FIRST PROBLEM OF GLOBALIZATION

in the theory of dynamical systems admitting the normal shift of hypersurfaces.

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Abstract. Formula for the force field of Newtonian dynamical systems admitting the normal shift of hypersurfaces in Riemannian manifolds is considered. Problem of globalization for geometric structures associated with this formula is studied.

1. Introduction.

Let M be a Riemannian manifold of the dimension n. Newtonian dynamical system in M is determined in local coordinates by n ordinary differential equations

$$\ddot{x}^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j=F^k(x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n),\tag{1.1}$$

where $k=1,\ldots,n$. Here $\Gamma^k_{ij}=\Gamma^k_{ij}(x^1,\ldots,x^n)$ are components of metric connection, while F^k are components of force vector $\mathbf{F}$. They determine force field of dynamical system (1.1). Let S be a hypersurface in M and let $p\in M$. Consider the following initial data for the system of equations (1.1):

$$x^k\,\Big|_{t=0}=x^k(p),\qquad\qquad \dot{x}^k\,\Big|_{t=0}=\nu(p)\cdot n^k(p).\tag{1.2}$$

Here $n^k(p)$ are components of unitary normal vector $\mathbf{n}$ to S at the point p. Initial data (1.2) determine the trajectory coming out from the point p in the direction of normal vector $\mathbf{n}(p)$. The quantity $\nu(p)$ in (1.2) is introduced to determine modulus of initial velocity for such trajectory.

Let's choose and fix some point $p_0\in S$, then consider a smooth function $\nu(p)$ defined in some neighborhood of the point p_0. Let

$$\nu(p_0)=\nu_0\neq 0.\tag{1.3}$$

Then in some (possibly smaller) neighborhood of p_0 this function $\nu(p)$ does not vanish and take values of some definite sign. Upon restricting $\nu(p)$ to such neighborhood we use it to determine initial velocity in (1.2). As a result we obtain a family of trajectories of dynamical system (1.1). Displacement of points of hypersurface S along these trajectories determines shift maps $f_t\colon S\to S_t$. Relying upon theorem on smooth dependence on initial data for the system of ODE's (see [1] and [2]), we can assume that shift maps $f_t\colon S'\to S'_t$ are defined in some neighborhood

S' of the point p_0 on S for all values of parameter t in some interval $(-\varepsilon, +\varepsilon)$ on real axis $\mathbb{R}$. At the expense of further restriction of the interval $(-\varepsilon, +\varepsilon)$ one can make maps $f_t : S' \to S'_t$ to be diffeomorphisms and make their images S'_t to be smooth hypersurfaces, disjoint union of which fills some neighborhood of the point p_0 in M. Moreover, at the expense of restriction of the neighborhood S' and the range of parameter t one can reach the situation in which shift trajectories would cross hypersurfaces S_t transversally at all points of mutual intersection. For such a case we state the following definitions.

Definition 1.1. Shift $f_t\colon S' \to S'_t$ of some part S' of hypersurface S along trajectories of Newtonian dynamical system (1.1) is called **a normal shift** if all hypersurfaces S'_t arising in the process of shifting are perpendicular to the trajectories of this shift.

Definition 1.2. Newtonian dynamical system (1.1) with force field $\mathbf{F}$ ia called **a system admitting normal shift** in strong sense if for any hypersurface S in M, for any point $p_0 \in S$, and for any real number $\nu_0 \neq 0$ one can find a neighborhood S' of the point p_0 on S, and a smooth function $\nu(p)$, which do not vanish in S' and which is normalized by the condition (1.3), such that the shift $f_t\colon S' \to S'_t$ defined by this function is a normal shift in the sense of definition 1.1.

First we used the definition without normalizing condition (1.3) for the function $\nu(p)$. Such definition was called the normality condition. Definition 1.2 strengthens this condition making it more restrictive with respect to the choice of force field $\mathbf{F}$ of dynamical system (1.1). Therefore it is called strong normality condition.

Definitions 1.1 and 1.2 form the base of the theory of dynamical systems admitting the normal shift. This theory was constructed in papers [3–18]. The results of these papers were used in preparing theses [19] and [20].

As it was shown in [19], Newtonian dynamical systems admitting the normal shift of hypersurfaces in Riemannian manifolds of the dimension $n \geqslant 3$ can be effectively described. Force field of such systems is given by explicit formula

$$F_k = \frac{h(W)\,N_k}{W_v} - v\sum_{i=1}^{n}\frac{\nabla_i W}{W_v}\left(2\,N^i\,N_k - \delta^i_k\right), \tag{1.4}$$

which contains one arbitrary function of one variable $h = h(w)$ and one arbitrary function of $(n+1)$ variables $W = W(x^1, \ldots, x^n, v)$ restricted by natural condition

$$W_v = \frac{\partial W}{\partial v} \neq 0. \tag{1.5}$$

Components of gradient ∇W in formula (1.4) are partial derivatives

$$\nabla_i W = \frac{\partial W}{\partial x^i}. \tag{1.6}$$

Here N^i and N_k are components of unitary vector $\mathbf{N}$ directed along velocity vector:

$$N^i = \frac{v^i}{|\mathbf{v}|}, \qquad\qquad N_k = \frac{v_k}{|\mathbf{v}|}. \tag{1.7}$$

Upon substituting (1.5), (1.6), and (1.7) into formula (1.4) independent variable v should be replaced by modulus of velocity vector: $v = |\mathbf{v}|$.

2. First problem of globalization.

If we fix a pair of functions (h, W), then formula (1.4) uniquely determines the force field $\mathbf{F}$ of Newtonian dynamical system (1.1). However, fixing force field (1.4), we cannot determine uniquely the corresponding pair of functions (h, W). In particular, global force field $\mathbf{F}$ can be represented by different pairs of functions in different local maps forming an atlas of the manifold M. Thus, we meet a problem of describing global geometric structures associated with such a way of defining force field $\mathbf{F}$. This problem was formulated by S. E. Kozlov and Yu. R. Romanovsky when I was reporting my thesis [19] in the seminar of N. Yu. Netsvetaev at Saint-Petersburg department of Steklov mathematical Institute.

There is another problem of globalization concerning the process of normal shift of some particular hypersurface S along trajectories of dynamical system (1.4). We shall call it second problem of globalization, though historically it arises earlier than first one. Second problem was formulated by A. S. Mishchenko when I was reporting thesis [19] in the seminar of the Chair of higher geometry and topology at Moscow State University. It's expedient to deal with second problem of globalization only upon solving first one. Therefore we shall consider it in separate paper.

3. Scalar ansatz and gauge transformations.

Let's consider the projection of force vector (1.4) onto the direction of the velocity vector. This projection can be calculated as a scalar product of vectors $\mathbf{F}$ and $\mathbf{N}$:

$$A = (\mathbf{F} \,|\, \mathbf{N}) = \sum_{k=1}^{n} F_k\, N^k. \tag{3.1}$$

Substituting (1.4) into (3.1), we get the following expression for A:

$$A = \frac{h(W)}{W_v} - \frac{v}{W_v}\,(\nabla W \,|\, \mathbf{N}). \tag{3.2}$$

A very important point is that force fields (1.4) can be recovered by corresponding scalar fields A. This recovery is given by formula, which is called **a scalar ansatz**:

$$F_k = A\, N_k - |\mathbf{v}| \sum_{i=1}^{n} P_k^i\, \tilde{\nabla}_i A. \tag{3.3}$$

Here $P_k^i = \delta_k^i - N^i\, N_k$ are components of orthogonal projector onto the hyperplane perpendicular to the vector $\mathbf{v}$. By $\tilde{\nabla}_i A$ we denote partial derivatives $\partial A/\partial v^i$. Scalar ansatz (3.3) was found in [18]. In thesis [19] it was used in deriving formula (1.4).

Formulas (3.1) and (3.3) set up one-to-one correspondence of vector fields $\mathbf{F}$ of the form (1.4) with scalar fields A of the form (3.2). Formula (3.2) uniquely determines the scalar field A by the pair of functions (h, W). But inverse correspondence is not univalent. This is confirmed by the existence of gauge transformations

$$\begin{aligned} &W(x^1, \ldots, x^n, v) \longrightarrow \rho(W(x^1, \ldots, x^n, v)),\\ &h(w) \longrightarrow h(\rho^{-1}(w)) \cdot \rho'(\rho^{-1}(w)) \end{aligned} \tag{3.4}$$

with one arbitrary function of one variable $\rho = \rho(w)$. Transformations (3.4) change h and W, but they don't change the scalar field A.

Let's investigate which part of information on h and W can be recovered by A. Suppose that the point $p \in M$ is fixed. The dependence of A on the direction of velocity vector at the point p is determined by the term $\mathbf{N}$ in scalar product $(\nabla W\,|\,\mathbf{N})$. Therefore if we change $\mathbf{v}$ by $-\mathbf{v}$, first summand in (3.2) remains unchanged, while second summand changes in sign. Hence

$$\frac{h(W)}{W_v} = \frac{A(\mathbf{v}) + A(-\mathbf{v})}{2}, \qquad\qquad \frac{(\nabla W\,|\,\mathbf{N})}{W_v} = \frac{A(-\mathbf{v}) - A(\mathbf{v})}{2\,|\mathbf{v}|}.$$

Keeping the value of $v = |\mathbf{v}|$ unchanged, we can change the direction of vector $\mathbf{N}$. This allows us to determine each component of vector $\nabla W/W_v$. Thus by A one can recover the scalar $h(W)/W_v$ and the vector $\nabla W/W_v$.

Let p be a point of the manifold M. Suppose that the field A is determined by two pairs of functions (h, W) and $(\tilde{h}, \tilde{W})$ in some neighborhood of p. Then

$$\frac{h(W)}{W_v} = \frac{\tilde{h}(\tilde{W})}{\tilde{W}_v}, \qquad\qquad \frac{\nabla W}{W_v} = \frac{\nabla \tilde{W}}{\tilde{W}_v}. \tag{3.5}$$

Being more accurate, one should note that functions W and $\tilde{W}$ are determined in some domain U in Cartesian product $M \times \mathbb{R}^{+}$, where by $\mathbb{R}^{+}$ we denote the set of positive numbers. Second relationship in (3.5) means that complete gradients of these two functions in U are collinear:

$$\operatorname{grad} W \parallel \operatorname{grad} \tilde{W}. \tag{3.6}$$

The conditions $W_v \neq 0$ and $\tilde{W}_v \neq 0$ mean that both gradients in (3.6) are nonzero. This situation is described by the following lemma.

Lemma 3.1. *If gradient of one smooth function $f(x^1, \ldots, x^n)$ is nonzero in some domain $U \subset \mathbb{R}^n$ and gradient of another smooth function $g(x^1, \ldots, x^n)$ is collinear to it in U, then functions f and g are functionally dependent in U. This means that for each point $p \in U$ one can find some neighborhood $O(p)$ and a smooth function of one variable $\rho(y)$ such that $g = \rho \circ f$ in $O(p)$.*

Lemma 3.1 is purely local fact following from the theory of implicit functions (see [21] and [22]). But, despite to this, it is worth while, since it describes the structure of non-uniqueness in inverse correspondence for $(h, W) \to A$.

Theorem 3.1. *Suppose that two pairs of functions (h, W) and $(\tilde{h}, \tilde{W})$ defined in some domain $U \subset M \times \mathbb{R}^{+}$ determine the same force field $\mathbf{F}$ of the form* (1.4)*. Then for each point $q \in U$ one can find some neighborhood $O(q)$ and a smooth function of one variable $\rho(y)$ such that (h, W) and $(\tilde{h}, \tilde{W})$ are bound by the gauge transformation* (3.4) *in $O(q)$.*

4. Projectivization of cotangent bundle.

Denote by $\mathcal{M}$ the Cartesian product of manifolds $M \times \mathbb{R}^{+}$. Let $T^{*}\mathcal{M}$ be cotangent bundle for $\mathcal{M}$. If we take the pair of functions h and W, which determine force field $\mathbf{F}$ of the form (1.4), then we see that derivatives

$$\nabla_1 W,\ \ \nabla_2,\ \ \ldots\ ,\ \ \nabla_n W,\ \ W_w$$

constitute the set of components of differential 1-form $\boldsymbol{\omega} = dW$. The domain, where this 1-form is defined, shouldn't coincide with the whole manifold $\mathcal{M}$. Hence we have a local section of the bundle $\mathcal{T}^*\mathcal{M}$. Second summand in formula (1.4) contain not the components of differential form $\boldsymbol{\omega}$ by themselves, but the quotients

$$b_i = -\frac{\nabla_i W}{W_v} = -\frac{\omega_i}{\omega_{n+1}}. \tag{4.1}$$

Let's factorize fibers of cotangent bundle $\mathcal{T}^*\mathcal{M}$ with respect to the action of multiplicative group of real numbers $\boldsymbol{\omega} \to \alpha\cdot\boldsymbol{\omega}$. In other words, we replace linear spaces $\mathcal{T}^*_q(\mathcal{M})$ over the points $q \in \mathcal{M}$ by corresponding projective spaces $\mathcal{P}^*_q(\mathcal{M})$. As a result we get projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$. This is locally trivial bundle $\mathcal{P}^*\mathcal{M}$, standard fiber of which is n-dimensional projective space $\mathbb{RP}^n$ (see definitions in books [23] or [24]).

Fibers of projective bundle $\mathcal{P}^*\mathcal{M}$ are parameterized by components of covectors $\boldsymbol{\omega}$ taken up to an arbitrary numeric factor:

$$\alpha\cdot\omega_1,\ \ \alpha\cdot\omega_2,\ \ \ldots\ ,\ \ \alpha\cdot\omega_n,\ \ \alpha\cdot\omega_{n+1}. \tag{4.2}$$

If $\omega_{n+1} \neq 0$, then we can choose numeric factor $\alpha = 1/\omega_{n+1}$. Then from (4.2) we obtain $-b_1,\ -b_2,\ \ldots\ ,\ -b_n,\ 1$. This means that quantities b_i from (4.1) are the local coordinates in one of affine maps in projective fiber of the bundle $\mathcal{P}^*\mathcal{M}$. Let's turn back to the problem of globalization formulated in section 2. From formulas (3.5) we derive the following proposition.

Lemma 4.1. *Each force field $\mathbf{F}$ of the form* (1.4) *determines some global section σ of projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$.*

But not all global sections of the bundle $\mathcal{P}^*\mathcal{M}$ can be obtained in this way. There is a restriction. The matter is that on the level of cotangent bundle $\mathcal{T}^*\mathcal{M}$ our section σ in lemma 4.1 is represented by closed differential forms $\boldsymbol{\omega}$, which possibly may be defined only locally. Let's study how this fact is reflected on the level of projective bundle $\mathcal{P}^*\mathcal{M}$? In order to recover components of the form $\boldsymbol{\omega}$ in (4.2) by $b_1,\ b_2,\ \ldots\ ,\ b_n$ we should take a proper factor $\varphi = \omega_{n+1}$. Then

$$\omega_i = \begin{cases} -b_i\,\varphi & \text{for}\ \ i = 1,\ \ldots\ ,\ n,\\ \ \ \varphi & \text{for}\ \ i = n+1.\end{cases} \tag{4.3}$$

Closedness of the form $\boldsymbol{\omega}$ is written in form of the following relationships:

$$\frac{\partial\omega_i}{\partial x^j} - \frac{\partial\omega_j}{\partial x^i} = 0. \tag{4.4}$$

Here we denote $v = x^{n+1}$. This is natural, since $\mathcal{M} = M \times \mathbb{R}^+$. Substituting (4.3) into the relationships (4.4), for $i \leqslant n$ and $j \leqslant n$ we get

$$\frac{\partial b_i}{\partial x^j}\,\varphi + \frac{\partial\varphi}{\partial x^j}\,b_i = \frac{\partial b_j}{\partial x^i}\,\varphi + \frac{\partial\varphi}{\partial x^i}\,b_j. \tag{4.5}$$

From the same relationships (4.4) for the case $i \leqslant n$ and $j = n+1$ we derive

$$\frac{\partial\varphi}{\partial x^i} = -\frac{\partial b_i}{\partial v}\,\varphi - \frac{\partial\varphi}{\partial v}\,b_i. \tag{4.6}$$

Now let's substitute the derivatives $\partial\varphi/\partial x^i$ and $\partial\varphi/\partial x^j$ calculated according to (4.6) into the equations (4.5). As a result we obtain the equations free of φ:

$$\left(\frac{\partial}{\partial x^j} + b_j\,\frac{\partial}{\partial v}\right) b_i = \left(\frac{\partial}{\partial x^i} + b_i\,\frac{\partial}{\partial v}\right) b_j. \tag{4.7}$$

Note that the equations (4.7) are already known (see [19], Chapter VII, § 4). However, the geometric interpretation of quantities b_i in [19] was quite different.

Lemma 4.2. *Each force field* $\mathbf{F}$ *of the form* (1.4) *determines some global section* σ *of the bundle* $\mathcal{P}^*\mathcal{M}$ *with components satisfying the equations* (4.7).

The equations (4.7) above arise as necessary condition for the existence of closed differential 1-form $\boldsymbol{\omega}$ corresponding to the section of projective bundle $\mathcal{P}^*\mathcal{M}$. But they are sufficient condition for the existence of such 1-form as well (certainly, only for local existence). Let's prove this fact. In order to integrate the equations (4.6) we use the auxiliary system of Pfaff equations

$$\frac{\partial V}{\partial x^i} = b_i(x^1, \ldots, x^n, V), \text{ where } i = 1, \ldots, n. \tag{4.8}$$

The relationships (4.7) are exactly the compatibility conditions for the equations (4.8). Remember that variables $x^1, \ldots, x^n, v$ are local coordinates in the manifold $\mathcal{M} = M \times \mathbb{R}^+$, while first n of them are local coordinates in M. Let's fix some point $p_0 \in M$. Without loss of generality we can assume that local coordinates of the point p_0 are equal to zero. For compatible system of Pfaff equations (4.8) we set up the following Cauchy problem at the point p_0:

$$V\Bigr|_{x^1 = \ldots = x^n = 0} = w. \tag{4.9}$$

Thereby we take $w > 0$. The solution of Cauchy problem (4.9) for the equations (4.8) does exist and it is unique in some neighborhood of the point p_0. It is smooth function of coordinates $x^1, \ldots, x^n$ and parameter w:

$$v = V(x^1, \ldots, x^n, w). \tag{4.10}$$

For $x^1 = \ldots = x^n = 0$ due to (4.9) we have $V(0, \ldots, 0, w) = w$. Therefore

$$\frac{\partial V}{\partial w}\Bigr|_{x^1 = \ldots = x^n = 0} = 1. \tag{4.11}$$

Let's consider the set of point $q = (p_0, v)$ in $\mathcal{M}$. They form a linear ruling in Cartesian product $\mathcal{M} = M \times \mathbb{R}^+$. Let's denote it by $l_0 = l(p_0)$. The equality (4.11) means that for any point $q_0 \in l_0$ there is some neighborhood of this point, where we have local coordinates $y^1, \ldots, y^n$, w related to initial coordinates $x^1, \ldots, x^n$, v as

$$\begin{cases} x^i = y^i \quad \text{for} \quad i = 1, \ldots, n, \\ v = V(y^1, \ldots, y^n, w). \end{cases} \tag{4.12}$$

Back transfer to initial coordinates is determined by a function $W(x^1, \ldots, x^n, v)$:

$$\begin{cases} y^i = x^i \quad \text{for} \quad i = 1, \ldots, n, \\ w = W(x^1, \ldots, x^n, v). \end{cases} \tag{4.13}$$

Function $W(x^1, \ldots, x^n, v)$ is calculated implicitly from the relationship (4.10) considered as the equation with respect to w.

Let's use (4.12) and (4.13) for to simplify the equations (4.6). Instead of function $\varphi(x^1, \ldots, x^n, v)$ in these equations we introduce another function

$$\psi(y^1, \ldots, y^n, w) = \varphi(y^1, \ldots, y^n, V(y^1, \ldots, y^n, w)). \tag{4.14}$$

The equations (4.6) are reduced to the following equations for the function (4.14):

$$\frac{\partial \psi}{\partial y^i} = -B_i\, \psi. \tag{4.15}$$

The quantities B_i are expressed through the derivatives of the function V:

$$B_i = \frac{1}{Z}\,\frac{\partial Z}{\partial y^i}, \ \text{ where } \ Z = \frac{\partial V}{\partial w}. \tag{4.16}$$

It is easy to see that (4.15) is a system of Pfaff equations being compatible due to (4.16). Moreover, it is explicitly integrable. General solution of the equations (4.15) is given by the following explicit formula:

$$\psi = \frac{C(w)}{Z(y^1, \ldots, y^n, w)}, \tag{4.17}$$

Here $C(w)$ is an arbitrary smooth function of one variable. Now let's use the local invertibility of the relationship (4.14):

$$\varphi(x^1, \ldots, x^n) = \psi(x^1, \ldots, x^n, W(x^1, \ldots, x^n, v)). \tag{4.18}$$

From (4.17) and (4.18) we derive general solution for the system of equations (4.6):

$$\varphi = C(W) \cdot W_v, \ \text{ where } \ W_v = \frac{\partial W}{\partial v}.$$

Similar to force field $\mathbf{F}$ in formula (1.4), it is determined by two functions $C(w)$ and $W(x^1, \ldots, x^n, v)$, the latter one satisfying the condition (1.5). This coincidence is not occasional. From (4.8) and from (4.18) for b_i we derive the relationship

$$b_i = -\frac{\nabla_i W}{W_v}, \tag{4.19}$$

being of the same form as (4.1). Certainly, function W in (4.19) obtained by inverting local change of variables (4.12) shouldn't coincide with initial function W in (4.1). The relation of these two functions is characterized by theorem 3.1 (see above). The calculations we have just made result in the following lemma, sharpening lemma 4.2.

Lemma 4.3. *The relationships* (4.7) *form necessary and sufficient condition for global section σ of projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$ given by its components $b_1, \ldots, b_n$ in local coordinates to be related to some force field $\mathbf{F}$ of the form* (1.4).

5. Involutive distributions.

Relying upon lemmas 4.2 and 4.3, now we consider some global section σ of projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$ that satisfies the equations (4.7). Let's reveal invariant meaning of these equations. In order to do it we consider vector fields

$$\mathbf{L}_i = \frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial v}, \quad \text{where} \quad i = 1, \ldots, n, \tag{5.1}$$

and some differential 1-form $\boldsymbol{\omega}$ with components (4.3). Values of vector fields (5.1) are linearly independent at each point of the domain, where they are defined. These values belong to the kernel of the form $\boldsymbol{\omega}$ for any choice of function φ in (4.3). The equations (4.7) are exactly the commutation conditions for vector fields (5.1):

$$[\mathbf{L}_i, \mathbf{L}_j] = 0. \tag{5.2}$$

Note that global sections of the bundle $\mathcal{P}^*\mathcal{M}$ are in one-to-one correspondence with n-dimensional distributions in the manifold $\mathcal{M}$, the dimension of which is equal to $n+1$. Indeed, in the neighborhood of each point $q \in \mathcal{M}$ the section σ of the bundle $\mathcal{P}^*\mathcal{M}$ is determined by some 1-form $\boldsymbol{\omega}$ fixed up to a scalar factor φ. But the kernel $U = \operatorname{Ker} \boldsymbol{\omega} \subset T_q(\mathcal{M})$ does not depend on this factor. Therefore we have global n-dimensional distribution $U = \operatorname{Ker} \sigma$. And conversely, if n-dimensional distribution U is given, then in the neighborhood of each point $q \in \mathcal{M}$ we have 1-form $\boldsymbol{\omega}$ such that $U = \operatorname{Ker} \boldsymbol{\omega}$. Form $\boldsymbol{\omega}$ defines local section of the bundle $\mathcal{P}^*\mathcal{M}$ in the neighborhood of the point q. The fact that form $\boldsymbol{\omega}$ is determined by U uniquely up to a scalar factor means that local sections of the bundle $\mathcal{P}^*\mathcal{M}$ are glued into one global section σ of this bundle.

The condition (5.2) means that the distribution $U = \operatorname{Ker} \sigma$ is involutive (see [24]). In this case in the neighborhood of each point $q \in \mathcal{M}$ the section σ can be represented by closed 1-form $\boldsymbol{\omega}$. Let's introduce the following terminology.

Definition 5.1. The section σ of projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$ is called **closed** if corresponding distribution $U = \operatorname{Ker} \sigma$ in $\mathcal{M}$ is involutive.

For the sections σ related to force fields (1.4) the manifold $\mathcal{M}$ is a Cartesian product $M \times \mathbb{R}^+$. In this case we have a restriction expressed by the condition (1.5). It can be written as $\omega_{n+1} \neq 0$. Therefore we have the following lemma.

Lemma 5.1. *Global section σ of projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$ with base manifold $\mathcal{M} = M \times \mathbb{R}^+$ satisfies the condition* (1.5) *if and only if corresponding distribution $U = \operatorname{Ker} \sigma$ is transversal to linear rulings of cylindric manifold $M \times \mathbb{R}^+$.*

For the sake of brevity we shall write the condition stated in lemma 5.1 as

$$\operatorname{Ker}\sigma = U \nparallel \mathbb{R}^{+}. \tag{5.3}$$

Results of lemmas 4.2, 4.3, and 5.1 can be summarized in the following theorem.

Theorem 5.1. *Each force field* $\mathbf{F}$ *of the form* (1.4) *determines some closed global section* σ *of projectivized cotangent bundle* $\mathcal{P}^{*}\mathcal{M}$ *over base manifold* $\mathcal{M} = M \times \mathbb{R}^{+}$ *such that it satisfies additional condition* (5.3). *And conversely, each such section of the bundle* $\mathcal{P}^{*}\mathcal{M}$ *corresponds to some force field* $\mathbf{F}$ *of the form* (1.4).

6. Normalizing vector fields.

Up to now we studied only the second summand in formula (3.2). And we have found that it gives rise to geometric structures mentioned in theorem 5.1. Now let's consider first summand in (3.2). Denote by a the following quotient:

$$a = \frac{h(W)}{W_v}. \tag{6.1}$$

Function $a = a(x^1, \ldots, x^n, v)$ in (6.1) is invariant with respect to gauge transformations (3.4). Due to the relationships (3.5) it can be continued through the region of overlapping of two maps, in which force field $\mathbf{F}$ is determined by two different pairs of functions (h, W) and $(\tilde{h}, \tilde{W})$. But, despite to this fact, it would be wrong to interpret a as scalar field on $\mathcal{M}$. The matter is that in local coordinates, for which formula (1.4) holds, the variable v plays exclusive role related with the expansion of $\mathcal{M}$ into Cartesian product $M \times \mathbb{R}^{+}$. Due to this reason we derive differential equations for the function $a = a(x^1, \ldots, x^n, v)$. Let's apply one of the differential operators (5.1) to a. This yields

$$\left(\frac{\partial}{\partial x^i} + b_i\,\frac{\partial}{\partial v}\right) a = h'(W)\,\frac{\nabla_i W + b_i\,W_v}{W_v} - \frac{h(W)}{W_v}\,\frac{\nabla_i W_v + b_i\,W_{vv}}{W_v}.$$

If we take into account (4.1), then this relationship can be brought to

$$\left(\frac{\partial}{\partial x^i} + b_i\,\frac{\partial}{\partial v}\right) a = \frac{\partial b_i}{\partial v}\,a. \tag{6.2}$$

Note that the equations (6.2) are also already known (see [19], Chapter VII, § 4). Formula (1.4) was derived as a result of integrating the equations (4.7) and (6.2). Following [19], we append vector fields (5.1) by the following one

$$\mathbf{L}_{n+1} = a\,\frac{\partial}{\partial v}. \tag{6.3}$$

The equations (6.2) are equivalent to the following commutational relationships:

$$[\mathbf{L}_i, \mathbf{L}_{n+1}] = 0, \ \ \text{where} \ \ i = 1, \ldots, n. \tag{6.4}$$

Now let's give invariant (coordianteless) interpretation for the relationships (6.4). Vector fields (5.1) by themselves have no invariant interpretation. But their linear span at each point q coincides with n-dimensional subspace $U_q \subset \mathcal{T}_q(\mathcal{M})$ defined by distribution $U = \operatorname{Ker}\sigma$. Consider one-dimensional factor-spaces

$$\Omega_q = \mathcal{T}_q(\mathcal{M})/U_q. \tag{6.5}$$

They are glued into one-dimensional vector bundle $\Omega\mathcal{M}$ over the base manifold $\mathcal{M} = M \times \mathbb{R}^+$. Let $x^1, \ldots, x^n, v$ be local coordinates in $\mathcal{M}$ not necessarily related to the structure of Cartesian product $M \times \mathbb{R}^+$, but such that vector $\partial/\partial v$ is transversal to U_q. Then vectors (5.1) form the base in subspace U_q, while elements of factor-space (6.5) are cosets of subspace U_q represented by vectors (6.3):

$$a = \operatorname{Cl}_U(a \cdot \partial/\partial v).$$

Sections of one dimensional vector bundle $\Omega\mathcal{M}$ in such local coordinates can be associated with functions $a(x^1, \ldots, x^n, v)$ or with vector fields

$$\mathbf{X} = a(x^1, \ldots, x^n, v) \cdot \frac{\partial}{\partial v}.$$

Definition 6.1. Vector field $\mathbf{X}$ is called **normalizing field** for smooth distribution U if for any vector field $\mathbf{Y}$ belonging to U the commutator $[\mathbf{X}, \mathbf{Y}]$ is also in U.

Theorem 6.1. *Smooth distribution U of codimension 1 in the manifold $\mathcal{M}$ possesses nonzero normalizing vector field transversal to U in the neighborhood of the point $q \in \mathcal{M}$ if and only if it is involutive in the neighborhood of this point.*

Proof. Without loss of generality we can take $\dim\mathcal{M} = n+1$ and $\dim U = n$. Since $\mathbf{X} \neq 0$, we can choose local coordinates $x^1, \ldots, x^n, v$ in $\mathcal{M}$ such that $\mathbf{X} = \partial/\partial v$. And since $\mathbf{X} \nparallel U$, the base in U can be formed by vector fields $\mathbf{L}_i$ of the form (5.1). Let's write the condition that $\mathbf{X}$ is normalizing vector field for the distribution U. For this purpose we calculate the commutator $[\mathbf{X}, \mathbf{L}_i]$:

$$[\mathbf{X}, \mathbf{L}_i] = -\frac{\partial b_i}{\partial v} \cdot \frac{\partial}{\partial v} = -\frac{\partial b_i}{\partial v} \cdot \mathbf{X}.$$

Recall that vector field $\mathbf{X}$ is transversal to U. Therefore from $[\mathbf{X}, \mathbf{L}_i] \in U$ we get

$$\frac{\partial b_i}{\partial v} = 0. \tag{6.6}$$

If we take into account (6.6), we find that the equations (4.7) turn to identities. But we know, that they are equivalent to commutational relationships (5.2). Hence the distribution U is involutive. Theorem 6.1 is proved. $\square$

Let $\mathbf{X}$ be normalizing vector field for involutive distribution U and let $\mathbf{Y}$ be in U. Then $\mathbf{X} + \mathbf{Y}$ is also normalizing vector field for U. Thus we can define normalizing sections of the bundle $\Omega\mathcal{M}$ obtained by factorization of tangent bundle $\mathcal{T}\mathcal{M}$ with respect to U.

Definition 6.2. Section s of factor-bundle $\Omega\mathcal{M} = \mathcal{T}\mathcal{M}/U$ is called **normalizing section** if in the neighborhood of each point $q \in \mathcal{M}$ it is represented by some normalizing vector field for the distribution U.

Now we can formulate the main result of this paper, characterizing global geometric structures associated with formula (1.4) for the force field $\mathbf{F}$. It follows from all what was said above.

Theorem 6.2. *Defining Newtonian dynamical system admitting the normal shift in Riemannian manifold M is equivalent to defining closed global section σ for projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$ with base $\mathcal{M} = M\times\mathbb{R}^+$, satisfying the condition $\operatorname{Ker}\sigma \nparallel \mathbb{R}^+$, and to defining normalizing global section s for one-dimensional factor-bundle $\Omega\mathcal{M} = \mathcal{T}\mathcal{M}/U$, where $U = \operatorname{Ker}\sigma$.*

7. Integration of geometric structures.

Formulating theorem 6.2, we have made a step forward in understanding global geometry associated with formula (1.4) for the force field $\mathbf{F}$. But as far as the effectiveness of calculations in coordinates is concerned, we came back to situation, in which scalar field A is expressed by formula

$$A = a + \sum_{i=1}^{n} b_i\, v^i, \tag{7.1}$$

where quantities a and $b_1, \ldots, b_n$ should be found as solutions of the equations (4.7) and (6.2). Formula (3.2) was more effective. Therefore we have a natural question: *can one integrate the equations* (4.7) *and* (6.2) *globally and find the pair of functions* (h, W) *that would define scalar field A by formula* (3.2) *and force field* $\mathbf{F}$ *by formula* (1.4) *on the whole manifold $\mathcal{M}$?*

According to theorem 6.2, each force field $\mathbf{F}$ of Newtonian dynamical system admitting the normal shift is related with some unique closed global section σ of the bundle $\mathcal{P}^*\mathcal{M}$. If such section is generated by closed global section $\boldsymbol{\omega}$ of cotangent bundle $\mathcal{T}^*\mathcal{M}$, then we can construct the function $W = W(q)$ on $\mathcal{M}$ by integrating 1-form $\boldsymbol{\omega}$ along the curve binding the point q with some fixed point q_0 on $\mathcal{M}$:

$$W(q) = \int_{q_0}^{q} \boldsymbol{\omega}. \tag{7.2}$$

Formula (7.2) yields the function $W(q)$ that possibly can be multivalued, since first homotopic group $\pi_1(\mathcal{M})$ of the manifold $\mathcal{M}$ can be non-trivial. This ambiguity is admissible. It can be eliminated by passing to universal covering of $\mathcal{M}$.

Apart from $\boldsymbol{\sigma}$, each force field $\mathbf{F}$ of Newtonian dynamical system admitting the normal shift determines some section of factor-bundle $\Omega\mathcal{M} = \mathcal{T}\mathcal{M}/U$, where $U = \operatorname{Ker}\sigma$. Let's use the structure of Cartesian product $M \times \mathbb{R}^+$ of $\mathcal{M}$. This yields the vector field $\mathbf{E}$ directed along linear rulings in $\mathcal{M}$. If $x^1, \ldots, x^n$ are local coordinates in M and if v is natural variable ranging in positive semiaxis $\mathbb{R}^+$, then in local coordinates $x^1, \ldots, x^n, v$ in $\mathcal{M}$ this field is given by formula $\mathbf{E} = \partial/\partial v$. According to theorem 6.2, we have $U = \operatorname{Ker}\sigma \nparallel \mathbb{R}^+$, i. e. $U \nparallel \mathbf{E}$. Therefore the section s of the bundle $\Omega\mathcal{M}$ can be represented by the vector field

$$\mathbf{X} = a \cdot \mathbf{E}. \tag{7.3}$$

This representation is unique, coefficient a in it is a scalar field (a function) on $\mathcal{M}$. The condition that s is normalizing section with respect to U in local coordinates

$x^1, \ldots, x^n, v$ is expressed by the equations (6.2) for the function a. It's easy to check that if a satisfies the equations (6.2), then the function $\varphi = 1/a$ satisfies the equations (4.6). Hence if section s is nonzero at all points $q \in \mathcal{M}$, then we can use $\varphi = 1/a$ as proper integrating factor in formula (4.3) determining components of closed 1-form $\boldsymbol{\omega}$. Contracting this form with vector field (7.3), we get

$$\boldsymbol{\omega}(\mathbf{X}) = C(\boldsymbol{\omega} \otimes \mathbf{X}) = 1. \tag{7.4}$$

Section σ of the bundle $\mathcal{P}^*\mathcal{M}$ determines 1-form $\boldsymbol{\omega}$ up to a scalar factor, formula (4.3) fixes this factor within the domain of local coordinates $x^1, \ldots, x^n, v$, while the condition (7.4) shows that 1-forms defined locally by this procedure is glued into one global closed 1-form $\boldsymbol{\omega}$. Substituting its components into (7.1), we get

$$A = \frac{1}{\omega_{n+1}} - \sum_{i=1}^{n} \frac{\omega_i\, v^i}{\omega_{n+1}}. \tag{7.5}$$

Scalar field (7.5) corresponds to the force field $\mathbf{F}$ with components

$$F_k = \frac{N_k}{\omega_{n+1}} - v \sum_{i=1}^{n} \frac{\omega_i}{\omega_{n+1}} \left(2\, N^i\, N_k - \delta^i_k\right), \tag{7.6}$$

Theorem 7.1. *If the section s of factor-bundle $\Omega\mathcal{M} = \mathcal{T}\mathcal{M}/U$ corresponding to the force field $\mathbf{F}$ of Newtonian dynamical system admitting the normal shift is nonzero at all points $q \in \mathcal{M} = M \times \mathbb{R}^+$, then there is a global closed 1-form $\boldsymbol{\omega}$ determining $\mathbf{F}$ according to the formula* (7.6).

In thesis [19] it was noted that if the function $h(w)$ in formula (1.4) is nonzero, then at the expense of gauge transformation (3.4) one can make it identically equal to unity. There this fact was understood as purely local. Theorem 7.1 shows that it is valid in global situation too.

8. Absence of topological obstructions.

It is well-known that some geometric structures cannot be realized in manifolds with non-trivial topology. Thus, on the sphere S^2 there are no smooth vector fields without special points, where they vanish. For geometric structures from theorem 6.2 we have no such obstructions. Indeed, on any manifold M one has a smooth function $w = w(p)$ which is not identically zero. Let $W(p, v) = w(p) + v$, where $v \in \mathbb{R}^+$. It's obvious that the function $W(p, v)$ on Cartesian product $M \times \mathbb{R}^+$ satisfies the condition (1.5). This function defines some global force field $\mathbf{F}$ of the form (1.4) and all geometric structures from theorem 6.2 as well.

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