A note on Newtonian, Lagrangian, and Hamiltonian dynamical systems in Riemannian manifolds.

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Abstract. Newtonian, Lagrangian, and Hamiltonian dynamical systems are well formalized mathematically. They give rise to geometric structures describing motion of a point in smooth manifolds. Riemannian metric is a different geometric structure formalizing concepts of length and angle. The interplay of Riemannian metric and its metric connection with mechanical structures produces some features which are absent in the case of general (non-Riemannian) manifolds. The aim of present paper is to discuss these features and develop special language for describing Newtonian, Lagrangian, and Hamiltonian dynamical systems in Riemannian manifolds.

1. Force field of Newtonian dynamical system.

The primary and most transparent way of describing real mechanical systems is based on Newton laws. Newton's second law yields differential equation for the motion of small particle with mass m under the action of force $\mathbf{F}$:

$$m \cdot \ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}). \tag{1.1}$$

Here $\mathbf{r} = \mathbf{r}(t)$ is a vector of three-dimensional geometric space marking position of moving particle. Formally, one can consider the equation (1.1) for $\mathbf{r} \in \mathbb{R}^n$ and can take $m = 1$ for the sake of simplicity. Further one can replace $\mathbb{R}^n$ by arbitrary smooth manifold M and write the equation (1.1) in local coordinates $x^1, \ldots, x^n$:

$$\ddot{x}^k = \Phi^k(x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n), \qquad k = 1, \ldots, n. \tag{1.2}$$

Once the equations (1.2) are written, we meet the problem if interpreting these equations. If $x^1, \ldots, x^n$ are coordinates of moving point $p = p(t)$ in M, then their first derivatives $\dot{x}^1, \ldots, \dot{x}^n$ are components of velocity vector $\mathbf{v} \in T_p(M)$. But second derivatives $\ddot{x}^1, \ldots, \ddot{x}^n$ are not components of a tangent vector of $T_p(M)$. Therefore we are to consider the pair $q = (p, \mathbf{v})$ being a point of tangent bundle TM, and then write the equations (1.2) as a system of first order ODE's:

$$\begin{aligned} &\dot{x}^k = v^k, \\ &\dot{v}^k = \Phi^k(x^1, \ldots, x^n, v^1, \ldots, v^n). \end{aligned} \tag{1.3}$$

Ordinary differential equations (1.3) correspond to the following vector field in TM:

$$\mathbf{\Phi} = v^1 \cdot \frac{\partial}{\partial x^1} + \ldots + v^n \cdot \frac{\partial}{\partial x^n} + \Phi^1 \cdot \frac{\partial}{\partial v^1} + \ldots + \Phi^n \cdot \frac{\partial}{\partial v^n}. \tag{1.4}$$

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If $q=(p,\mathbf{v})$ is a point of tangent bundle TM and if $\pi\colon TM\to M$ is a map of canonical projection, then, applying associated linear map $\pi_*\colon T_q(TM)\to T_p(M)$ to the above vector (1.4), we obtain the equality

$$\pi_*\mathbf{\Phi}=\mathbf{v}. \tag{1.5}$$

Definition 1.1. Vector field $\mathbf{\Phi}$ in tangent bundle TM satisfying the condition (1.5) is called *Newtonian vector field*.

Definition 1.2. *Newtonian dynamical system* in smooth manifold M is a dynamical system determined by some Newtonian vector field in TM.

For the motion of real particle both vectors $\mathbf{v}$ and $\mathbf{F}$ are in the same space. We can measure their lengths and the angle between them. Passing to general case of n-dimensional smooth manifold M, we loose this opportunity. Indeed, vector $\mathbf{\Phi}$ is $2n$-dimensional vector tangent to TM, while $\mathbf{v}$ is n-dimensional vector tangent to M. This situation changes crucially if we take Riemannian manifold M. In this case we can consider vector $\mathbf{F}$ with components

$$F^k=\Phi^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,v^i\,v^j,\qquad k=1,\,\ldots\,,\,n. \tag{1.6}$$

It is tangent to M at the point $p=\pi(q)$. But its components (1.6) are functions of double set of arguments $x^1,\,\ldots\,,\,x^n,\,v^1,\,\ldots\,,\,v^n$. In other words, $\mathbf{F}$ is a vector in $T_p(M)$ depending on the point $q=(p,\mathbf{v})\in TM$.

Definition 1.3. *Extended* vector field $\mathbf{F}$ in M is a vector-valued function that maps each point $q\in G\subseteq TM$ to a vector of tangent space $T_p(M)$, where $p=\pi(q)$. Subset G of TM is a domain of extended vector field $\mathbf{F}$. If $G=TM$, then $\mathbf{F}$ is called *global* extended vector field in M.

Vector $\mathbf{F}$ with components (1.6) is called *force vector*. It determines *force field* of Newtonian dynamical system in Riemannian manifold. Force field $\mathbf{F}$ of Newtonian dynamical system is an extended vector field in the sense of definition 1.3. Velocity vector $\mathbf{v}$ can also be treated as extended vector field. Indeed, if $q=(p,\mathbf{v})$ is a point of tangent bundle TM, then one can map it to the vector $\mathbf{v}\in T_p(M)$. Now we can calculate modulus of velocity vector $\mathbf{v}$ and scalar product of vectors $\mathbf{v}$ and $\mathbf{F}$. In terms of force field $\mathbf{F}$ differential equations (1.3) are written as:

$$\begin{aligned}&\dot{x}^k=v^k,\\&\nabla_t v^k=F^k(x^1,\ldots,x^n,\,v^1,\ldots,v^n).\end{aligned} \tag{1.7}$$

Here $\nabla_t v^k$ are components of vector $\nabla_t\mathbf{v}$, where ∇_t is a covariant derivative with respect to time variable t along trajectory:

$$\nabla_t v^k=\dot{v}^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,v^i\,v^j. \tag{1.8}$$

Definition 1.4. *Newtonian dynamical system* in smooth Riemannian manifold M is a dynamical system determined by some extended vector field $\mathbf{F}$ in M.

Note that Newtonian dynamical systems (1.7) in Riemannian manifolds are not purely artificial objects obtained as mathematical generalizations of the equation (1.1). As shown in Chapter II of thesis [1], they arise in describing constrained mechanical systems with holonomic constraints. Riemannian metric in configuration space of such systems is given by quadratic form of *kinetic energy*.

2. Extended tensor fields.

Extended tensor fields are defined in a similar way as extended vector fields in definition 1.3. Let's denote by $T^r_s(p, M)$ the following tensor product:

$$T^r_s(p, M) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r \text{ times}} \otimes \underbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}_{s \text{ times}}$$

Tensor product $T^r_s(p, M)$ is the space of tensors of type (r, s) at the point $p \in M$.

Definition 2.1. *Extended* tensor field $\mathbf{X}$ of type (r, s) in M is a tensor-valued function that maps each point $q \in G \subseteq TM$ to a tensor of the space $T^r_s(p, M)$, where $p = \pi(q)$. Subset G of TM is a domain of extended tensor field $\mathbf{X}$. If $G = TM$, then $\mathbf{X}$ is called *global* extended tensor field in M.

Traditional tensor fields of type (r, s) in M are sections of tensor bundle $T^r_s M$. Extended tensor fields of type (r, s) are sections of pull-back tensor bundle $\pi_*(T^r_s M)$ induced by the map of canonical projection $\pi\colon TM \to M$. Below we recall some facts concerning extended tensor fields. Detailed explanation of the theory of such fields can be found in Chapters II, III, and IV of thesis [1].

The most important fact of the theory of extended tensor fields in Riemannian manifolds is the presence of two covariant differentiations

$$\nabla\colon\ T^r_s(M) \to T^r_{s+1}(M), \qquad\qquad \tilde\nabla\colon\ T^r_s(M) \to T^r_{s+1}(M).$$

First covariant differentiation ∇ is called *spatial differentiation* or *spatial gradient*. In local coordinates it is represented by formula

$$\begin{aligned}\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} &= \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^q} - \sum_{a=1}^{n}\sum_{b=1}^{n} v^a\,\Gamma^b_{qa}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^b} +\\ &+\sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{q\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{q\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned} \tag{2.1}$$

Second covariant differentiation $\tilde\nabla$ is given by much more simple formula:

$$\tilde\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^q}. \tag{2.2}$$

It is called *velocity differentiation* or *velocity gradient*. Velocity gradient $\tilde\nabla$ is defined in arbitrary smooth manifold. Unlike ∇, it doesn't require the presence of Riemannian metric in the manifold.

3. Covariant representation of extended tensor fields.

Note, that if we replace tangent bundle TM by cotangent bundle T^*M, we obtain another definition of extended tensor fields in M.

Definition 3.1. *Extended* tensor field $\mathbf{X}$ of type (r,s) in M is a tensor-valued function that maps each point $q \in G \subseteq T^*M$ to a tensor of the space $T^r_s(p,M)$, where $p=\pi(q)$. Subset G of T^*M is a domain of extended tensor field $\mathbf{X}$. If $G = T^*M$, then $\mathbf{X}$ is called *global* extended tensor field in M.

In the case of arbitrary smooth manifold M definitions 2.1 and 3.1 lead to different theories. But for Riemannian manifold M tangent bundle TM and cotangent bundle T^*M are bound with each other by duality maps

$$\mathbf{g}\colon TM \to T^*M, \qquad\qquad \mathbf{g}^{-1}\colon T^*M \to TM. \tag{3.1}$$

In local coordinates duality maps (3.1) are represented as index lowering and index raising procedures applied to the components of velocity vector $\mathbf{v}$:

$$v_a = \sum_{c=1}^{n} g_{ac}\,v^c, \qquad\qquad v^c = \sum_{a=1}^{n} g^{ca}\,v_a.$$

Due to duality maps (3.1) two objects introduced by definitions 2.1 and 2.2 are the same in essential. We call them *contravariant* and *covariant* representations of extended tensor field $\mathbf{X}$. If we take covariant representation of $\mathbf{X}$, then formula (2.1) for spatial covariant differentiation ∇ is rewritten as

$$\begin{aligned} \nabla_q X^{i_1\dots\,i_r}_{j_1\dots\,j_s} &= \frac{\partial X^{i_1\dots\,i_r}_{j_1\dots\,j_s}}{\partial x^q} + \sum_{a=1}^{n}\sum_{b=1}^{n} v_a\,\Gamma^a_{qb}\,\frac{\partial X^{i_1\dots\,i_r}_{j_1\dots\,j_s}}{\partial v_b} +\\ &+\sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{q\,a_k}\,X^{i_1\dots\,a_k\dots\,i_r}_{j_1\dots\dots\dots\,j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{q\,j_k}\,X^{i_1\dots\dots\dots\,i_r}_{j_1\dots\,b_k\dots\,j_s}. \end{aligned}\tag{3.2}$$

Formula (2.2) for velocity gradient $\tilde\nabla$ now is written as follows:

$$\tilde\nabla_q X^{i_1\dots\,i_r}_{j_1\dots\,j_s} = \sum_{k=1}^{n} g_{qk}\,\frac{\partial X^{i_1\dots\,i_r}_{j_1\dots\,j_s}}{\partial v_k}. \tag{3.3}$$

In order to make formulas (2.2) and (3.3) more similar to each other we raise index q in (3.3). As a result we get the following formula for $\tilde\nabla$:

$$\tilde\nabla^q X^{i_1\dots\,i_r}_{j_1\dots\,j_s} = \frac{\partial X^{i_1\dots\,i_r}_{j_1\dots\,j_s}}{\partial v_q}. \tag{3.4}$$

4. Differentiation along curves.

Let $p=p(t)$ be some parametric curve (e.g. the trajectory of Newtonian dynamical system (1.7)). Suppose that at each point $p(t)$ of this curve some tensor $\mathbf{X}=\mathbf{X}(t)$ of type (r,s) is given. If $\mathbf{X}(t)$ is smooth function of t, then one can differentiate it with respect to parameter t along the curve. This is done by means of covariant derivative ∇_t. As a result we get another tensor-valued function $\nabla_t\mathbf{X}$ on the curve. Its components are given by the following well-known formula:

$$\begin{aligned}\nabla_t X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\frac{dX^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{dt}+\sum^n_{q=1}\sum^r_{k=1}\sum^n_{a_k=1}\Gamma^{i_k}_{q\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}\,\dot x^q\,-\\-\sum^n_{q=1}\sum^s_{k=1}\sum^n_{b_k=1}\Gamma^{b_k}_{q\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}\,\dot x^q.\end{aligned}\tag{4.1}$$

Formula (1.8) is a special case of formula (4.1) with $\mathbf{X}=\mathbf{v}(t)$, and with time derivatives $\dot x^q$ being replaced by v^q.

Now suppose again that some curve $p=p(t)$ in M is given. Its tangent vector $\mathbf{v}$ with components $\dot x^1,\,\ldots\,,\,\dot x^n$ is vector-valued function of parameter t. Taking pairs $q=(p,\mathbf{v})$, where $p=p(t)$ and $\mathbf{v}=\mathbf{v}(t)$, we construct a parametric curve $q=q(t)$ in TM. This curve is called *natural lift* of initial curve $p=p(t)$.

Suppose that $\mathbf{X}$ is some extended tensor field of type (r,s) in M. According to the definition 2.1, it is tensor-valued function with argument $q\in TM$. Substituting $q=q(t)$ into the argument of extended tensor field $\mathbf{X}(q)$, we get tensor-valued function $\mathbf{X}(t)$. If $q=q(t)$ is natural lift of curve $p=p(t)$, then tensor-function $\mathbf{X}(t)$ is called *natural restriction* of extended tensor field $\mathbf{X}$ to the curve $p=p(t)$. Let's apply ∇_t to $\mathbf{X}(t)$. As a result for components of tensor field $\nabla_t\mathbf{X}$ we get

$$\nabla_t X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum^n_{k=1}\nabla_k X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\cdot v^k+\sum^n_{k=1}\tilde\nabla_k X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\cdot\nabla_t v^k.\tag{4.2}$$

One can easily write formula (4.2) in coordinate free form. Here it is:

$$\nabla_t\mathbf{X}=C(\nabla\mathbf{X}\otimes\mathbf{v})+C(\tilde\nabla\mathbf{X}\otimes\nabla_t\mathbf{v}).\tag{4.3}$$

Note that ∇ and $\tilde\nabla$ in right hand sides of formulas (4.2) and (4.3) are spatial and velocity gradients respectively, while C is the operation of contraction.

5. Lagrangian dynamical systems.

Lagrangian dynamical system is a special case of Newtonian dynamical system. The equations of dynamics (1.3) for them are given in implicit form by equations

$$\frac{d}{dt}\frac{\partial L}{\partial\dot x^k}-\frac{\partial L}{\partial x^k}=0,\qquad k=1,\,\ldots\,,\,n.\tag{5.1}$$

Here $L=L(x^1,\ldots,x^n,\dot x^1,\ldots,\dot x^n)$ is Lagrange function. Differential equations (5.1) are known as *Euler-Lagrange equations*. Differentiating composite function, we can rewrite Euler-Lagrange equations as follows

$$\sum^n_{s=1}\frac{\partial^2 L}{\partial v^k\,\partial v^s}\cdot\dot v^s+\sum^n_{s=1}\frac{\partial^2 L}{\partial v^k\,\partial x^s}\cdot v^s-\frac{\partial L}{\partial x^k}=0.\tag{5.2}$$

It's clear that L is a scalar function in TM. In other words, it is extended scalar field. Therefore we can rewrite (5.2) in terms of covariant differentiations determined by formulas (2.1) and (2.2). Let's use (1.8) for to express $\dot v^s$ in (5.2) through covariant derivative $\nabla_t v^s$. Then let's use formula (2.1) applied to scalar field L for to express partial derivative $\partial L/\partial x^k$ through spatial gradient $\nabla_k L$. This yields

$$\begin{aligned}&\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^k\,\partial v^s}\cdot\nabla_t v^s-\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n}\Gamma^s_{ij}\,\frac{\partial^2 L}{\partial v^k\,\partial v^s}\,v^i\,v^j\,+\\&+\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^k\,\partial x^s}\,v^s-\nabla_k L+\sum_{s=1}^{n}\sum_{j=1}^{n}v^j\,\Gamma^s_{kj}\,\frac{\partial L}{\partial v^j}=0.\end{aligned}\tag{5.3}$$

Gathering second, third, and fifth terms in (5.3) and using formulas (2.1) and (2.2), we can transform (5.3) to the following form:

$$\sum_{s=1}^{n}\tilde{\nabla}_s\tilde{\nabla}_k L\cdot\nabla_t v^s+\sum_{s=1}^{n}\nabla_s\tilde{\nabla}_k L\cdot v^s-\nabla_k L=0.\tag{5.4}$$

Now, if we recall formula (4.2), we can further simplify our equations (5.4):

$$\nabla_t\bigl(\tilde{\nabla}_k L\bigr)-\nabla_k L=0.\tag{5.5}$$

This form of Euler-Lagrange equations is quite similar to initial one. But now these equations are written in terms of covariant derivatives (2.1) and (2.2).

6. Legendre transformation.

In what case Lagrangian dynamical system determined by Euler-Lagrange equations (5.5) can be written in Newtonian form (1.7)? The answer to this question depend on the value of determinant of the matrix A with the following components:

$$A_{ij}=\tilde{\nabla}_i\tilde{\nabla}_j L=\frac{\partial L}{\partial v^i\,\partial v^j}.\tag{6.1}$$

If $\det A\neq 0$ then, using (5.4), we can express $\nabla_t v^s$ in explicit form, thus obtaining expression for the force field of corresponding Newtonian dynamical system. Lagrangian dynamical system for which the condition

$$\det A\neq 0\tag{6.2}$$

is fulfilled is called *regular*. Now suppose that L is Lagrange function for regular Lagrangian dynamical system (5.5). Then let's denote $\mathbf{p}=\tilde{\nabla}L$. It is clear that $\mathbf{p}$ is an extended covector field with components

$$p_k=\tilde{\nabla}_k L=\frac{\partial L}{\partial v^k},\qquad 1,\,\dots\,,\,n.\tag{6.3}$$

Extended covector field $\mathbf{p}$ with components (6.3) is used to define nonlinear map $\lambda\colon TM\to T^*M$. Indeed, if $q=(p,v)$ is a point of TM, then, taking $\mathbf{p}=\mathbf{p}(q)$, we can construct another pair $\tilde{q}=(p,\mathbf{p})$ being a point of T^*M:

$$\lambda(q)=\tilde{q}=(\pi(q),\mathbf{p}(q)).\tag{6.4}$$

Nonlinear map $\lambda\colon TM \to T^*M$ defined by formula (6.4) is called *Legendre transformation*. The above condition (6.2) provides local invertibility of Legendre transformation. Traditionally matrix (6.1) is assumed to be a positive matrix (see [2]):

$$A > 0. \tag{6.5}$$

This means that A is a matrix of positive quadratic form. Under the condition (6.5) Legendre transformation (6.4) is globally invertible, i.e. it is nonlinear bijective map $T_p(M) \to T_p^*(M)$ at each point $p \in M$. The whole set of maps binding tangent and cotangent bundles (including linear maps (3.1)) is shown on diagram below:

$$\begin{tikzcd}
TM \arrow[d, shift left, "\mathbf{g}"] \arrow[drr, shift left, "\lambda"] & & TM \arrow[d, shift right, "\mathbf{g}"'] \\
T^*M \arrow[u, shift left, "\mathbf{g}^{-1}"] & & T^*M \arrow[ull, shift left, "\lambda^{-1}"] \arrow[u, shift right, "\mathbf{g}^{-1}"']
\end{tikzcd}$$

In section 3 above we agreed to make no difference between covariant and contravariant representations of extended tensor fields. This means that we consider $\mathbf{X}$ and $\mathbf{X}\circ\mathbf{g}$ as two forms of the same object. Differentiations ∇ and $\tilde{\nabla}$ for these two representations of $\mathbf{X}$ are defined by formulas (2.1), (2.2), (3.2), and (3.4). Differentiations ∇ and $\tilde{\nabla}$ satisfy the following identities:

$$\nabla(\mathbf{X}\circ\mathbf{g}) = (\nabla\mathbf{X})\circ\mathbf{g}, \qquad \tilde{\nabla}(\mathbf{X}\circ\mathbf{g}) = (\tilde{\nabla}\mathbf{X})\circ\mathbf{g}. \tag{6.6}$$

Presence of nonlinear maps λ and λ^{-1} on diagram above increases the number of representations of extended tensor field $\mathbf{X}$. If $\mathbf{Y} = \mathbf{X}\circ\lambda^{-1}$, then we say that $\mathbf{Y}$ is $\mathbf{p}$-representation or *momentum* representation for $\mathbf{X}$, while $\mathbf{X}$ is $\mathbf{v}$-representation or velocity representation for $\mathbf{Y}$. For instance, ∇L is a $\mathbf{v}$-representation for covector field $\mathbf{p}$, which is called the field of *momentum*. In $\mathbf{p}$-representation components of covector field $\mathbf{p}$ are treated as independent variables $p_1, \ldots, p_n$.

Legendre transformation λ does not commute with differentiations ∇ and $\tilde{\nabla}$. Unlike (6.6), here we have the following equalities:

$$\tilde{\nabla}(\mathbf{Y}\circ\lambda) = C(\tilde{\nabla}\mathbf{Y}\circ\lambda\otimes\tilde{\nabla}\tilde{\nabla}L), \tag{6.7}$$

$$\nabla(\mathbf{Y}\circ\lambda) = \nabla\mathbf{Y}\circ\lambda + C(\tilde{\nabla}\mathbf{Y}\circ\lambda\otimes\nabla\tilde{\nabla}L). \tag{6.8}$$

In local coordinates the equalities (6.7) and (6.8) are written as follows:

$$\tilde{\nabla}_r X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum_{k=1}^{n} \tilde{\nabla}_r\tilde{\nabla}_k L\cdot\tilde{\nabla}^k Y^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}, \tag{6.9}$$

$$\nabla_r X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \nabla_r Y^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} + \sum_{k=1}^{n} \nabla_r\tilde{\nabla}_k L\cdot\tilde{\nabla}^k Y^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}. \tag{6.10}$$

Here $\mathbf{X} = \mathbf{Y} \circ \lambda$. Similar to (4.2), formulas (6.9) and (6.10) express the rule of differentiation for composite functions. They are proved by direct calculations.

Further let's define the following extended scalar field in $\mathbf{v}$-representation:

$$h = \sum_{k=1}^{n} v^k \cdot \tilde{\nabla}_k L - L. \tag{6.11}$$

Then let's take composition of h with Legendre map λ^{-1}:

$$H = h \circ \lambda^{-1}. \tag{6.12}$$

As a result we get another extended scalar field H. It is called Hamilton function. Hamilton function is traditionally used in covariant $\mathbf{p}$-representation. This means that its argument is a point $q = (p, \mathbf{p})$ of cotangent bundle T^*M:

$$H = H(x^1, \ldots, x^n, p_1, \ldots, p_n).$$

Due to (6.12) we have $h = H \circ \lambda$. First let's calculate $\nabla_i h$ directly, using formula (6.11), then let's apply formula (6.9) for the same purpose. This yields

$$\tilde{\nabla}_i h = \sum_{k=1}^{n} (\tilde{\nabla}_i v^k) \cdot \tilde{\nabla}_k L + \sum_{k=1}^{n} v^k \cdot \tilde{\nabla}_i \tilde{\nabla}_k L - \tilde{\nabla}_i L = \sum_{k=1}^{n} v^k \cdot \tilde{\nabla}_i \tilde{\nabla}_k L,$$
$$\tilde{\nabla}_i h = \sum_{k=1}^{n} \tilde{\nabla}_i \tilde{\nabla}_k L \cdot \tilde{\nabla}^k H.$$

Comparing these two formulas for $\tilde{\nabla}_i h$ and taking into account that matrix A with components (6.1) is non-degenerate, we obtain

$$v^k = \tilde{\nabla}^k H, \qquad k = 1, \ldots, n. \tag{6.13}$$

Formula (6.13) is analogous to (6.3), it yields an explicit expression for inverse Legendre map λ^{-1} in local coordinates. Moreover, this formula means that $\tilde{\nabla} H$ is a $\mathbf{p}$-representation for vector field of velocity $\mathbf{v}$.

Matrix A with components (6.1) is Jacobi matrix for Legendre map λ. Using (6.13), we can calculae components of Jacobi matrix B for inverse Legendre map:

$$B^{ij} = \tilde{\nabla}^i \tilde{\nabla}^j H. \tag{6.14}$$

Matrix B with components (6.14) inherits properties of matrix A, i. e. $\det B \neq 0$, and if $A > 0$, then $B > 0$.

Let's denote $\mathbf{p}$-representation of L by l. Then let's transform (6.11) to $\mathbf{p}$-representation. Note that $\mathbf{p}$-representation for $\mathbf{v}$ is $\tilde{\nabla} H$ and $\mathbf{p}$-representation for $\tilde{\nabla} L$ is $\mathbf{p}$. Therefore, combining (6.3), (6.13), (6.11), and (6.12), we derive

$$l = \sum_{k=1}^{n} p_k \cdot \tilde{\nabla}^k H - H. \tag{6.15}$$

Since l by definition is $\mathbf{p}$-representation of L, we can write the equality

$$L = l \circ \lambda. \tag{6.16}$$

Formulas (6.15) and (6.16) are quite similar to (6.11) and (6.12). This reflects symmetry of direct and inverse Legendre maps.

Let's calculate $\nabla_i h$. First let's do it directly, using formula (6.11). Then let's apply formula (6.10). As a result we get

$$\nabla_i h = \sum_{k=1}^{n} v^k \cdot \nabla_i \tilde{\nabla}_k L - \nabla_i L,$$

$$\nabla_i h = \nabla_i H + \sum_{k=1}^{n} \nabla_r \tilde{\nabla}_k L \cdot \tilde{\nabla}^k H.$$

Comparing these two formulas for $\nabla_i h$, let's take into account (6.13). This yields

$$\nabla_i H = -\nabla_i L. \tag{6.17}$$

In coordinate free form the equality (6.17) is written as

$$\nabla H = -(\nabla L) \circ \lambda^{-1}. \tag{6.18}$$

Formula (6.18) also reflects the symmetry of direct and inverse Legendre maps.

7. Hamiltonian dynamical systems.

Legendre transformation is used in order to write Lagrangian dynamical system (5.5) in Hamiltonian form. Note that v^k is time derivative of x^k. Therefore the above equations (6.13) are the equations of dynamics by themselves:

$$\dot{x}^k = \tilde{\nabla}^k H(x^1, \ldots, x^n, p_1, \ldots, p_n). \tag{7.1}$$

But they are not complete. In order to complete these equations (7.1) we are to calculate time derivatives for $p_1, \ldots, p_n$. Let's do it using formulas (6.3) and (4.2):

$$\nabla_t p_k = \sum_{s=1}^{n} \tilde{\nabla}_s \tilde{\nabla}_k L \cdot \nabla_t v^s + \sum_{s=1}^{n} \nabla_s \tilde{\nabla}_k L \cdot v^s. \tag{7.2}$$

Comparing (7.2) and (5.4) and using formula (6.17), we obtain

$$\nabla_t p_k = -\nabla_k H(x^1, \ldots, x^n, p_1, \ldots, p_n). \tag{7.3}$$

Now we see that the equations (7.1) and (7.3) form complete system of ODE's. They are called Hamilton equations. We gather them into a system:

$$\dot{x}^k = \tilde{\nabla}^k H, \qquad\qquad \nabla_t p_k = -\nabla_k H. \tag{7.4}$$

Hamiltonian dynamical system given by the equations (7.4) is called *regular* if matrix B with components (6.14) is non-degenerate. Each regular Hamiltonian dynamical system (7.4) is locally equivalent to some regular Lagrangian dynamical system (5.5), and vice versa, each regular Lagrangian dynamical system (5.5) is locally equivalent to some Hamiltonian dynamical system (7.4). If Legendre map defined by (6.13) is bijective, then this equivalence is global.

8. Fiberwise spherically symmetric Lagrangians.

Extended tensor field $\mathbf{X}$ is called *fiberwise spherically symmetric* if its components depend only on modulus of velocity vector:

$$X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(x^1,\,\ldots\,,\,x^n,v),\ \ \text{where}\ \ v=|\mathbf{v}|.$$

This means that $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ is spherically symmetric function within each fiber of tangent bundle TM for each fixed point $p\in M$. Such fields were considered in Chapter VII of thesis [1]. Now suppose that Lagrange function L of some Lagrangian dynamical system (5.5) is fiberwise spherically symmetric scalar field. Let's write this field in Newtonian form and let's calculate its force field $\mathbf{F}$. Components of $\mathbf{F}$ should be obtained from the equations (5.4) written as follows:

$$\sum^n_{s=1}\tilde\nabla_s\tilde\nabla_kL\cdot F^s+\sum^n_{s=1}\nabla_s\tilde\nabla_kL\cdot v^s-\nabla_kL=0. \tag{8.1}$$

Let's calculate covariant derivatives $\tilde\nabla_s\tilde\nabla_kL$ and $\nabla_s\tilde\nabla_kL$, assuming L to be fiberwise spherically symmetric. For first order derivative $\tilde\nabla_kL$ we have

$$\tilde\nabla_kL=L'\cdot\frac{v_k}{|\mathbf{v}|}, \tag{8.2}$$

Here by L' we denote partial derivative of the function $L(x^1,\ldots,x^n,v)$ with respect to its last argument v, which is interpreted as modulus of velocity vector $\mathbf{v}$. Let's apply covariant derivatives ∇_s and $\tilde\nabla_s$ to (8.2). This yields

$$\nabla_s\tilde\nabla_kL=\nabla_sL'\cdot\frac{v_k}{|\mathbf{v}|}, \tag{8.3}$$

$$\tilde\nabla_s\tilde\nabla_kL=L''\cdot\frac{v_s\,v_k}{|\mathbf{v}|^2}+\frac{L'}{|\mathbf{v}|}\cdot\left(g_{sk}-\frac{v_s\,v_k}{|\mathbf{v}|^2}\right). \tag{8.4}$$

Note that (8.4) are components of matrix A (see (6.2) above). In order to invert this matrix let's consider two operators of orthogonal projection $\mathbf{Q}$ and $\mathbf{P}$:

$$Q^i_k=\frac{v^i\,v_k}{|\mathbf{v}|^2},\qquad\qquad P^i_k=\delta^i_k-\frac{v^i\,v_k}{|\mathbf{v}|^2}. \tag{8.5}$$

First of them is a projector to the direction of velocity vector $\mathbf{v}$, second is a projector to hyperplane perpendicular to $\mathbf{v}$. Projection operators $\mathbf{Q}$ and $\mathbf{P}$ with components (8.5) are complementary to each other, this means that

$$\mathbf{Q}+\mathbf{P}=\mathbf{1}\quad\text{and}\quad\mathbf{Q}\circ\mathbf{P}=\mathbf{P}\circ\mathbf{Q}=\mathbf{0}. \tag{8.6}$$

Comparing (8.4) with (8.5), we find that

$$A_{sk} = \tilde{\nabla}_s \tilde{\nabla}_k L = L'' \cdot Q_{sk} + \frac{L'}{|\mathbf{v}|} \cdot P_{sk}. \tag{8.7}$$

Matrix A with components (8.7) is non-degenerate if and only if $L' \neq 0$ and $L'' \neq 0$ simultaneously. In this case matrix $B = A^{-1}$ has the following components:

$$B^{rk} = \frac{1}{L''} \cdot Q^{rk} + \frac{|\mathbf{v}|}{L'} \cdot P^{rk}. \tag{8.8}$$

Now, combining (8.1), (8.3), and (8.8), we derive formula for components of $\mathbf{F}$:

$$F_r = -\sum_{s=1}^{n} \left(\frac{\nabla_s L'}{L''} - \frac{\nabla_s L}{|\mathbf{v}| \cdot L''} \right) \cdot \frac{v^s \, v_r}{|\mathbf{v}|} - |\mathbf{v}| \sum_{s=1}^{n} \frac{\nabla_s L}{L'} \cdot \left(\frac{v^s \, v_r}{|\mathbf{v}|^2} - \delta^s_r \right). \tag{8.9}$$

This formula (8.9) is quite similar to the following one:

$$F_r = -|\mathbf{v}| \sum_{s=1}^{n} \frac{\nabla_s W}{W'} \cdot \left(\frac{2 \, v^s \, v_r}{|\mathbf{v}|^2} - \delta^s_r \right). \tag{8.10}$$

Here W is some fiberwise spherically symmetric scalar field with $W' \neq 0$. Force fields of the form (8.10) arise in the theory of Newtonian dynamical systems *admitting normal shift* (see Chapter VII of thesis [1]). Let's find in which case formulas (8.9) and (8.10) do coincide. This occurs if the following equations hold:

$$\frac{\nabla_s L'}{L''} - \frac{\nabla_s L}{|\mathbf{v}| \cdot L''} + \frac{\nabla_s L}{L'} = \frac{2 \, \nabla_s W}{W'}, \qquad \frac{\nabla_s L}{L'} = \frac{\nabla_s W}{W'}. \tag{8.11}$$

We consider nontrivial case, when $\nabla W \neq 0$ and $\nabla L \neq 0$. In this case equations (8.11) mean that spatial gradients of scalar fields L and L' are collinear. This occurs if and only if $L' = f(L, v)$, where $f = f(u, v)$ is some smooth function of two variables. This means that we deal with a class of functions $L = L(x^1, \ldots, x^n, v)$, each of which is a solution of partial differential equation

$$\frac{\partial L}{\partial v} = f(L, v) \tag{8.12}$$

with some particular function $f = f(u, v) \neq 0$. Note that we should not solve the equation (8.12) for particular function f. We should describe the whole set of solutions for all equations of the form (8.12). This is done by formula

$$L = \beta(C(x^1, \ldots, x^n), v). \tag{8.13}$$

Here $\beta = \beta(u, v)$ is a smooth function of two variables with $\beta'_v \neq 0$ and $\beta''_{vv} \neq 0$, while $C = C(x^1, \ldots, x^n)$ is a function of spatial variables only, i. e. this is traditional (not extended) scalar field in M. Let's substitute (8.13) into the equation

$$\frac{\nabla_s L'}{L''} - \frac{\nabla_s L}{|\mathbf{v}| \cdot L''} = \frac{\nabla_s L}{L'}.$$

derived from (8.11). This leads to the following differential equation for $\beta(u,v)$:

$$\frac{\beta''_{uv}}{\beta''_{vv}}-\frac{\beta'_u}{v\cdot\beta''_{vv}}=\frac{\beta'_u}{\beta'_v}.$$

This equation can be transformed so that it can be further integrated:

$$\frac{\beta''_{uv}}{\beta'_u}=\frac{1}{v}+\frac{\beta''_{vv}}{\beta'_v}. \tag{8.14}$$

Integrating (8.14) with respect to v, we obtain

$$\log(\beta'_u)=\log(v)+\log(\beta'_v)+\log(c),\ \text{ where }\ c=c(u). \tag{8.15}$$

Note that, ultimately, in formula (8.13) for L we substitute $u=C(x^1,\dots,x^n)$, where $C(x^1,\dots,x^n)$ is arbitrary smooth function. This means that varying function $c(u)$ in (8.15), we do not change class of Lagrange functions L. Let's choose $c(u)=1/u$ for the sake of further convenience. Then we get

$$u\cdot\beta'_u=v\cdot\beta'_v.$$

This equation is explicitly integrable. Its general solution is determined by one arbitrary smooth function of one variable $\phi=\phi(z)$:

$$\beta(u,v)=\phi(u\cdot v). \tag{8.16}$$

Substituting (8.16) into (8.13) and further into (8.9), we obtain

$$F_r=-\sum_{s=1}^{n}\frac{\nabla_s C}{C}\cdot\left(2\,v^s\,v_r-|\mathbf{v}|^2\cdot\delta^s_r\right). \tag{8.17}$$

If we substitute $C=e^{-f}$, where $f=f(x^1,\dots,x^n)$, we can rewrite (8.17) as

$$F_r=-|\mathbf{v}|^2\cdot\nabla_r f+\sum_{s=1}^{n}2\cdot(\nabla_s f\,v^s)\cdot v_r \tag{8.18}$$

Newtonian dynamical system (1.7) with force field $\mathbf{F}$ given by formula (8.18) coincides with geodesic flow of metric $\tilde{g}=e^{-2f}\cdot\mathbf{g}$, which is conformally equivalent to basic metric $\mathbf{g}$ of Riemannian manifold M. Thus we have proved a theorem.

Theorem 8.1. *Newtonian dynamical system with force field of the form* (8.10) *possess Lagrangian structure with fiberwise spherically symmetric Lagrange function L if and only if its force field is given by formula* (8.18)*, which is special case of formula* (8.10) *with $W=v\cdot e^{-f}$, where $f=f(x^1,\dots,x^n)$.*

9. Inverse problem of Lagrangian dynamics.

Theorem 8.1 is not an ultimate result concerning Lagrangian structures of Newtonian dynamical systems admitting normal shift. First reason is that formula (8.10) does not cover general case (see Chapter VII of thesis [1]). General formula for the force field of Newtonian dynamical system admitting normal shift of hypersurfaces in Riemannian manifold M with $\dim M \geqslant 3$ looks like

$$F_r = \frac{h(W)}{W'} \cdot \frac{v_r}{|\mathbf{v}|} - |\mathbf{v}| \sum_{s=1}^{n} \frac{\nabla_s W}{W'} \cdot \left(\frac{2\; v^s\, v_r}{|\mathbf{v}|^2} - \delta^s_r\right), \tag{9.1}$$

where $h = h(w)$ is an arbitrary smooth function of one variable. And second reason is that in theorem 8.1 we restrict ourselves to the case of fiberwise spherically symmetric Lagrange functions.

In order to study general case we should substitute (9.1) into (8.1) and consider (8.1) as a system of PDE's for unknown Lagrange function. Problem of determining whether the Newtonian dynamical system with a given force field $\mathbf{F}$ admits Lagrangian structure (and finding Lagrange function if it admits) is known as *inverse problem of Lagrangian dynamics*. As known to me, this problem is not solved in general case (see more details and references in [3–7]). Even for special force fields given by explicit formula (9.1) it remains unsolved. Solving this problem for force field (9.1) is very important since it would open a way for applying theory from [1] to the description of wave front dynamics and to some related problems arising in analysis of partial differential equations (see [8] and [9]).

10. Resume.

Concept of extended tensor field arisen in [10] and [11], and used in [1] for describing Newtonian dynamical systems is applicable to Lagrangian and Hamiltonian dynamical systems in Riemannian manifolds as well. It gives a method (or a language) for describing these systems in terms of their configuration space M instead of using geometric structures in tangent bundle TM (exception is Hamiltonian dynamical systems in abstract simplectic manifolds, when one cannot separate configuration space within phase space of dynamical system). As an example of applying suggested method I consider inverse problem of Lagrangian dynamics for Newtonian dynamical systems admitting normal shift, and I give partial solution of this problem in class of fiberwise spherically symmetric Lagrange functions.

References

1. Sharipov R. A., *Dynamical systems admitting the normal shift*, thesis for the degree of Doctor of Sciences in Russia, Ufa, 1999; English version of thesis is submitted to Electronic Archive at LANL[1], see archive file ArXiv:math.DG/0002202 in the section of Differential Geometry[2].
2. Arnold V. I., *Mathematical methods of classical mechanics*, Nauka publishers, Moscow, 1979.
3. Filippov V. M., Savchin V. M., Shorohov S. G., *Variational principles for non-potential operators*, Modern problems in mathematics. Recent achievements, vol. 40, VINITI, Moscow, 1992, pp. 3–176.

[1]Electronic Archive at Los Alamos National Laboratory of USA (LANL). Archive is accessible through Internet **http://arXiv.org**, it has mirror site **http://ru.arXiv.org** at the Institute for Theoretical and Experimental Physics (ITEP, Moscow).

[2]For the convenience of reader we give direct reference to archive file. This is the following URL address: **http://arXiv.org/eprint/math.DG/0002202**.

4. Morandi G., Ferrario C., Lo Vecchio G., Marmo G., Rubano C., *The inverse problem in the calculus of variations and the geometry of the tangent bundle*, Phys. Reports **188** (1990), no. 3–4, 147–284.
5. Crampin M., *On the differential geometry of the Euler–Lagrange equations and the inverse problem of Lagrangian dynamics*, Journ. of Phys. A. **14** (1981), no. 10, 2567–2575.
6. Sarlet W., *Contribution to the study of symmetries, first integrals and inverse problem of the calculus of variations in theoretical mechanics*, Acad. Analecta **49** (1987), no. 1, 27–57.
7. Carinena W. F., Lopez C., Martinez E., *A geometrical characterization of Lagrangian second-order differential equations*, Inverse Problems **5** (1989), no. 5, 691–705.
8. Fedoryuk M. V., *The equations with fast oscillating solutions*, Summaries of Science and Technology. Modern problems of Mathematics. Fundamental researches. Vol. 34, VINITI, Moscow, 1988.
9. Arnold V. I., *Singularities of caustics and wave fronts*, Phazis publishers, Moscow, 1996.
10. Finsler P., *Über Kurven and Flächen in algemeinen Raumen*, Dissertation, Göttingen, 1918.
11. Cartan E., *Les espaces de Finsler*, Actualites 79, Paris, 1934.

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