

$$A = \sum_{i=1}^{n} F^i\,N_i. \tag{3.2}$$

Substituting (3.1) into the second equation in the system (1.4), we obtain the following equation for scalar field A:

$$\begin{aligned}&\sum_{s=1}^{n}\left(\nabla_s A + |\mathbf{v}|\sum_{q=1}^{n}\sum_{r=1}^{n} P^{qr}\,\tilde{\nabla}_q A\,\tilde{\nabla}_r\tilde{\nabla}_s A - \right.\\ &\left. - \sum_{r=1}^{n} N^r\,A\,\tilde{\nabla}_r\tilde{\nabla}_s A - |\mathbf{v}|\sum_{r=1}^{n} N^r\,\nabla_r\tilde{\nabla}_s A\right) P^s_k = 0.\end{aligned} \tag{3.3}$$

Formulas (3.1) and (3.2) establishes one-to-one correspondence between solutions of the equations (1.4) and (3.3). If force field $\mathbf{F}$ satisfies the equations (1.4) and (1.5) simultaneously, then it is expressed by formula (1.6). Scalar field A corresponding to such force field is given by formula

$$A = \frac{h(W)}{W_v} - v\sum_{i=1}^{n}\frac{N^i\,\nabla_i W}{W_v}. \tag{3.4}$$

Formula (3.4) admits gauge transformations, which change functions $h = h(w)$ and $W = W(x^1,\ldots,x^n,v)$, but which don't change A:

$$\begin{aligned}&W(x^1,\ldots,x^n,v) \longrightarrow \rho(W(x^1,\ldots,x^n,v)),\\ &h(w) \longrightarrow h(\rho^{-1}(w))\;\rho'(\rho^{-1}(w)).\end{aligned} \tag{3.5}$$

If the function $h(w)$ is nonzero, then by means of gauge transformations (3.5) it can be made identically equal to unity (see thesis [19] and succeeding paper [21]).

Therefore, instead of formula (3.4) with two arbitrary functions, we can use two formulas with one arbitrary function:

$$A=\begin{cases}\dfrac{1}{W_v}-v\displaystyle\sum_{i=1}^{n}\frac{N^i\,\nabla_i W}{W_v} & \text{for}\quad h=1,\\[2ex] -v\displaystyle\sum_{i=1}^{n}\frac{N^i\,\nabla_i W}{W_v} & \text{for}\quad h=0.\end{cases}\tag{3.6}$$

Now the problem stated in section 1 is reformulated as follows: one should find a solution of the equations (3.3) that cannot be expressed by formula (3.6) neither for $h=1$ nor $h=0$.

4. Spatially homogeneous force field with axial symmetry.

Let M be a space $\mathbb{R}^n$ with standard Euclidean metric. We shall construct the required solution of the equations (3.3) in this simples case. In the space $\mathbb{R}^n$ covariant derivatives ∇_m and $\tilde{\nabla}_m$ given by formulas (2.1) and (2.2) turn to partial derivatives: $\nabla_m=\partial/\partial x^m$ and $\tilde{\nabla}_m=\partial/\partial v^m$. We restrict our consideration to spatially homogeneous force fields, in $\mathbb{R}^n$ they depend only on velocity vector, but don't depend on coordinates $x^1,\,\ldots\,,\,x^n$. For the corresponding function A this yields $A=A(v^1,\ldots,v^n)$. When substituting this function into the equation (3.3), this equation is reduced to the following one:

$$\sum_{s=1}^{n}\left(|\mathbf{v}|\sum_{q=1}^{n}\sum_{r=1}^{n}P^{qr}\,\tilde{\nabla}_qA\,\tilde{\nabla}_r\tilde{\nabla}_sA-\sum_{r=1}^{n}N^r\,A\,\tilde{\nabla}_r\tilde{\nabla}_sA\right)P^s_k=0.\tag{4.1}$$

Let's mark some direction in$\mathbb{R}^n$ determined by some constant unitary vector $\mathbf{m}$. Without loss of generality we can assume $\mathbf{m}$ to be directed along n-th coordinate axis. Let's expand $\mathbf{v}$ into a sum

$$\mathbf{v}=u\cdot\mathbf{n}+w\cdot\mathbf{m},\tag{4.2}$$

where $\mathbf{n}\perp\mathbf{m}$ and $|\mathbf{n}|=1$ All directions perpendicular to $\mathbf{m}$ are assumed to be equivalent. Therefore we choose function A depending only on two variables: $A=A(u,w)$. One should study whether such choice is compatible with normality equations (4.1). For this purpose we calculate partial derivatives

$$\tilde{\nabla}_iA=\frac{\partial A}{\partial v^i}=\begin{cases}A_u\cdot\dfrac{v_i}{u} & \text{for}\quad i<n,\\[1ex] A_w & \text{for}\quad i=n.\end{cases}\tag{4.3}$$

Formula (4.3) shows that vector of velocity gradient $\tilde{\nabla}A$ belong to the linear span of vectors $\mathbf{m}$ and $\mathbf{n}$ from the expansion (4.2):

$$\tilde{\nabla}A=A_u\cdot\mathbf{n}+A_w\cdot\mathbf{m}.\tag{4.4}$$

Denote by $\mathbf{B}$ the vector with the following components:

$$B^r=\sum_{q=1}^{n}P^{qr}\,\tilde{\nabla}_q A.$$

This is the projection of velocity gradient $\tilde{\nabla}A$ to the hyperplane perpendicular to velocity vector. One can obtain explicit formula for the vector $\mathbf{B}$. For this purpose let's denote by θ the angle between vectors $\mathbf{m}$ and $\mathbf{v}$. Then let's expand projections $\mathbf{Pm}$ and $\mathbf{Pn}$ in the base composed by unitary vectors $\mathbf{m}$ and $\mathbf{n}$:

$$\begin{aligned}\mathbf{Pm}&=\sin^2\theta\cdot\mathbf{m}-\sin\theta\,\cos\theta\cdot\mathbf{n},\\ \mathbf{Pn}&=\cos^2\theta\cdot\mathbf{n}-\sin\theta\,\cos\theta\cdot\mathbf{m}.\end{aligned}\tag{4.5}$$

From (4.4) and (4.5) we derive the following expression for $\mathbf{B}$:

$$\begin{aligned}\mathbf{B}=&(A_u\,\cos^2\theta-A_w\,\sin\theta\,\cos\theta)\cdot\mathbf{n}\,+\\ &+\,(A_w\,\sin^2\theta-A_u\,\sin\theta\,\cos\theta)\cdot\mathbf{m}.\end{aligned}\tag{4.6}$$

Components of the vector $\mathbf{B}$ are present in the equation (4.1). Now we can write this equation in the following form:

$$\sum_{s=1}^{n}\sum_{r=1}^{n}(v\,B^r-A\,N^r)\;\tilde{\nabla}_r\tilde{\nabla}_s A\;P^s_k=0.\tag{4.7}$$

Now let's calculate the derivatives $\tilde{\nabla}_r\tilde{\nabla}_s A$, which are present in the equation (4.7). In order to do it we rewrite formula (4.3) as follows:

$$\tilde{\nabla}_s A=A_u\,\frac{v_s-m_s\,(\mathbf{m}\,|\mathbf{v})}{u}+A_w\,m_s.$$

Here $(\mathbf{m}\,|\mathbf{v})$ is the scalar product of vectors $\mathbf{m}$ and $\mathbf{v}$. Now for second order derivatives $\tilde{\nabla}_r\tilde{\nabla}_s A$ by direct calculations we get

$$\begin{gathered}\tilde{\nabla}_r\tilde{\nabla}_s A=A_{uu}\,\frac{v_r-m_r\,(\mathbf{m}\,|\mathbf{v})}{u}\cdot\frac{v_s-m_s\,(\mathbf{m}\,|\mathbf{v})}{u}\,+\\ +\,A_{uw}\left(m_r\,\frac{v_s-m_s\,(\mathbf{m}\,|\mathbf{v})}{u}+\frac{v_r-m_r\,(\mathbf{m}\,|\mathbf{v})}{u}\,m_s\right)+A_{ww}\,m_r\,m_s.\end{gathered}$$

The above expression is rather complicated. In order to simplify it let's note that $(\mathbf{m}\,|\mathbf{v})=v\,\cos\theta$, $(\mathbf{n}\,|\mathbf{v})=v\,\sin\theta=u$. Hence

$$\frac{v_r-m_r\,(\mathbf{m}\,|\mathbf{v})}{u}=\frac{n_r\,(\mathbf{n}\,|\mathbf{v})}{u}=n_r.$$

When applied to second order derivatives $\tilde{\nabla}_r\tilde{\nabla}_s A$, this equality yields:

$$\tilde{\nabla}_r\tilde{\nabla}_s A=A_{uu}\,n_r\,n_s+A_{uw}\,(m_r\,n_s+n_r\,m_s)+A_{ww}\,m_r\,m_s.$$

Denote $\mathbf{b} = \cos\theta \cdot \mathbf{n} - \sin\theta \cdot \mathbf{m}$. This is unitary vector belonging to the linear span of vectors $\mathbf{v}$ and $\mathbf{m}$ and being perpendicular to $\mathbf{v}$. Using this vector we can rewrite the relationships (4.5) as follows:

$$\mathbf{P}\mathbf{n} = \cos\theta \cdot \mathbf{b}, \qquad\qquad \mathbf{P}\mathbf{m} = -\sin\theta \cdot \mathbf{b}.$$

Now we are able to contract $\tilde{\nabla}_r \tilde{\nabla}_s A$ with the components of projector $\mathbf{P}$:

$$\begin{aligned}\sum_{s=1}^{n} \tilde{\nabla}_r \tilde{\nabla}_s A \; P^s_k &= A_{uu}\,\cos\theta\, n_r\, b_k - \,A_{uw}\,\sin\theta\, n_r\, b_k + \\ &+ A_{uw}\,\cos\theta\, m_r\, b_k - A_{ww}\,\sin\theta\, m_r\, b_k.\end{aligned} \tag{4.8}$$

Let's use (4.8) for to rewrite (4.7) in more explicit form:

$$\begin{gathered}\sum_{r=1}^{n} (A_{uu}\,\cos\theta - A_{uw}\,\sin\theta)\,(v\,B^r - A\,N^r)\,n_r\,b_k + \\ + \sum_{r=1}^{n} (A_{uw}\,\cos\theta - A_{ww}\,\sin\theta)\,(v\,B^r - A\,N^r)\,m_r\,b_k = 0.\end{gathered} \tag{4.9}$$

Vector $\mathbf{b}$ is nonzero. Its components cannot vanish simultaneously. Hence in left hand side of (4.9) we can collect common multiple b_k and cancel it. Sums in r are scalar products. Therefore we get

$$\begin{gathered}(A_{uu}\,\cos\theta - A_{uw}\,\sin\theta)\,(v\,\mathbf{B} - A\,\mathbf{N}\,|\,\mathbf{n}) + \\ + (A_{uw}\,\cos\theta - A_{ww}\,\sin\theta)\,(v\,\mathbf{B} - A\,\mathbf{N}\,|\,\mathbf{m}) = 0.\end{gathered} \tag{4.10}$$

In order to calculate scalar products in (4.10) we use the expansion (4.6) for the vector $\mathbf{B}$ and the expansion $\mathbf{N} = \cos\theta \cdot \mathbf{m} + \sin\theta \cdot \mathbf{n}$ for the vector $\mathbf{N}$:

$$\begin{gathered}(A_{uu}\,\cos\theta - A_{uw}\,\sin\theta)(v\,A_u\,\cos^2\theta - \\ - v\,A_w\,\sin\theta\,\cos\theta - A\,\sin\theta) + (A_{uw}\,\cos\theta - A_{ww}\,\sin\theta)\times \\ \times(v\,A_w\,\sin^2\theta - v\,A_u\,\sin\theta\,\cos\theta - A\,\cos\theta) = 0.\end{gathered} \tag{4.11}$$

Theorem 4.1. *System weak normality equations* (3.3) *written with respect to the function* $A = A(x^1, \ldots, x^n, v^1, \ldots, v^n)$ *in flat Euclidean case* $M = \mathbb{R}^n$ *admits the substitution* $A = A(u, w)$*, where* $u = \sqrt{(v^1)^2 + \ldots + (v^{n-1})^2}$ *and* $w = v^n$*. Thereby it is reduced to the single differential equation* (4.11)*, where* $v = \sqrt{(u)^2 + (w)^2}$ *and* $\theta = \arccos(w/v)$*.*

The equation (4.11) is still rather complicated. In order to simplify it we transform it to polar coordinates v and θ in the plane of variables u and w, i. e. we do the following change of variables

$$u = v\,\sin\theta, \qquad\qquad w = v\,\cos\theta. \tag{4.12}$$

Let's calculate whether how partial derivatives are transformed under the change of variables (4.12). For the first order derivatives we have

$$A_u = A_v\,\sin\theta + A_\theta\,\frac{\cos\theta}{v}, \qquad\qquad A_w = A_v\,\cos\theta - A_\theta\,\frac{\sin\theta}{v}.$$

Then let's calculate second order partial derivatives:

$$\begin{aligned}
A_{uu} &= A_{vv}\,\sin^2\theta + A_{v\theta}\,\frac{\sin 2\theta}{v} + A_{\theta\theta}\,\frac{\cos^2\theta}{v^2} + A_v\,\frac{\cos^2\theta}{v} - A_\theta\,\frac{\sin 2\theta}{v},\\
A_{uw} &= A_{vv}\,\frac{\sin 2\theta}{2} + A_{v\theta}\,\frac{\cos 2\theta}{v} - A_{\theta\theta}\,\frac{\sin 2\theta}{2\,v^2} - A_v\,\frac{\sin 2\theta}{2\,v} - A_\theta\,\frac{\cos 2\theta}{v^2},\\
A_{ww} &= A_{vv}\,\cos^2\theta - A_{v\theta}\,\frac{\sin\theta}{v} + A_{\theta\theta}\,\frac{\sin^2\theta}{v^2} + A_v\,\frac{\sin^2\theta}{v} + A_\theta\,\frac{\sin 2\theta}{v^2}.
\end{aligned}$$

And finally, let's substitute all the above expressions for partial derivatives into the equation (4.11). Thereby the equation (4.11) crucially simplifies and takes the form

$$\frac{A\,A_\theta}{v} + \frac{A_\theta\,A_{\theta\theta}}{v} + A_\theta\,A_v = A\,A_{v\theta}. \tag{4.13}$$

Theorem 4.2. *System weak normality equations* (3.3) *written with respect to the function $A = A(x^1,\dots,x^n,v^1,\dots,v^n)$ in flat Euclidean case $M = \mathbb{R}^n$ admits the substitution $A = A(v,\theta)$, where $v = |\mathbf{v}|$ and $\theta =$* $\arccos(v^n/|\mathbf{v}|)$*. Thereby it is reduced to the single differential equation* (4.13)*.*

Note that the equation (4.13) do not depend on the dimension of the space $M = \mathbb{R}^n$. It holds either in two-dimensional case $n = 2$, and in multidimensional case $n \geqslant 3$ as well. Moreover, this equation is well known in the theory of dynamical systems admitting the normal shift (see paper [10] and thesis [20]). In paper [10] was shown that the equation (4.13) is integrable in quadratures. For this purpose it was first transformed to the following form:

$$\frac{A_\theta}{A}\cdot\left(\frac{A_\theta}{A}\right)'_\theta - v\cdot\left(\frac{A_\theta}{A}\right)'_v + \frac{A_\theta}{A} + \left(\frac{A_\theta}{A}\right)^3 = 0.$$

This form of the equation (4.13) says that we should denote $A_\theta/A = b$. Then for $b = b(v,\theta)$ we obtain quasilinear partial differential equation of the first order:

$$b\,b_\theta - v\,b_v + b + b^3 = 0. \tag{4.14}$$

In the equation (4.14) it is convenient to do another one change of variables, taking $b = \operatorname{cotan} z$, where $z = z(v,\theta)$. This brings the equation (4.14) to the form

$$z_\theta - v\,\frac{\sin z}{\cos z}\,z_v - 1 = 0. \tag{4.15}$$

The equation (4.15) with the use of method of characteristics (see [25]). Characteristics of the equation (4.15) are the solutions of the following system of ODEs:

$$\begin{cases} \dot\theta = 1,\\ \dot z = 1,\\ \dot v = -v\,\dfrac{\sin z}{\cos z}. \end{cases} \tag{4.16}$$

Note, that from (4.16) one can derive the following equalities:

$$\dot{\theta} - \dot{z} = 0, \qquad \frac{\cos z}{v^2}\,\dot{v} + \frac{\sin z}{v}\,\dot{z} = 0.$$

These equalities can be integrated. They mean that the system of equations (4.16) has the pair of first integrals I_1 and I_2:

$$I_1 = \theta - z, \qquad I_2 = \frac{\cos z}{v}. \tag{4.17}$$

General solution of the equation (4.15) is determined by first integrals (4.17) in implicit form by means of functional equation

$$\Phi(I_1, I_2) = 0, \tag{4.18}$$

where Φ is some arbitrary function of two variables.

Formula (4.18) proves the integrability of the equation (4.13) in quadratures. But in general it yields only local solution of this equation. Our goal is to construct global solution of the equation (4.13). It should be a smooth function in direct product of two intervals: closed interval $[0, \pi]$ for the variable θ and open interval $(0, +\infty)$ for the variable v. At the ends of the interval $[0, \pi]$ one should provide the boundary conditions

$$A_\theta\Big|_{\theta=0} = 0, \qquad A_\theta\Big|_{\theta=\pi} = 0. \tag{4.19}$$

They appear because the function $A(v, \theta)$ should correspond to to the function of several variables $A(v^1, \ldots, v^n)$ with axial symmetry, being its restriction to the plane passing through the axis of symmetry.

Now let's proceed with constructing the required solution of the equation (4.13). Note that functional relation of two first integrals I_1 and I_2 written as (4.18) for some cases can be given by the function of one variable $y = f(w)$. Let's write (4.18) as $I_1 = f(I_2)$, i. e. let's consider the following functional equation:

$$\theta - z = f\left(\frac{\cos z}{v}\right). \tag{4.20}$$

For $f(w)$ we choose smooth increasing function with decreasing derivative; we assume that $f(w)$ is defined in semiopen interval $[0, +\infty)$ and $f(0) = \pi/2$. Let's also assume that $f(w)$ is not restricted and grows to infinity as $w \to +\infty$. Its graph is shown on Fig. 4.2. By means of function $y = f(w)$ we construct a family of functions $y = F_{[v]}(z)$ depending on v as parameter:

$$y = F_{[v]}(z) = f\left(\frac{\cos z}{v}\right). \tag{4.21}$$

Graphs of the functions are shown on Fig. 4.3. We use them in order to solve the equation (4.20) graphically. For this purpose we consider a family of straight lines being graphs of the following functions:

$$y = F_{[\theta]}(z) = \theta - z. \tag{4.22}$$

Denote by $v_{\min}$ the value of the derivative $f'(w)$ at the point $w=0$:

$$v_{\min}=f'(w)\bigr|_{w=0}=0.$$

Suppose that the values of parameters v and θ satisfy the following inequalities:

$$v_{\min}<v<+\infty, \qquad\qquad 0\leqslant\theta\leqslant\pi. \tag{4.23}$$

From Fig. 4.2 and Fig. 4.3 we see that for these values of parameters graphs of functions (4.21) and (4.22) intersect at unique point and determine smooth function $z=z(v,\theta)$ satisfying the equation (4.15). For each fixed value of v from the domain determined by inequalities (4.23) the function $z(v,\theta)$ is increasing in θ, it takes all values from closed interval $[-\pi/2,\,+\pi/2]$:

$$z\bigr|_{\theta=0}=-\frac{\pi}{2}, \qquad\qquad z\bigr|_{\theta=\pi}=\frac{\pi}{2}. \tag{4.24}$$

Exactly at one point $\theta_0=\theta_0(v)$ in the interval $[0,\,\pi]$ this function vanishes, while its derivative in θ at this point is equal to unity:

$$z_\theta(v,\theta_0(v))=1. \tag{4.25}$$

Note also that any fixed value of θ the function $z(v,\theta)$ is increasing function in v, though the interval of its values here is more narrow, and it depends on θ.

Function $z(v,\theta)$, which is constructed graphically, determines the function $b=\operatorname{cotan}z=b(v,\theta)$. For the fixed value of v it is decreasing function in θ, but it has an infinite break at the point $\theta_0=\theta_0(v)$. From the equality (4.25) we derive

$$b(v,\theta)=\frac{1}{\theta-\theta_0}+O(1)\quad\text{for}\quad\theta\to\theta_0. \tag{4.26}$$

From (4.24) we obtain that $b(v,\theta)$ vanishes at both ends of interval $[-\pi/2,\,+\pi/2]$:

$$b\bigr|_{\theta=0}=0, \qquad\qquad b\bigr|_{\theta=\pi}=0. \tag{4.27}$$

Using $b(v, \theta)$, now we define the function $A(v, \theta)$ by the following formula

$$A(v, \theta) = \exp\left(\mathrm{v.\,p.}\int\limits_{-\pi/2}^{\theta} b(v, \tau)\,d\tau\right). \tag{4.28}$$

From (4.26) it follows that the function (4.28) vanishes at the point $\theta_0 = \theta_0(v)$, while from (4.27) we derive boundary conditions (4.19) for this function. By construction the function (4.28) is a solution of the equation (4.13). However its domain (4.23) do not embrace the whole phase space. In order to expand its domain one should note that if $A(v, \theta)$ is the solution of the equation (4.13), then the product $C(v) \cdot A(v, \theta)$, where $C = C(v)$ is an arbitrary smooth function, is also the solution of this equation. Let's choose the function $C(v)$ such that

$$\begin{aligned} &C(v) = 0 \quad \text{for} \quad v \leqslant v_{\text{min}}, \\ &C(v) = 1 \quad \text{for} \quad v \geqslant v_0 > v_{\text{min}}. \end{aligned}$$

Now, instead of formula (4.28), we define the function $A(v, \theta)$ by formula

$$A(v, \theta) = C(v) \cdot \exp\left(\mathrm{v.\,p.}\int\limits_{-\pi/2}^{\theta} b(v, \tau)\,d\tau\right). \tag{4.29}$$

Function (4.29) is determined in all phase space, except for those points, where $v = |\mathbf{v}| = 0$. It is the solution of the equation (4.13) and it satisfies boundary conditions (4.19).

5. Theorem on non-coincidence of classes.

The solution of the equation (4.13) constructed by formula (4.29) determines the function $A(v^1, \ldots, v^n)$, which, in turn, determines force field $\mathbf{F}$ of Newtonian dynamical system in $\mathbb{R}^n$ admitting the normal blow-up of points. The construction of this field has functional arbitrariness due to the function $f = f(w)$ in (4.20) and the function $C = C(v)$ in (4.29). Formula (3.6) for scalar field A corresponding to dynamical systems admitting the normal shift of hypersurfaces also has functional arbitrariness due to the function $W = W(x^1, \ldots, x^n, v)$. Let's study which part of this arbitrariness remains if we assume A to be spatially homogeneous function with axially symmetric dependence on $\mathbf{v}$. The dependence on the direction of velocity vector $\mathbf{v}$ in (3.6) is completely determined by the sum

$$B = v\sum_{i=1}^{n}\frac{N^i\,\nabla_i W}{W_v} = \sum_{i=1}^{n}\frac{v^i\,\nabla_i W}{W_v} = \frac{(\mathbf{v}\,|\,\nabla W)}{W_v} \tag{5.1}$$

Changing v by $-v$, we change the sign of this sum, but the value of $v = |\mathbf{v}|$, which is the argument of function W, remains unchanged. For the case $h = 1$ this yields

$$\frac{1}{W_v} = \frac{A(\mathbf{v}) + A(-\mathbf{v})}{2}. \tag{5.2}$$

In spatially homogeneous case right hand side of (5.2) doesn't depend on coordinates $x^1, \ldots, x^n$. Hence for $h = 1$ the quantity $1/W_v$ depends only on v. Let's denote it by $H(v)$. For the quantity B this yields

$$B = \begin{cases} A(\mathbf{v}) - H(v) & \text{for } h = 1, \\ A(\mathbf{v}) & \text{for } h = 0. \end{cases} \tag{5.3}$$

Thus, for both cases the quantity $B = B(\mathbf{v})$ in (5.3) do not depend on coordinates $x^1, \ldots, x^n$. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be unitary vectors directed along coordinate axes. Substituting $\mathbf{v} = v \cdot \mathbf{e}_i$ into the sum (5.1), we get

$$m_i = \frac{\nabla_i W}{W_v} = \frac{B(v \cdot \mathbf{e}_i)}{v}. \tag{5.4}$$

The quantities m_i in (5.4) determine some vector $\mathbf{m}$. Due to (5.3) they do not depend on $x^1, \ldots, x^n$. Hence $\mathbf{m} = \mathbf{m}(v)$. Substituting the quantities (5.4) into the sum (5.1) and further into the formula (3.6) for scalar field A, we get

$$A = \begin{cases} H(v) + (\mathbf{v} \,|\, \mathbf{m}) & \text{for } h = 1, \\ (\mathbf{v} \,|\, \mathbf{m}) & \text{for } h = 0. \end{cases} \tag{5.5}$$

Conclusion: *the condition of spatial homogeneity reduces functional arbitrariness in* (3.6) *to the choice of* $(n + 1)$ *functions of one variable. These are the function* $H(v)$ *and components of the vector* $\mathbf{m}(v)$ *in formula* (5.5).

For the fixed value of $v = |\mathbf{v}|$ the function $A(\mathbf{v})$ in (5.5) possess axial symmetry with the axis directed along the vector $\mathbf{m}(v)$. While for the function (4.29) the axis of symmetry doesn't depend on v, it is directed along the vector $\mathbf{e}_n$. Aiming to express the function (4.29) by formula (5.5), we should choose

$$\mathbf{m}(v) = \frac{C(v)}{v} \cdot \mathbf{e}_n.$$

In variables v and θ in two-dimensional plane of axial section this yields

$$A = \begin{cases} H(v) + C(v) \cos\theta & \text{for } h = 1, \\ C(v) \cos\theta & \text{for } h = 0. \end{cases} \tag{5.6}$$

Similar to (4.29), formula (5.6) contain functional arbitrariness determined by two functions of one variable. But this functional arbitrariness does not affect the dependence of A upon angular variable θ. While the dependence on θ in formula (4.29) is much more complicated. It is determined by the choice of function $f(w)$ in the equation (4.20); in general case it is not reduced to trigonometric function $y = \cos\theta$. Thus, formula (4.29) determines some solution of the system of weak normality equations (1.4), which is not the solution for additional normality equations (1.5). Therefore we can formulate the main result of present paper.

Theorem 5.1. *In multidimensional case* $n \geqslant 3$ *class of Newtonian dynamical systems admitting normal blow-up of points is the expansion of the class of systems admitting the normal shift of hypersurfaces, and it doesn't coincide with the latter one.*

Note that in thesis [20] for two-dimensional case $n = 2$ Andrey Boldin has constructed another (more explicit) solution for the equation (4.13). It is expressed through elliptic functions. However, it is local and it doesn't satisfy the conditions (4.19). Therefore this solution cannot be used in multidimensional case $n \geqslant 3$ for proving theorem 5.1.

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