SECOND PROBLEM OF GLOBALIZATION
in the theory of dynamical systems admitting the normal shift of hypersurfaces.

R. A. Sharipov

Abstract. Problem of global integration of geometric structures arising in the theory of dynamical systems admitting the normal shift is considered. In the case when such integration is possible the problem of globalization for shift maps is studied.

1. Introduction.

Let M be Riemannian manifold of the dimension n. Newtonian dynamical system in M in local coordinates is determined by a system of n ODE's

$$\ddot{x}^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j=F^k(x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n),\tag{1.1}$$

where $k=1,\,\ldots\,,n$. Here $\Gamma^k_{ij}=\Gamma^k_{ij}(x^1,\ldots,x^n)$ are components of metric connection and F^k are components of force vector $\mathbf{F}$. They determine force field of dynamical system (1.1).

The theory of Newtonian dynamical systems admitting the normal shift of hypersurfaces was constructed in papers [1–16]; on the base of these papers two theses [17] and [18] were prepared. We shall consider some details of this theory a little bit later. Now we note only that this theory describes special class of force fields, which in the case of higher dimensions $n\geqslant 3$ locally (in some neighborhood of any point $p\in M$) can be given by explicit formula

$$F_k=\frac{h(W)\,N_k}{W_v}-v\sum_{i=1}^{n}\frac{\nabla_i W}{W_v}\left(2\,N^i\,N_k-\delta^i_k\right),\tag{1.2}$$

where $W=W(x^1,\ldots,x^n,v)$ is some function of $(n+1)$ variables and $h(w)$ is some function of one variable. The variable v in (1.2) denotes the modulus of velocity vector: $v=|\mathbf{v}|$. While by N^i and N_k we denote contravariant and covariant components of unitary vector $\mathbf{N}$ directed along velocity vector:

$$N^i=\frac{v^i}{|\mathbf{v}|},\qquad\qquad N_k=\frac{v_k}{|\mathbf{v}|}.$$

1991 *Mathematics Subject Classification.* Primary 53B20, 53C15; secondary 57R55, 53C12.
Key words and phrases. Newtonian dynamics, Normal shift.

Function $W(x^1, \ldots, x^n, v)$ in formula (1.2) should satisfy the condition

$$W_v = \frac{\partial W}{\partial v} \neq 0. \tag{1.3}$$

This is quite natural, since partial derivative W_v is in denominators of two fractions in formula (1.2).

Suppose that Riemannian manifold M is equipped with some Newtonian dynamical system admitting the normal shift of hypersurfaces. Functions W and h determine force field $\mathbf{F}$ of such system locally in a neighborhood of some point $p \in M$. In the neighborhood of another point $\tilde{p} \in M$ force field $\mathbf{F}$ in general case is determined by another pair of functions $\tilde{W}$ and $\tilde{h}$. In the region of overlapping of two neighborhoods (if they do really overlap) the force field $\mathbf{F}$ can be determined by each of these two pairs of functions. This gives an idea that force field $\mathbf{F}$ is related to some global geometric structures on M, which are locally represented by pairs of functions (h, W). The problem of revealing such structures was called **the first problem of globalization**. It is solved by the following theorem from [19].

Theorem 1.1. *Defining Newtonian dynamical system admitting the normal shift in Riemannian manifold M is equivalent to defining closed global section σ for projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$, where $\mathcal{M} = M \times \mathbb{R}^+$, satisfying the condition* $\operatorname{Ker}\sigma \nparallel \mathbb{R}^+$*, and normalizing global section s for one-dimensional factor-bundle $\varOmega\mathcal{M} = \mathcal{TM}/U$, where $U = \operatorname{Ker}\sigma$.*

Any section σ of the bundle $\mathcal{P}^*\mathcal{M}$ in the neighborhood of each point $q = (p, v)$ of $\mathcal{M} = M \times \mathbb{R}^+$ is determined by some differential 1-form $\boldsymbol{\omega}$, which is unique up to scalar factor $\boldsymbol{\omega} \to \varphi \cdot \boldsymbol{\omega}$. Closedness of σ means that the form $\boldsymbol{\omega}$ can be chosen closed. Each closed 1-form is locally exact, it is a differential of some function:

$$\boldsymbol{\omega} = dW, \ \text{ where } \ W = W(x^1, \ldots, x^n, v).$$

The condition $\operatorname{Ker}\sigma \nparallel \mathbb{R}^+$ means that $(n+1)$-th component of the form $\boldsymbol{\omega}$ in local coordinates $x^1, \ldots, x^n, v$ is nonzero:

$$W_v = \omega_{n+1} \neq 0. \tag{1.4}$$

In other words, $\operatorname{Ker}\sigma \nparallel \mathbb{R}^+$ is simply an invariant (non-coordinate) form of the condition (1.3). When condition (1.4) is fulfilled, we can consider the quotients

$$b_i = -\frac{\nabla_i W}{W_v} = -\frac{\omega_i}{\omega_{n+1}}. \tag{1.5}$$

The quantities $b_1, \ldots, b_n$ do not change if we replace $\boldsymbol{\omega}$ by $\varphi \cdot \boldsymbol{\omega}$, they are local coordinates in fibers of projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$. If a section σ of this bundle is given, we have n functions $b_i(x^1, \ldots, x^n, v)$, where $i = 1, \ldots, n$. The condition of closedness for σ is written in form of the following relationships:

$$\left(\frac{\partial}{\partial x^j} + b_j \frac{\partial}{\partial v}\right) b_i = \left(\frac{\partial}{\partial x^i} + b_i \frac{\partial}{\partial v}\right) b_j. \tag{1.6}$$

The kernel $\operatorname{Ker}\sigma = \operatorname{Ker}\boldsymbol{\omega}$ determines n-dimensional distribution U in the manifold $\mathcal{M} = M \times \mathbb{R}^+$, whose dimension is $n + 1$. It also determines 1-dimensional vector-bundle obtained by factorization of cotangent bundle $\mathcal{TM}$ with respect to U. Due to the condition $\operatorname{Ker}\sigma \nparallel \mathbb{R}^+$ vector field $\partial/\partial v$ is transversal to U. Therefore each section s of factor-bundle $\Omega\mathcal{M} = \mathcal{TM}/U$ in local coordinates can be determined by vector field of the form

$$a(x^1, \ldots, x^n, v) \cdot \frac{\partial}{\partial v}, \tag{1.7}$$

or by function $a(x^1, \ldots, x^n, v)$, which arises as a coefficient in formula (1.7). The concept of normalizing section is introduced by the following two definitions from paper [19].

Definition 1.1. Vector field $\mathbf{X}$ is called **normalizing field** for smooth distribution U if for any vector field $\mathbf{Y}$ belonging to U the commutator $[\mathbf{X}, \mathbf{Y}]$ is also in U.

Definition 1.2. Section s of factor-bundle $\Omega\mathcal{M} = \mathcal{TM}/U$ is called **normalizing section** if in the neighborhood of each point $q \in \mathcal{M}$ it is represented by some normalizing vector field for the distribution U.

The fact that section s of factor-bundle $\mathcal{TM}/\operatorname{Ker}\sigma$ is normalizing is expressed by the following equations for the function a in (1.7):

$$\left(\frac{\partial}{\partial x^i} + b_i\,\frac{\partial}{\partial v}\right) a = \frac{\partial b_i}{\partial v}\,a. \tag{1.8}$$

Note that the concept of normalizing section s of the bundle $\mathcal{TM}/U$ is correctly determined only for involutive distribution U. In this case s is a coset of vector $\mathbf{X}$ respective to subspace U, i. e. $s = \operatorname{Cl}_U(\mathbf{X})$. The choice of vector field representing such coset doesn't matter, since if $\mathbf{X}$ is normalizing vector field for U and $\mathbf{Y} \in U$, then the sum $\mathbf{X} + \mathbf{Y}$ is also normalizing vector field for U. In our case $U = \operatorname{Ker}\sigma$ is involutive. This follows from closedness of σ.

2. Integration of geometric structures.

Theorem 1.1 determines global geometric structures related to force fields of Newtonian dynamical systems admitting the normal shift, thus solving first problem of globalization. As for calculation of components of force vector, it yields formula

$$F_k = a\,N_k + v \sum_{i=1}^{n} b_i\left(2\,N^i\,N_k - \delta^i_k\right), \tag{2.1}$$

where a and $b_1, \ldots, b_n$ should satisfy the equations (1.6) and (1.8). It's clear that formula (2.1) is much less effective than formula (1.2). The passage from (2.1) to (1.2) consists in integrating the equations (1.6) and (1.8). These equations are compatible and locally integrable, this is shown in Chapter VII of thesis [17] (see also paper [19]). Here we are interested in those cases, when they are globally integrable.

First step in global integration of the structures σ and s from theorem 1.1 consist in exploiting the closedness of the section σ. On the manifold $\mathcal{M} = M \times \mathbb{R}^+$ (or possibly on universal cover for $\mathcal{M}$) one should find global closed 1-form $\boldsymbol{\omega}$ that should satisfy the condition $\operatorname{Ker}\boldsymbol{\omega} = \operatorname{Ker}\sigma$. This form would determine the quantities $b_1, \ldots, b_n$ according to the formula (1.5). If such formula is found, we say that **first level of global integration** of structures σ and s is reached.

Assuming that first level of global integration of structures σ and s is already reached, let's integrate the 1-form $\boldsymbol{\omega}$ just found along the path binding some fixed initial point q_0 with ending point q:

$$W(q)=\int_{q_0}^{q}\boldsymbol{\omega}. \tag{2.2}$$

Passing from $\mathcal{M}=M\times\mathbb{R}^{+}$ to universal cover for $\mathcal{M}$, if necessary, we warranty that formula yields single-valued function W on such cover. Then $dW=\boldsymbol{\omega}$.

Let $\widehat{M}$ be universal cover for M. Then universal cover $\widehat{\mathcal{M}}$ for the manifold $\mathcal{M}=M\times\mathbb{R}^{+}$ can be identified with $\widehat{M}\times\mathbb{R}^{+}$. The structure of Cartesian product in $\widehat{\mathcal{M}}=\widehat{M}\times\mathbb{R}^{+}$ provides vector field $\mathbf{V}=\partial/\partial v$ directed along linear rulings in this manifold. Applying this field to the function (2.2) we get the function $W_v=\mathbf{V}W$. In local coordinates $x^1, \ldots, x^n, v$ this function coincides with partial derivative:

$$W_v=\frac{\partial W}{\partial v}=\omega_{n+1}\neq 0.$$

Let's define the function $\widetilde{W}=a\cdot W_v$ and let's calculate its differential:

$$d\widetilde{W}=\sum_{i=1}^{n}\frac{\partial\widetilde{W}}{\partial x^i}\cdot dx^i+\frac{\partial\widetilde{W}}{\partial v}\cdot dv. \tag{2.3}$$

For first n components in 1-form (2.3) we have

$$\begin{aligned}\frac{\partial\widetilde{W}}{\partial x^i}&=\frac{\partial a}{\partial x^i}\,\omega_{n+1}+a\,\frac{\partial\omega_{n+1}}{\partial x^i}=\left(\frac{\partial b_i}{\partial v}\,a-b_i\,\frac{\partial a}{\partial v}\right)\omega_{n+1}+a\,\frac{\partial\omega_{n+1}}{\partial x^i}=\\&=\left(-\frac{\partial\omega_i}{\partial v}\,a-b_i\,a\,\frac{\partial\omega_{n+1}}{\partial v}-b_i\,\frac{\partial a}{\partial v}\,\omega_{n+1}\right)+a\,\frac{\partial\omega_{n+1}}{\partial x^i}=\frac{\omega_i}{\omega_{n+1}}\,\frac{\partial\widetilde{W}}{\partial v}.\end{aligned}$$

In these calculations we used closedness of the form $\boldsymbol{\omega}$ and the relationships (1.5) and (1.8). The result of calculations can be formulated as follows: the ratio of i-th and $(n+1)$-th components of 1-form (2.3) is equal to the ratio of ω_i and ω_{n+1}. This means that forms $d\widetilde{W}$ and $\boldsymbol{\omega}=dW$ are collinear. This situation is described by the following lemma, which was used in paper [19].

Lemma 2.1. *If gradient of one smooth function $f(x^1,\ldots,x^n)$ is nonzero in some domain $U\subset\mathbb{R}^n$ and gradient of another smooth function $g(x^1,\ldots,x^n)$ is collinear to it in U, then functions f and g are functionally dependent in U. This means that for each point $p\in U$ one can find some neighborhood $O(p)$ and a smooth function of one variable $\rho(y)$ such that $g=\rho\circ f$ in $O(p)$.*

As an immediate consequence of lemma 2.1 we find that locally in the neighborhood of each point $q\in\widehat{\mathcal{M}}$ there is some function $h=h(w)$ such that components of force vector $\mathbf{F}$ are determined by formula (1.2). If such function is unique, i. e. one for all points q overall the manifold $\widehat{\mathcal{M}}$, then we say that **second level of global**

integration of structures σ and s is reached. One particular case, when both levels of global integration are reached, was found in paper [19]. It is described by the following theorem.

Theorem 7.1. *If the section s of factor-bundle $\Omega\mathcal{M} = T\mathcal{M}/U$ corresponding to the force field $\mathbf{F}$ of Newtonian dynamical system admitting the normal shift is nonzero at all points $q \in \mathcal{M} = M \times \mathbb{R}^+$, then there is a global closed 1-form $\boldsymbol{\omega}$ determining $\mathbf{F}$ according to the following formula*

$$F_k = \frac{N_k}{\omega_{n+1}} - v\sum_{i=1}^{n}\frac{\omega_i}{\omega_{n+1}}\left(2\,N^i\,N_k - \delta^i_k\right). \tag{2.4}$$

Formula (2.4) corresponds to the choice of $h(w)$ being identically equal to unity. Note also that we need not to pass to universal cover $\widehat{\mathcal{M}}$ in this case. Below we consider other cases when both levels of integration of structures σ and s are reached, the restriction $s \neq 0$ there is eliminated.

3. Extended tensor fields.

Theorem 1.1 relates formula 2.1 and parameters $b_1, \ldots, b_n$, and a in it with structures σ and s on Cartesian product $\mathcal{M} = M \times \mathbb{R}^+$. However, initially these quantities were interpreted in a quite different way (see papers [6–16] and thesis [17]). Function a was interpreted as extended scalar field, while $b_1, \ldots, b_n$ are components of extended covectorial field on $\mathbf{M}$. Let's recall appropriate definition.

Definition 3.1. The function $\mathbf{X}$ that to each point $q = (p, \mathbf{v})$ of tangent bundle TM puts into correspondence some tensor from the space $T^r_s(p, M)$ at the point $p = \pi(q)$ of M is called **extended tensor field** of the type (r, s) on M.

In order to compare note that traditional tensor field $\mathbf{X}$ of the type (r, s) on M is a function that maps a point p of M, but not a point of tangent bundle TM as in definition 3.1 above, to some tensor from the space $T^r_s(p, M)$ at that point $p \in M$. The idea to extend the concept of tensor field in the sense of definition 3.1 goes back to Finsler and Cartan (see [20] and [21]). In book [22] the class of **semibasic tensor fields**, being subclass in the class of traditional tensor fields on tangent bundle TM, is considered. Theory of **semibasic tensor fields** constructed in [22] appears to be isomorphic to the theory of **extended tensor fields** based on definition 3.1. This fact was discovered by N. S. Dairbekov when I was making report in the seminar of Yu. G. Reshetnyak at the Institute of Mathematics of Siberian Branch of Russian Academy of Sciences (IM SB RAS) in October, 2000. Despite to the presence of alternative approach, below we use extended tensor fields, theory of which was especially developed (see thesis [17]) for the problems related to Newtonian dynamical systems in Riemannian and Finslerian manifolds.

4. Norm of covectorial field b and global integration of the section σ.

As we already mentioned above, the quantities $b_1, \ldots, b_n$ in (2.1) are interpreted as components of extended covectorial field $\mathbf{b}$ in $\mathbf{M}$. But this is extended field of special form, its components depend on components of velocity vector $\mathbf{v}$ in fibers of tangent bundle TM only through their dependence on modulus of velocity vector

$v = |\mathbf{v}|$. In paper [23] such fields were called **fiberwise spherically symmetric**. It is the property of fiberwise spherical symmetry that gave us the opportunity in [19] to introduce the manifold $\mathcal{M} = M \times \mathbb{R}^+$ and find simple geometric interpretation of the equations (1.6) and (1.8). There the field $\mathbf{b}$ was associated with the section σ of projectivized cotangent bundle $\mathcal{P}^*\mathcal{M}$.

Let's consider the field $\mathbf{b}$ in its initial interpretation and define $|\mathbf{b}|$ as a length of covector $\mathbf{b}$ in Riemannian metric of the manifold M:

$$|\mathbf{b}| = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}\,b_i\,b_j}.$$

The quantity $|\mathbf{b}|$ does not depend on local coordinates $x^1, \ldots, x^n$, it depends only upon the point $p \in M$ and upon variable $v \in \mathbb{R}^+$, which is interpreted as modulus of velocity vector. Let $f = f(v)$ be a positive function defined on semiaxis $\mathbb{R}^+$ and such that the following conditions are fulfilled:

$$F(v) = \int\limits_{v_0}^{v} \frac{dv}{f(v)} \longrightarrow +\infty \quad \text{for} \quad v \longrightarrow +\infty, \tag{4.1}$$

$$F(v) = \int\limits_{v_0}^{v} \frac{dv}{f(v)} \longrightarrow -\infty \quad \text{for} \quad v \longrightarrow 0. \tag{4.2}$$

Here v_0 is some arbitrary positive number from positive semiaxis $\mathbb{R}^+$. Let's define f-norm of covectorial field $\mathbf{b}$ by the following formula:

$$\|\mathbf{b}\| = \sup_{p\in M,\ v\in\mathbb{R}^+}\left(\frac{|\mathbf{b}|}{f(v)}\right). \tag{4.3}$$

Theorem 4.1. *If manifold M is connected and if f-norm of covectorial field* $\mathbf{b}$ *defined by formula* (4.3) *is finite, then first levels of integration of structures σ and s related to the force field of dynamical system is reached.*

Theorem 4.1 means that if $\|\mathbf{b}\| < \infty$, then the section σ is determined by some global 1-form $\boldsymbol{\omega} = dW$. This form can be multivalued, but becomes single-valued upon passage to universal cover $\widehat{\mathcal{M}} = \widehat{M} \times \mathbb{R}^+$ for the manifold $\mathcal{M} = M \times \mathbb{R}^+$.

Let's start proving theorem 4.1 by studying problem of local existence of the form $\boldsymbol{\omega}$. Thereby we partially resume the content of paper [19]. Let's choose local coordinates $x^1, \ldots, x^n, v$ in $\mathcal{M} = M \times \mathbb{R}^+$. Components of the form $\boldsymbol{\omega}$ to be found are bound with components of the field $\mathbf{b}$ by relationships (1.5). Therefore in order to find 1-form $\boldsymbol{\omega}$ it is sufficient to choose proper factor φ:

$$\omega_i = \begin{cases} -b_i\,\varphi & \text{for} \quad i = 1, \ldots, n, \\ \varphi & \text{for} \quad i = n+1. \end{cases} \tag{4.4}$$

The condition of closedness of 1-form $\boldsymbol{\omega}$ is written in form of relationships

$$\frac{\partial\omega_i}{\partial x^j} - \frac{\partial\omega_j}{\partial x^i} = 0. \tag{4.5}$$

Here $v = x^{n+1}$. From (4.4) and (4.5) for $i \leqslant n$ and $j \leqslant n$ we derive

$$\frac{\partial b_i}{\partial x^j}\,\varphi + \frac{\partial \varphi}{\partial x^j}\,b_i = \frac{\partial b_j}{\partial x^i}\,\varphi + \frac{\partial \varphi}{\partial x^i}\,b_j. \tag{4.6}$$

From the same relationships (4.4) and (4.5) for $i \leqslant n$ and $j = n + 1$ we derive

$$\frac{\partial \varphi}{\partial x^i} = -\frac{\partial b_i}{\partial v}\,\varphi - \frac{\partial \varphi}{\partial v}\,b_i. \tag{4.7}$$

Now let's substitute the derivatives $\partial\varphi/\partial x^i$ and $\partial\varphi/\partial x^j$ calculated according to (4.7) into the equations (4.6). As a result we get the equations without entries of φ. They coincide with (1.6) exactly. Thus, the equations (1.6) form necessary condition for local existence of closed form $\boldsymbol{\omega}$ with components (4.4). As it was shown in [19], these equations constitute sufficient condition as well. We prove this fact by constructing the solution for the equations (4.7). Let's consider an auxiliary system of partial differential equations

$$\frac{\partial V}{\partial x^i} = b_i(x^1, \ldots, x^n, V), \text{ where } i = 1, \ldots, n. \tag{4.8}$$

This is complete system of Pfaff equations with respect to function $V(x^1, \ldots, x^n)$. It is compatible. The compatibility condition for the equations (4.8) coincides with (1.6) exactly. Let's fix some point $p_0 \in M$. Without loss of generality we can assume that local coordinates of the point p_0 are equal to zero. For compatible system of Pfaff equations (4.8) we set up the Cauchy problem

$$V\Bigr|_{x^1 = \ldots = x^n = 0} = w, \tag{4.9}$$

where $w > 0$. Solution of Cauchy problem (4.9) exists and is unique in some neighborhood of the point p_0. It is smooth function of coordinates $x^1, \ldots, x^n$ and parameter w from right hand side of (4.9):

$$v = V(x^1, \ldots, x^n, w). \tag{4.10}$$

For $x^1 = \ldots = x^n = 0$ due to (4.9) we get $V(0, \ldots, 0, w) = w$. Therefore

$$\frac{\partial V}{\partial w}\Bigr|_{x^1 = \ldots = x^n = 0} = 1. \tag{4.11}$$

Let's consider the set of points $q = (p_0, v)$ in $\mathcal{M}$. They constitute linear ruling in Cartesian product $\mathcal{M} = M \times \mathbb{R}^{+}$. Denote it $l_0 = l(p_0)$. The equality (4.10) means that for any point $q_0 \in l_0$ there is some neighborhood where we have local coordinates $y^1, \ldots, y^n, w$ related to $x^1, \ldots, x^n, v$ as follows:

$$\begin{cases} x^i = y^i \text{ for } i = 1, \ldots, n, \\ v = V(y^1, \ldots, y^n, w). \end{cases} \tag{4.12}$$

Inverse passage to $x^1, \ldots, x^n, v$ is determined by the function $W(x^1, \ldots, x^n, v)$:

$$\begin{cases} y^i = x^i \text{ for } i = 1, \ldots, n, \\ w = W(x^1, \ldots, x^n, v). \end{cases} \tag{4.13}$$

Function $W(x^1,\ldots,x^n,v)$ is determined in implicit form from (4.10) if one treat this equality as an equation with respect to w.

Let's use (4.12) and (4.13) in order to simplify the equations (4.7). Instead of function $\varphi(x^1,\ldots,x^n,v)$ in these equations we introduce another function

$$\psi(y^1,\ldots,y^n,w)=\varphi(y^1,\ldots,y^n,V(y^1,\ldots,y^n,w)). \tag{4.14}$$

The equations (4.7) are reduced to the following ones with respect to ψ:

$$\frac{\partial\psi}{\partial y^i}=-B_i\,\psi. \tag{4.15}$$

The quantities B_i are expressed through partial derivatives of $V=V(y^1,\ldots,y^n,w)$:

$$B_i=\frac{1}{Z}\,\frac{\partial Z}{\partial y^i},\ \text{ where }\ Z=\frac{\partial V}{\partial w}. \tag{4.16}$$

It's easy to see that (4.15) is a system of Pfaff equations, being compatible due to (4.16). Moreover, it is explicitly integrable. General solution of the system of differential equations (4.15) has the following form:

$$\psi=\frac{C(w)}{Z(y^1,\ldots,y^n,w)}. \tag{4.17}$$

Here $C(w)$ is an arbitrary function of one variable. Now let's use local invertibility of the relationship (4.14):

$$\varphi(x^1,\ldots,x^n)=\psi(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v)). \tag{4.18}$$

From (4.17) and (4.18) we can derive general solution for the equations (4.7):

$$\varphi=C(W)\cdot W_v,\ \text{ where }\ W_v=\frac{\partial W}{\partial v}. \tag{4.19}$$

Let's turn back to the equations (4.8) and let's write them with more details:

$$b_i(x^1,\ldots,x^1,V(x^1,\ldots,x^n,w))=\frac{\partial V(x^1,\ldots,x^n,w)}{\partial x^i}. \tag{4.20}$$

The relationships (4.20) are the identities, which are fulfilled since the function $V(x^1,\ldots,x^n,w)$ is a solution for the system of equations (4.8). Let's substitute $w=W(x^1,\ldots,x^n,v)$ into (4.20) in order to express the variable w trough v:

$$b_i(x^1,\ldots,x^1,v)=\frac{\partial V(x^1,\ldots,x^n,w)}{\partial x^i}\biggr|_{w=W(x^1,\ldots,x^n,v)}. \tag{4.21}$$

If we take into account that V and W determines mutually inverse changes of variables (4.12) and (4.13), then we can express right hand side of (4.21) through partial derivatives of the function $W(x^1,\ldots,x^n,v)$. This yields

$$b_i(x^1,\ldots,x^1,v)=-\frac{\nabla_i W}{W_v},\ \text{ where }\ \nabla_i W=\frac{\partial W}{\partial x^i}. \tag{4.22}$$

Now let's substitute (4.22) and (4.19) into (4.4) and calculate components of $\boldsymbol{\omega}$:

$$\omega_i = \begin{cases} C(W)\cdot\nabla_i W & \text{for } \ i = 1,\,\ldots\,,\,n,\\ C(W)\cdot W_v & \text{for } \ i = n+1.\end{cases} \tag{4.23}$$

If we take the function $C(W)$ being identically equal to unity, then we get $\boldsymbol{\omega} = dW$. This means that form ω just constructed is closed. However, any other choice of $C(W)$ also yields closed form $\boldsymbol{\omega}$.

The above method for constructing 1-form $\boldsymbol{\omega}$ is purely local yet. The possibility to make it global depends on the answer to the question — **how big** is the neighborhood of the point p_0, where the solution of Cauchy problem (4.9) for the equations (4.8) is defined? Let $O(p_0)$ be some neighborhood of the point p_0, where such solution does exist, and let p_1 be the point on the boundary of this neighborhood. Let's bind p_0 and p_1 by a smooth curve γ in M. It's clear that neighborhood $O(p_0)$ is within the chart where local coordinates $x^1,\,\ldots\,,\,x^n$ are defined. Suppose, that the point p_1 is also within this chart. Then curve γ can be represented by smooth functions $x^1(t),\ \ldots\,,\ x^n(t)$. Coordinates of the point p_0 are zero $x^1 = \ldots = x^n = 0$, as we have took them above in (4.9). Let $t = t_0$ and $t = t_1$ be the values of parameter t for the points p_0 and p_1 on γ. Consider the restriction of the function $V(x^1,\ldots,x^n)$ to γ:

$$V(t) = V(x^1(t),\ldots,x^n(t)). \tag{4.24}$$

Let's differentiate the function (4.24) with respect to parameter t and take into account the equations (4.8). This yields the equation

$$\dot V = \sum_{i=1}^{n} b_i\,\dot x^i. \tag{4.25}$$

Since norm (4.3) is finite, for the derivative $\dot V$ in (4.25) we have the estimate:

$$\left|\frac{\dot V}{f(V)}\right| = \left|\sum_{i=1}^{n}\frac{b_i\,\dot x^i}{f(V)}\right| \leqslant \frac{|\mathbf{b}|}{f(V)}\cdot|\mathbf{K}| \leqslant \|\mathbf{b}\|\cdot|\mathbf{K}|. \tag{4.26}$$

Here $\mathbf{K}$ is a vector with components $\dot x^1,\ldots,\dot x^n$. It is tangent to γ. We do not denote it by $\mathbf{v}$, since parameter t on the curve γ is not a time. It does not relate to Newtonian dynamics in (1.1). Let $V(t_0)$ and $V(t)$ be the values of the function (4.24) at two points on the curve γ. Then we have

$$F(V(t)) - F(V(t_0)) = \int\limits_{t_0}^{t}\frac{\dot V}{f(V)}\,dt.$$

The function $V(t)$ for $t = t_1$ is not defined. But one can consider the limit

$$\lim_{t \to t_1} F(V(t)) = F(V(t_0)) + \int\limits_{t_0}^{t_1} \frac{\dot{V}}{f(V)}\,dt. \tag{4.27}$$

Integral in right hand side of (4.27) is understood as an improper integral. From the inequality (4.26) it follows that such integral absolutely converges:

$$\int\limits_{t_0}^{t_1} \left|\frac{\dot{V}}{f(V)}\right| dt \leqslant \|\mathbf{b}\| \cdot \int\limits_{t_0}^{t_1} |\mathbf{K}|\,dt = \|\mathbf{b}\| \cdot L_\gamma(t_1, t_0).$$

Here $L_\gamma(t_1, t_0)$ is the length of the segment of curve γ with ending points p_1 and p_0. It is finite. Hence the integral in (4.27) converges, this implies the existence of finite limit in left hand side of (4.27).

Further let's apply some properties of the function $F(v)$. Let's remember that it is defined in $\mathbb{R}^+$, it's monotonic and increasing, and it satisfies the conditions (4.1) and (4.2). Graph of such function is drawn on Fig. 4.2. Due to the above properties of the function $F(v)$ we can assert that the existence of finite limit in (4.27) implies the existence and finiteness of limit

$$\lim_{t \to t_1} V(t) = \tilde{v}, \tag{4.28}$$

the value $\tilde{v}$ of this limit being positive number from real semiaxis $\mathbb{R}^+$:

$$0 < \tilde{v} < +\infty.$$

Existence and finiteness of the limit (4.28) is very important fact. Now remember that the point p_1 corresponding to the value $t = t_1$ of parameter t is within the chart where local coordinates $x^1, \ldots, x^n$ are defined. At the point p_1 we can set up the Cauchy problem similar to (4.9):

$$V\Big|_{x^1 = x^1(p_1), \ldots, \, x^n = x^n(p_1)} = \tilde{v}. \tag{4.29}$$

The solution of Cauchy problem (4.29) for Pfaff equations (4.8) does exist ant it is unique in some neighborhood $O(p_1)$ of the point p_1. It is smooth function of parameter $\tilde{v}$ and coordinates $x^1, \ldots, x^n$. Let's denote it $V = \tilde{V}(x^1, \ldots, x^n, \tilde{v})$. This function also can be restricted to the curve γ, where we have the equality

$$\lim_{t \to t_1} \tilde{V}(t) = \lim_{t \to t_1} V(t) = \tilde{v}. \tag{4.30}$$

From (4.30) it follows that $V(x^1, \ldots, x^n, w)$ is equal to $V(x^1, \ldots, x^n, \tilde{v})$ on the curve in the region of overlapping of neighborhoods $O(p_0)$ and $O(p_1)$.

Lemma 4.1. *If f-norm of covectorial field* $\mathbf{b}$ *is finite, then the solution of Cauchy problem* (4.9) *for the equations* (4.8) *can be continued to any point of chart where local coordinates* $x^1, \ldots, x^n$ *are defined.*

The equations (4.8) possess the property of **coordinate covariance**. This means that their shape doesn't change under the transition from one set of local coordinates to another. Therefore the solution of Cauchy problem (4.9) is a scalar field $V = V(p, w)$ depending on auxiliary parameter w. If $\|\mathbf{b}\| < \infty$ and if we have two overlapping charts, then scalar field V can be continued from one chart to another along any curve passing through the region of overlapping. Now we can strengthen lemma 4.1 as follows.

Lemma 4.2. *If f-norm of covectorial field* $\mathbf{b}$ *is finite, then the solution of Cauchy problem* (4.9) *for the equations* (4.8) *can be continued to any point p of manifold M along any curve binding p with the point p_0.*

Note that the result of continuation of scalar field V along the curve γ from the neighborhood of p_0 to the point p doesn't change under continuous deformations of the curve γ. Therefore each Cauchy problem for the equations (4.8) determines some global scalar field V on universal cover $\widehat{M}$. There is simple invariant (non-coordinate) interpretation of scalar field V. Indeed, $V(p, w)$ is a numeric function of the point $p \in \widehat{M}$ and positive numeric parameter $w \in \mathbb{R}^+$, values of this function also being positive numbers. Its graph is a hypersurface in Cartesian product $\widehat{\mathcal{M}} = \widehat{M} \times \mathbb{R}^+$. It appears that this hypersurface coincides with integral manifold for involutive distribution $U = \operatorname{Ker} \sigma$. If we take into account that the point p_0, where Cauchy problem (4.9) is set up, can be taken for an arbitrary point in M or, which is more convenient, for an arbitrary point in universal cover $\widehat{M}$, then we can reformulate lemma 4.2 as follows.

Lemma 4.3. *If f-norm of covectorial field* $\mathbf{b}$ *is finite, then Cartesian product* $\widehat{\mathcal{M}} = \widehat{M} \times \mathbb{R}^+$ *foliates into the disjoint union of integral manifolds of involutive distribution* $U = \operatorname{Ker} \sigma$*, each of which being graph for some real-valued function on* $\widehat{M}$ *with the values in* $\mathbb{R}^+$*.*

Thus, the function $v = V(p, w)$ is defined and is single-valued function on universal cover $\widehat{M}$ for M. Let's define a function $W(p, v)$ such that the conditions $W(p, V(p, w)) = w$ and $V(p, W(p, v)) = v$ would be fulfilled. These conditions are equivalent to requirement that in local coordinates the changes of variables (4.12) and (4.13) are inverse to each other. Does such function $W(p, v)$ exist on the whole manifold $\widehat{M}$? The answer to this question depends on solvability of the equation

$$v = V(p, w) \tag{4.31}$$

with respect to variable w at each fixed point $p = p_1$ in $\widehat{M}$. In our case, when $\|\mathbf{b}\| < \infty$, the equation (4.31) appears to be solvable. Let's prove this fact using lemma 4.3. Suppose that $v_1 \in \mathbb{R}^+$. Consider the point $q_1 = (p_1, v_1)$ of Cartesian product $\widehat{\mathcal{M}} = \widehat{M} \times \mathbb{R}^+$. Some integral manifold I of distribution $U = \operatorname{Ker} \sigma$ passes through this point, it is a graph for some function $\psi(p)$. Then $\psi(p_1) = v_1$. Denote by w the value of this function at the point p_0, where Cauchy problem (4.9) is set up. This means that we take $w = \psi(p_0)$. Then submanifold I is a graph for the function $V(p,w)$, where parameter w is fixed to be equal to $\psi(p_0)$. Therefore

$$\psi(p)=V(p,\psi(p_0)). \tag{4.32}$$

Substituting $p=p_1$ into the equality (4.32), we get $v_1=V(p_1,\psi(p_0))$. This means that $w=\psi(p_0)$ is a solution for the equation $v_1=V(p,w)$ at the point $p=p_1$. Solvability of the equation (4.31) means that required function $W(p,v)$ does exist. Now, similar to $V(p,w)$, it is global, since it is defined on the whole manifold $\widehat{M}$.

Let's prove that $W(p,v)$ is smooth function. According to the theory of implicit functions (see [24] or [25]), it is sufficient to show that the derivative

$$V_w(p,w)=\frac{\partial V(p,w)}{\partial w} \tag{4.33}$$

does not vanish. At the point $p=p_0$ the derivative (4.33) is equal to unity:

$$V_w\Bigr|_{p=p_0}=1 \tag{4.34}$$

(see relationship (4.11)). From (4.8) one can easily derive the differential equations for the function V_w in local coordinates. They are the following ones:

$$\frac{\partial V_w}{\partial x^i}=\frac{\partial b_i}{\partial v}\,V_w. \tag{4.35}$$

Similar to $V(p,w)$, the function V_w can be restricted to the curve γ. Here we get

$$W_w(t)=W_w(x^1(t),\ldots,x^n(t),w). \tag{4.36}$$

For the function (4.36) from the equations (4.35) we derive the differential equation

$$\frac{dV_w}{dt}=\left(\sum_{i=1}^{n}\frac{\partial b_i}{\partial v}\,\dot x^i\right)\cdot V_w. \tag{4.37}$$

If the curve γ passes through the point p_0, then the condition (4.34) sets up the Cauchy problem for linear ordinary differential equation (4.37). Its solution does exist and is unique. It is given by the following formula:

$$V_w=\exp\left(\int\limits_{t_0}^{t}\sum_{i=1}^{n}\frac{\partial b_i}{\partial v}\,\dot x^i\,dt\right)=\exp\left(\int\limits_{t_0}^{t}\mathbf{b}_v(\mathbf{K})\,dt\right). \tag{4.38}$$

Here, as in formula (4.26), $\mathbf{K}$ is the tangent vector of the curve γ, its components are $\dot x^1,\ldots,\dot x^n$. The integral in argument of exponential function in (4.38) is a smooth function of parameter of t, it has no singular points. Therefore the value of exponent (4.38) is nonzero. Hence $V_w\neq 0$. This provides smoothness of the above function $W(p,v)$. The differential of this function dW is a required 1-form

$\boldsymbol{\omega}$ (see formula (4.23) and calculations preceding it). Thus, theorem 4.1 is proved. This means that under the assumption that $\|\mathbf{b}\| < \infty$ we reached **first level of global integration** of geometric structures σ and s determining force field $\mathbf{F}$ of Newtonian dynamical system that we consider.

5. Global integration of the section σ.

Suppose that the condition $\|\mathbf{b}\| < \infty$ is fulfilled. Let's consider the section s of one-dimensional factor-bundle $\varOmega\mathcal{M} = \mathcal{T}\mathcal{M}/U$, where $U = \operatorname{Ker}\sigma$. Passing to universal cover $\widehat{\mathcal{M}} = \widehat{M} \times \mathbb{R}^{+}$ we can consider the section of factor-bundle $\varOmega\widehat{\mathcal{M}} = \mathcal{T}\widehat{\mathcal{M}}/U$. Due to the condition $\operatorname{Ker}\sigma \nparallel \mathbb{R}^{+}$ (see theorem 1.1 above) such section can be defined by vectorial field

$$\mathbf{X} = a(p,v) \cdot \frac{\partial}{\partial v}$$

in $\widehat{\mathcal{M}}$, or by scalar field $a(p,v)$ in $\widehat{\mathcal{M}}$. Let's consider the product $\widetilde{W} = a \cdot W_v$, where $W = W(p,v)$ is the function which was constructed above in proving theorem 4.1. Covectors dW and $d\widetilde{W}$ are collinear and $\boldsymbol{\omega} = dW \neq 0$ (see relationship (2.3) and calculations preceding lemma 2.1). Let $q_0 = (p_0, v_0)$ and $q_1 = (p_1, v_1)$ be two points of the manifold $\widehat{\mathcal{M}}$ lying on the same level hypersurface of the function $W = W(p,v)$, i. e. such that $W(p_0,v_0) = W(p_1,v_1)$. Suppose that these points are connected by a curve γ lying on the same level hypersurface as q_0 and q_1. Then for the difference of $\widetilde{W}(p_1,v_1)$ and $\widetilde{W}(p_0,v_0)$ we get the expression

$$\widetilde{W}(p_1,v_1) - \widetilde{W}(p_0,v_0) = \int\limits_{t_0}^{t_1} d\widetilde{W}(\mathbf{K})\,dt = 0. \tag{5.1}$$

Here K is the tangent vector of curve γ, while t_1 and t_0 are the values of parameter t on this curve corresponding to the points p_1 and p_0 respectively. Vector $\mathbf{K}$ belongs to the kernel of the form $\boldsymbol{\omega} = dW$, therefore $dW(\mathbf{K}) = 0$. Due to collinearity of covectors $d\widetilde{W}$ and dW it follows that the expression $d\widetilde{W}(\mathbf{K})$ and the integral (5.1) in whole do vanish.

Lemma 5.1. *If level hypersurfaces of the function $W(p,v)$ are connected, then $W(p_0,v_0) = W(p_1,v_1)$ implies $\widetilde{W}(p_0,v_0) = \widetilde{W}(p_1,v_1)$.*

Level hypersurfaces of the function $W(p,v)$ are exactly the integral manifolds of involutive distribution $U = \operatorname{Ker}\sigma = \operatorname{Ker}\boldsymbol{\omega}$, since $\boldsymbol{\omega} = dW$. Due to lemma 4.3 each of these hypersurfaces is diffeomorphic to the manifold $\widehat{M}$. If M is connected, then $\widehat{M}$ is also connected. In such situation let's consider the point p_0 where Cauchy problem (4.9) for the equations (4.8) is set up. Here $V(p_0,w) = w$, hence $W(p_0,v) = v$. Let's define the following function of one variable:

$$h(v) = \widetilde{W}(p_0,v). \tag{5.2}$$

Since $W(p_0,v) = v$, the equality (5.2) can be rewritten as follows:

$$\widetilde{W}(p_0,v) = h(W(p_0,v)). \tag{5.3}$$

Relying upon lemma 5.1, we can replace p_0 in (5.3) by an arbitrary point p of universal cover $\widehat{M}$. Then the equality (5.3) looks like

$$\widetilde{W}(p, v) = h(W(p, v)). \tag{5.4}$$

From the equality (5.4) for the extended scalar field a in (2.1) we get

$$a = \frac{h(W)}{W_v}. \tag{5.5}$$

While components of covectorial field $\mathbf{b}$, as we have found above, are expressed by formula (4.22). Substituting (5.5) and (4.22) into the formula (2.1), we bring it to the form (1.2). Thereby the functions h and W are now globally defined for all points $p \in \widehat{M}$. This means that we reached **second level of global integration** of geometric structures σ and s.

Theorem 5.1. *If Riemannian manifold M is connected and if f-norm of covectorial field $\mathbf{b}$ is finite, then for geometric structures determining force field of Newtonian dynamical system admitting the normal shift of hypersurfaces in M both levels of global integration are reached.*

6. Monodromy transformations.

Force field F and geometric structures σ and s determining this field are related to the manifold $\mathcal{M}$. However, in integrating these structures we are to pass to universal cover $\widehat{M}$. Therefore the functions W and h, which were constructed above, should contain a discrete symmetry determined by first fundamental group $\pi_1(M)$. Group $\pi_1(M)$ acts in $\widehat{M}$ by discrete transformations, and M coincides with the result of factorization of $\widehat{M}$ with respect to such action: $M = \widehat{M}/\pi_1(M)$. Take $g \in \pi_1(M)$. Let's compare two functions $V(p, w)$ and $V(g^{-1}(p), v)$. If we localize them in the neighborhood of the point p_0, where Cauchy problem (4.9) is set up, then, upon passing from $\widehat{M}$ to M by canonical projection, these functions appears to be the solutions of the same system of differential equations (4.8). Now let's consider the following function of one variable:

$$\rho(w) = V(g^{-1}(p_0), w).$$

Then for the values of functions $V(p, w)$ and $V(g^{-1}(p), v)$ at the point p_0 we get

$$V(p, w)\Bigr|_{p=p_0} = w, \qquad\qquad V(g^{-1}(p), w)\Bigr|_{p=p_0} = \rho(w). \tag{6.1}$$

Each of the relationships (6.1) can be treated as Cauchy problem for the equations (4.8) in the neighborhood of the point p_0. Due to the uniqueness of solutions of such Cauchy problems, we get the following relationship:

$$V(g^{-1}(p), w) = V(p, \rho(w)). \tag{6.2}$$

Initially the relationship (6.2) is fulfilled in some neighborhood of the point p_0 on M. However, since we can continue functions $V(p, w)$ and $V(g^{-1}(p), v)$ along any curve γ in M, it is fulfilled as an identity on universal cover $\widehat{M}$. Function $W(p,v)$ is defined as the solution of the equation $v = V(p,w)$ with respect to w for fixed p (see above). Therefore from (6.2) we derive

$$\rho(W(g^{-1}(p),v)) = W(p,v). \tag{6.3}$$

The relationship (6.3) can be rewritten as follows:

$$W(g(p),v) = \rho(W(p,v)). \tag{6.4}$$

So, each element g from first fundamental group $\pi_1(M)$ appears to be related to some function $\rho = \rho_g(w)$. From (6.4) it's easy to derive the relationship

$$\rho_{g_1\,\cdot\,g_2} = \rho_{g_1} \circ \rho_{g_1}.$$

This means that we have a representation of the group $\pi_1(M)$ by transformations of real semiaxis $\mathbb{R}^+$ given by smooth strictly monotonic increasing functions $\rho_g(w)$. Such transformations are usually called **monodromy transformations**.

Extended scalar field $a(p,v)$ on universal cover $\widehat{M}$ is obtained by lifting the corresponding scalar field a from M. Therefore

$$a(g(p),v) = a(p,v). \tag{6.5}$$

Let's differentiate the equality (6.4) with respect to v for fixed $p \in M$. This yields

$$W_v(g(p),v) = \rho'(W(p,v)) \cdot W_v(p,v). \tag{6.6}$$

If we take into account (6.5) and (6.6), then formula (5.5) can be written as

$$a = \frac{h(\rho^{-1}(W^\bullet)) \cdot \rho'(\rho^{-1}(W^\bullet))}{W^\bullet_v}, \text{ where } W^\bullet = W(g(p),v). \tag{6.7}$$

Due to the relationships (6.4) and (6.7) we can associate each element g of first fundamental group $\pi_1(M)$ with transformations of the form

$$\begin{aligned} &W(p,v) \longrightarrow \rho(W(p,v)),\\ &h(w) \longrightarrow h(\rho^{-1}(w)) \cdot \rho'(\rho^{-1}(w)), \end{aligned} \tag{6.8}$$

where $\rho = \rho_g$. In the framework of local approach transformations of the form (6.8) were obtained in thesis [17] (see § 5 in Chapter VII) as transformations changing the pair of functions (h,W), but not changing the force field $\mathbf{F}$ given by formula (1.2). They were called **gauge transformations**.

Theorem 6.1. *Suppose that M is connected Riemannian manifold equipped with Newtonian dynamical system admitting the normal shift of hypersurfaces. In this situation if f-norm of covectorial field $\mathbf{b}$ corresponding to force field $\mathbf{F}$ of such system is finite, then*

1) *the set of pairs of functions (h,W) determining $\mathbf{F}$ by formula (1.2) globally at all points of $\widehat{M}$ is not empty;*

2) *group* $\pi_1(M)$ *acts in the set of such pairs of functions by means of gauge transformations of the form* (6.8).

First proposition in theorem 6.1 is direct consequence of theorem 5.1. Second proposition of this theorem was proved above.

7. Second problem of globalization.

Let S be a hypersurface in M, and suppose that $p \in S$. Consider the following initial data for the system of equations (1.1):

$$x^k\Big|_{t=0} = x^k(p), \qquad\qquad \dot{x}^k\Big|_{t=0} = \nu(p) \cdot n^k(p). \tag{7.1}$$

Here $n^k(p)$ are components of unitary normal vector $\mathbf{n}$ to S at the point p. Initial data (7.1) define the trajectory of dynamical system (1.1) coming out from the point p in the direction of normal vector $\mathbf{n}(p)$, while the quantity $\nu(p)$ in (7.1) determines modulus of initial velocity for such trajectory.

Let's choose and fix some point $p_0 \in S$, then consider a smooth function $\nu(p)$ defined on S in some neighborhood of the point p_0. Suppose that

$$\nu(p_0) = \nu_0 > 0. \tag{7.2}$$

Then in some (possibly smaller) neighborhood of the point p_0 the function $\nu(p)$ is positive. Restricting $\nu(p)$ to such neighborhood, we use it for to determine initial velocity in (7.1). As a result we get the whole family of trajectories of dynamical system (1.1). The displacement of points of hypersurface S along such trajectories determines shift maps $f_t : S' \to S'_t$. Relying upon the theorem on existence, uniqueness, and smooth dependence on initial data for the systems of ordinary differential equations (see [26] and [27]), we can take shift maps $f_t : S' \to S'_t$ to be defined in some neighborhood S' of the point p_0 on S for all values of parameter t from some interval $(-\varepsilon, +\varepsilon)$ on real axis. At the expense of further restriction of neighborhood S' and the interval $(-\varepsilon, +\varepsilon)$ one can achieve the situation, when shift maps would become diffeomorphisms, while their images S'_t would become smooth hypersurfaces, disjoint union of which would fill some neighborhood of the point p_0 in M. Moreover, at the expense of restricting the neighborhood S' and the interval $(-\varepsilon, +\varepsilon)$ one can achieve the transversality of hypersurfaces S_t and shift trajectories at all pints of their intersection.

Definition 7.1. Shift $f_t\colon S' \to S'_t$ of a part S' of hypersurface S along trajectories of Newtonian dynamical system (1.1) is called **a normal shift** if all hypersurfaces S'_t arising in the process of shifting are perpendicular to shift trajectories.

Definition 7.2. Newtonian dynamical system (1.1) with force field $\mathbf{F}$ is called a system **admitting normal shift** in strong sense[1] if for any hypersurface S in M, for any point $p_0 \in S$, and for any real number $\nu_0 > 0$ there exists a neighborhood S'

[1]First we used the definition without normalizing condition (7.2) for the function $\nu(p)$. Such definition was called the normality condition. Definition 7.2 strengthens this condition making it more restrictive with respect to the choice of force field $\mathbf{F}$ of dynamical system (1.1). Therefore it is called strong normality condition.

of the point p_0 on S and there exits a smooth positive in S' function $\nu(p)$ normalized by the condition (7.2), and such that the shift $f_t\colon S' \to S'_t$ defined by this function is a normal shift in the sense of definition 7.1.

Definitions 1.1 and 1.2 appeared to be very fruitful. On the base of these definitions in papers [1–16] the theory of dynamical systems admitting the normal shift was constructed. However, in these definitions we observe the series restrictions making theory very local. The most displeasing is the necessity to replace whole hypersurface by by a neighborhood S' of marked point p_0. So we meet the problem of finding situations, when one could provide the possibility to define a function $\nu(p)$ and shift maps $f_t : S \to S_t$ globally on the whole hypersurface S. This problem was called **a second problem of globalization**. It was formulated by A. S. Mishchenko when I was reporting the results of thesis [17] and succeeding papers [23], [28], and [29] in his seminar at Moscow State University.

Note that second problem of globalization is closely related to the **first problem of globalization**, which was considered in paper [19]. First problem of globalization was formulated by S. E. Kozlov and Yu. R. Romanovsky when we were discussing the results of thesis [17] and succeeding papers [23], [28], and [29] in the seminar of N. Yu. Netsvetaev at Saint-Petersburg department of Steklov Mathematical Institute.

8. Choosing initial velocity in the construction of normal shift.

Let M be connected Riemannian manifold equipped with a Newtonian dynamical system (1.1) admitting the normal shift of hypersurfaces. Suppose that f-norm of covectorial field $\mathbf{b}$ corresponding to the force field $\mathbf{F}$ of this system is finite. Let's choose and fix some hypersurface S and some point p_0 on it. We choose local coordinates $u^1, \ldots, u^{n-1}$ on S in some neighborhood of the p_0 and local coordinates $x^1, \ldots, x^n$ in the manifold M in a neighborhood of the same point p_0. Without loss of generality one can assume that coordinates of the point p_0 are zero: $u^1 = \ldots = u^{n-1} = 0$ and $x^1 = \ldots = x^n = 0$. Now hypersurface S in a neighborhood of the point p_0 can be represented parametrically by the following functions:

$$\begin{aligned} x^1 &= x^1(u^1, \ldots, u^{n-1}), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ x^n &= x^n(u^1, \ldots, u^{n-1}). \end{aligned} \tag{8.1}$$

The choice of local coordinates $u^1, \ldots, u^{n-1}$ determines coordinate tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ forming a base in tangent hyperplane to hypersurface S. They can be determined by the relationships

$$\boldsymbol{\tau}_k = \sum_{i=1}^{n} \frac{\partial x^i}{\partial u^k} \cdot \frac{\partial}{\partial x^i}, \text{ where } k = 1, \ldots, n-1. \tag{8.2}$$

Second problem of globalization is related to the problem of constructing smooth positive function $\nu(p)$ on S which would be normalized by the condition (7.2) and would define the normal shift of hypersurface S along trajectories of dynamical system (1.1) by fixing the value of initial velocity in (7.1). In the framework of local approach an algorithm of constructing such function $\nu(p)$ was found in [3] and [7] (see also Chapter V of thesis [17]). Omitting details, we shall only use the fact that $\nu(p) = \nu(u^1, \ldots, u^n)$ is constructed as a solution of the equations

$$\frac{\partial\nu}{\partial u^k} = -\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_k)}{\nu}, \text{ where } k = 1, \ldots, n-1. \tag{8.3}$$

In calculating scalar product $(\mathbf{F}\,|\,\boldsymbol{\tau}_k)$ in (8.3) now we can use explicit formula (2.1) for components of force field $\mathbf{F}$. Moreover, let's take into account the relationships (8.2) which determine components of vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$. This yields

$$\frac{\partial\nu}{\partial u^k} = \sum_{i=1}^{n} b_i(x^1, \ldots, x^n, \nu)\,\frac{\partial x^i}{\partial u^k} \tag{8.4}$$

where $k = 1, \ldots, n-1$. In deriving (8.4) we take into account that at initial instant of time $t = 0$ the velocity vector $\mathbf{v}$ is directed along normal vector to S (see initial data (7.1)). Therefore $\mathbf{v} \perp \boldsymbol{\tau}_k$ and $\mathbf{N} \perp \boldsymbol{\tau}_k$. Modulus of velocity vector for $t = 0$ coincides with ν. The dependence of $x^1, \ldots, x^n$ on $u^1, \ldots, u^{n-1}$ in the equations (8.4) is determined by functions (8.1). The same functions determine partial derivatives $\partial x^i/\partial u^k$ in right hand side of these equations.

The equations (8.4) form complete system of Pfaff equations for the function ν. It is compatible. Its compatibility follows from the relationships (1.6). Normalizing condition (7.2) sets up the Cauchy problem for Pfaff equations (8.4). Such Cauchy problem has unique solution in some neighborhood of the point p_0 on S. Now we are to study whether it's possible to continue this solution to whole hypersurface S. Let's compare the equations (8.4) with the equations (4.8), for which the Cauchy problem (4.9) at the point p_0 is set up. The equations (8.4) can be treated as the restrictions of the equations (4.8) from M to S. If $V(x^1, \ldots, x^n, w)$ is the solution of Cauchy problem (4.9) for the equations (4.8), then, substituting $w = \nu_0$ and substituting the functions (8.1) for $x^1, \ldots, x^n$, we get the solution of Cauchy problem (7.2) for the equations (8.4). This fact indicates the way for solving second problem of globalization.

Theorem 8.1. *Suppose that M is connected Riemannian manifold equipped with Newtonian dynamical system admitting the normal shift of hypersurfaces. In this situation if f-norm of covectorial field $\mathbf{b}$ corresponding to force field $\mathbf{F}$ of such system is finite, then for any hypersurface S in M there is a function $\nu(p)$ normalized by the condition (7.2) such that it determines modulus of initial velocity in the construction of normal shift for S. This function is continued globally to any point $p \in S$ along any curve lying on S, though thereby it may appear to be multivalued.*

The multivalued function $\nu(p)$ may arise since function $V(p, w)$ is defined not in M, but in universal cover $\widehat{M}$. Remember that first fundamental group $\pi_1(M)$ acts as a group of discrete transformations in $\widehat{M}$. Let's define the following subgroup:

$$G_{\mathbf{F}} = \{g \in \pi_1(M) \text{ such that } W(g(p), w) \equiv W(p, w)\}.$$

Subgroup $G_{\mathbf{F}}$ is a characteristic (topological invariant) of the force field $\mathbf{F}$ in $\mathbf{M}$. It is formed by elements monodromy transformations ρ_g for which are identical.

Let S be a hypersurface in M. It is known that the immersion $S \subset M$ determines homomorphism of fundamental groups $\pi_1(S) \to \pi_1(M)$.

Theorem 8.2. *Under the assumption of theorem 8.1 the function ν on S is single-valued if and only if the image of the group $\pi_1(S)$ under the immersion homomorphism $\pi_1(S) \to \pi_1(M)$ is contained in subgroup $G_{\mathbf{F}}$.*

Note that if M is simply connected or hypersurface S is simply connected, then the condition providing univalence of ν is fulfilled.

9. Acknowledgements.

I am grateful to A. S. Mishchenko for the invitation to visit Moscow and for the opportunity to report the results of thesis [17] and succeeding papers [23], [28], and [29] in his seminar at Moscow State University. I am grateful to N. Yu. Netsvetaev for the invitation to visit Saint-Petersburg and for the opportunity to report the same results in the seminar at Saint-Petersburg department of Steklov Mathematical Institute. I am grateful to all participants of both seminars mentioned above and to my colleague E. G. Neufeld from Bashkir State University for fruitful discussions which stimulated preparing this paper.

This work is supported by grant from Russian Fund for Basic Research (project No. 00-01-00068, coordinator Ya. T. Sultanaev), and by grant from Academy of Sciences of the Republic Bashkortostan (coordinator N. M. Asadullin). I am grateful to these organizations for financial support.

References

1. Boldin A. Yu., Sharipov R. A., *Dynamical systems accepting the normal shift*, Preprint No. 0001-M of Bashkir State University, Ufa, April, 1993.
2. Boldin A. Yu., Sharipov R. A., *Dynamical systems accepting the normal shift*, Theoretical and Mathematical Physics (TMF) **97** (1993), no. 3, 386–395; see also chao-dyn/9403003 in Electronic Archive at LANL[1].
3. Boldin A. Yu., Sharipov R. A., *Multidimensional dynamical systems accepting the normal shift*, Theoretical and Mathematical Physics (TMF) **100** (1994), no. 2, 264–269; see also patt-sol/9404001 in Electronic Archive at LANL.
4. Boldin A. Yu., Sharipov R. A., *Dynamical systems accepting the normal shift*, Reports of Russian Academy of Sciences (Dokladi RAN) **334** (1994), no. 2, 165–167.
5. Sharipov R. A., *Problem of metrizability for the dynamical systems accepting the normal shift*, Theoretical and Mathematical Physics (TMF) **101** (1994), no. 1, 85–93; see also solv-int/9404003 in Electronic Archive at LANL.
6. Boldin A. Yu., Dmitrieva V. V., Safin S. S., Sharipov R. A., *Dynamical systems accepting the normal shift on an arbitrary Riemannian manifold*, Theoretical and Mathematical Physics (TMF) **105** (1995), no. 2, 256–266; see also "Dynamical systems accepting the normal shift", Collection of papers, Bashkir State University, Ufa, 1994, pp. 4–19; see also hep-th/9405021 in Electronic Archive at LANL.
7. Boldin A. Yu., Bronnikov A. A., Dmitrieva V. V., Sharipov R. A., *Complete normality conditions for the dynamical systems on Riemannian manifolds*, Theoretical and Mathematical Physics (TMF) **103** (1995), no. 2, 267–275; see also "Dynamical systems accepting the normal shift", Collection of papers, Bashkir State University, Ufa, 1994, pp. 20–30; see also astro-ph/9405049 in Electronic Archive at LANL.
8. Boldin A. Yu., *On the self-similar solutions of normality equation in two-dimensional case*, "Dynamical systems accepting the normal shift", Collection of papers, Bashkir State University, Ufa, 1994, pp. 31–39; see also patt-sol/9407002 in Electronic Archive at LANL.
9. Sharipov R. A., *Metrizability by means of conformally equivalent metric for the dynamical systems*, Theoretical and Mathematical Physics (TMF) **105** (1995), no. 2, 276–282; see also "Integrability in dynamical systems", Institute of Mathematics, Bashkir Scientific Center of Ural branch of Russian Academy of Sciences (BNC UrO RAN), Ufa, 1994, pp. 80–90.
10. Sharipov R. A., *Dynamical systems accepting normal shift in Finslerian geometry,* (November, 1993), unpublished[2].
11. Sharipov R. A., *Normality conditions and affine variations of connection on Riemannian manifolds,* (December, 1993), unpublished.
12. Sharipov R. A., *Dynamical system accepting the normal shift (report at the conference)*, see in Progress in Mathematical Sciences (Uspehi Mat. Nauk) **49** (1994), no. 4, 105.
13. Sharipov R. A., *Higher dynamical systems accepting the normal shift*, "Dynamical systems accepting the normal shift", Collection of papers, Bashkir State University, Ufa, 1994, pp. 41–65.
14. Dmitrieva V. V., *On the equivalence of two forms of normality equations in* $\mathbb{R}^n$, "Integrability in dynamical systems", Institute of Mathematics, Bashkir Scientific Center of Ural branch of Russian Academy of Sciences (BNC UrO RAN), Ufa, 1994, pp. 5–16.
15. Bronnikov A. A., Sharipov R. A., *Axially symmetric dynamical systems accepting the normal shift in* $\mathbb{R}^n$, "Integrability in dynamical systems", Institute of Mathematics, Bashkir Scientific Center of Ural branch of Russian Academy of Sciences (BNC UrO RAN), Ufa, 1994, pp. 62–69.
16. Boldin A. Yu., Sharipov R. A., *On the solution of normality equations in the dimension* $n \geqslant 3$, Algebra and Analysis (Algebra i Analiz) **10** (1998), no. 4, 37–62; see also solv-int/9610006 in Electronic Archive at LANL.

[1] Electronic Archive at Los Alamos national Laboratory of USA (LANL). Archive is accessible through Internet **http://xxx.lanl.gov**, it has mirror site **http://xxx.itep.ru** at the Institute for Theoretical and Experimental Physics (ITEP, Moscow).

[2] Papers [3–18] are arranged here in the order they were written. However, the order of publication not always coincides with the order of writing.

17. Sharipov R. A., *Dynamical systems admitting the normal shift*, Thesis for the degree of Doctor of Sciences in Russia, 1999; English version of thesis is submitted to Electronic Archive at LANL, see archive file math.DG/0002202 in the section of Differential Geometry[1].
18. Boldin A. Yu., *Two-dimensional dynamical systems admitting the normal shift*, Thesis for the degree of Candidate of Sciences in Russia, 2000; English version of thesis is submitted to Electronic Archive at LANL, see archive file math.DG/0011134 in the section of Differential Geometry.
19. Sharipov R. A., *First problem of globalization in the theory of dynamical systems admitting the normal shift of hypersurfaces*, Paper math.DG math.DG/0101150 in Electronic Archive at LANL (2001).
20. Finsler, *Über Kurven and Flachen in algemeinen Raumen*, Dissertation, Göttingen, 1918.
21. Cartan E., *Les espaces de Finsler*, Actualites 79, Paris, 1934.
22. Sharafutdinov V. A., *Integral geometry of tensor fields*, VSP, Utrecht, The Netherlands, 1994.
23. Sharipov R. A., *Newtonian normal shift in multidimensional Riemannian geometry*, Paper math.DG/0006125 in Electronic Archive at LANL (2000).
24. Kudryavtsev L. D., *Course of mathematical analysis, Vol. I, II*, "Nauka" publishers, Moscow, 1985.
25. Ilyin V. A., Sadovnichiy V. A., Sendov B. H., *Mathematical analysis*, "Nauka" publishers, Moscow, 1979.
26. Petrovsky I. G., *Lectures on the theory of ordinary differential equations*, Moscow State University publishers, Moscow, 1984.
27. Fedoryuk M. V., *Ordinary differential equations*, "Nauka" publishers, Moscow, 1980.
28. Sharipov R. A., *Newtonian dynamical systems admitting normal blow-up of points*, Paper math.DG/0008081 in Electronic Archive at LANL (2000).
29. Sharipov R. A., *On the solutions of weak normality equations in multidimensional case*, Paper math.DG/0012110 in Electronic Archive at LANL (2000).

Rabochaya street 5, 450003, Ufa, Russia
E-mail address: `R_Sharipov@ic.bashedu.ru`
`ruslan-sharipov@usa.net`
URL: `http://www.geocities.com/CapeCanaveral/Lab/5341`

[1]For the convenience of reader we give direct reference to archive file. This is the following URL address: **http://xxx.lanl.gov/eprint/math.DG/0002202**.

This figure "pst-05a.gif" is available in "gif" format from:

http://arXiv.org/ps/math/0102141v1