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NORMAL SHIFT IN GENERAL LAGRANGIAN DYNAMICS.

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Abstract. It is well known that Lagrangian dynamical systems naturally arise in describing wave front dynamics in the limit of short waves (which is called pseudoclassical limit or limit of geometrical optics). Wave fronts are the surfaces of constant phase, their points move along lines which are called rays. In non-homogeneous anisotropic media rays are not straight lines. Their shape is determined by modified Lagrange equations. An important observation is that for most usual cases propagating wave fronts are perpendicular to rays in the sense of some Riemannian metric. This happens when Lagrange function is quadratic with respect to components of velocity vector. The goal of paper is to study how this property transforms for the case of general (non-quadratic) Lagrange function.

1. A simple preliminary example.

Description of most wave phenomena is based on wave equation. This is second order partial differential equation of the following form:

$$\frac{1}{c^2}\,\frac{\partial^2\psi}{\partial t^2}-\sum_{i=1}^{3}\frac{\partial^2\psi}{\partial {x^i}^2}=0. \tag{1.1}$$

Here t is time variable, while x^1, x^2, and x^3 are spatial Cartesian coordinates. Parameter c in first term is the velocity of wave process described by the equation (1.1). This is sound velocity for sound waves in gases, liquids, or solid materials, and this is light velocity for light waves in refracting media. For homogeneous media c is constant, but below we consider non-homogeneous media, where $c=c(t,x^1,x^2,x^3)$.

Function $\psi=\exp(i\,(\omega\,t-k_1\,x^1-k_2\,x^2-k_3\,x^3))$ is a solution of wave equation (1.1) for the case $c=\text{const}$. It describes a plane wave. Here ω is a frequency of wave, while k_1, k_2, and k_3 are components of wave vector $\mathbf{k}$. Frequency ω and wave vector $\mathbf{k}$ are related with each other as follows:

$$\omega=c\cdot|\mathbf{k}|. \tag{1.2}$$

The relationship (1.2) is called *dispersion law*. Short wave limit corresponds to the case of high frequency, when $\omega\to\infty$. Below we consider this case for non-homogeneous media with $c\neq\text{const}$. Therefore we cannot use simple exponential solution $\psi=\exp(i\,(\omega\,t-k_1\,x^1-k_2\,x^2-k_3\,x^3))$ of wave equation (1.1). However, we can look for the exponential solution with large parameter $\lambda\to\infty$:

$$\psi=\sum_{\alpha=0}^{\infty}\frac{\psi_{(\alpha)}}{(i\,\lambda)^{\alpha}}\cdot e^{i\lambda S}. \tag{1.3}$$

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Substituting (1.3) into the equation (1.1), we get the following equation for S:

$$\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2-\sum_{i=1}^{3}\left(\frac{\partial S}{\partial x^i}\right)^2=0. \tag{1.4}$$

This is well-known *eikonal equation* (see Chapter VII in [1]). Suppose that refracting properties of medium do not change in time. Then $c=c(x^1,x^2,x^3)$. In this case we can consider a wave with constant frequency $\omega=\lambda$. For such wave function S in eikonal equation (1.4) is taken to be linear function in time variable t:

$$S=t-\varphi(x^1,x^2,x^3). \tag{1.5}$$

Eikonal equation (1.4) then is written as the equation for gradient of φ:

$$\sum_{i=1}^{3}\left(\frac{\partial\varphi}{\partial x^i}\right)^2-\frac{1}{c^2}=0. \tag{1.6}$$

Let's denote $-1/(2c^2)=U$ and then let's write the equation (1.6) as

$$\sum_{i=1}^{3}\frac{(\nabla_i\varphi)^2}{2}+U(x^1,x^2,x^3)=0. \tag{1.7}$$

Here $\nabla_1\varphi$, $\nabla_2\varphi$, and $\nabla_3\varphi$ are components of gradient $\nabla\varphi$. We denote it by $\mathbf{p}$ and treat as a vector field in tree-dimensional space $\mathbb{R}^3$:

$$\mathbf{p}=\nabla\varphi=\begin{Vmatrix}\partial\varphi/\partial x^1\\ \\ \partial\varphi/\partial x^2\\ \\ \partial\varphi/\partial x^3\end{Vmatrix} \tag{1.8}$$

If we substitute components of vector (1.8) into (1.7), then we can write (1.7) as

$$H(\nabla_1\varphi,\nabla_2\varphi,\nabla_3\varphi,x^1,x^2,x^3)=0, \tag{1.9}$$

where function $H=H(p_1,p_2,p_3,x^1,x^2,x^3)$ looks like Hamilton function of a particle of unit mass $m=1$ in potential field $U=U(x^1,x^2,x^3)$:

$$H=\sum_{i=1}^{3}\frac{(p_i)^2}{2}+U. \tag{1.10}$$

Let $\varphi=\varphi(x^1,x^2,x^3)$ be a solution of the equation (1.7) and let $\mathbf{p}=\nabla\varphi$ be corresponding momentum vector field (1.8). Let's consider integral curves of vector field $\mathbf{p}$. They form two-parametric family of curves in $\mathbb{R}^3$

$$\begin{cases}x^1=x^1(t,y^1,y^2),\\ x^2=x^2(t,y^1,y^2),\\ x^3=x^3(t,y^1,y^2)\end{cases} \tag{1.11}$$

defined by solutions of the following system of ordinary differential equations:

$$\dot{x}^1 = p_1, \qquad \dot{x}^2 = p_2, \qquad \dot{x}^3 = p_3. \tag{1.12}$$

Differential equations (1.12) can be written as Hamilton equations:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad i = 1,\, 2,\, 3. \tag{1.13}$$

Now let's calculate time derivative $\dot{\mathbf{p}}$ for the momentum vector (1.8) due to the dynamics determined by differential equations (1.12):

$$\dot{p}_i = \sum_{k=1}^{3} \frac{\partial p_i}{\partial x^k} \cdot \dot{x}^k = \sum_{k=1}^{3} \frac{\partial^2 \varphi}{\partial x^i\, \partial x^k} \cdot \dot{x}^k = \sum_{k=1}^{3} \frac{\partial H}{\partial p_k} \cdot \frac{\partial^2 \varphi}{\partial x^i\, \partial x^k}.$$

Remember that φ is a solution of the equation (1.9). Differentiating (1.9), we get

$$\frac{\partial H(\nabla_1\varphi, \nabla_2\varphi, \nabla_3\varphi, x^1, x^2, x^3)}{\partial x^i} = \sum_{k=1}^{3} \frac{\partial H}{\partial p_k} \cdot \frac{\partial^2 \varphi}{\partial x^i\, \partial x^k} + \frac{\partial H}{\partial x^i} = 0.$$

Comparing the above two equalities, we derive differential equations

$$\dot{p}_i = -\frac{\partial H}{\partial x^i}, \qquad i = 1,\, 2,\, 3. \tag{1.14}$$

Both (1.13) and (1.14) form complete system of Hamilton equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i} \tag{1.15}$$

with Hamilton function (1.10).

Note that Hamilton equations (1.15) is a system of 6 first order ODE's. Its solutions define five-parametric family of curves in $\mathbb{R}^3$:

$$\begin{cases} x^1 = x^1(t, y^1, y^2, y^3, y^4, y^5),\\ x^2 = x^2(t, y^1, y^2, y^3, y^4, y^5),\\ x^3 = x^3(t, y^1, y^2, y^3, y^4, y^5). \end{cases} \tag{1.16}$$

Curves (1.11) form two-parametric subfamily in five-parametric family of curves (1.16). They are distinguished by the following two properties:

1) curves (1.11) correspond to zero level of energy $H = 0$;
2) curves (1.11) are perpendicular to level surfaces of the function $\varphi(x^1, x^2, x^3)$.

First property follows from (1.9). Second is obvious, since curves (1.11) are directed along gradient vector (1.8). One can calculate complete derivative of the function $\varphi(x^1, x^2, x^3)$ with respect to parameter t along these curves:

$$\frac{d\varphi}{dt} = \Omega = \sum_{i=1}^{3} p_i\, \frac{\partial H}{\partial p_i}. \tag{1.17}$$

Curves (1.16) defined by Hamilton equations (1.15) and restricted by the above conditions 1) and 2) are called *characteristic lines* for nonlinear first order partial differential equation (1.9). They are used in order to construct solutions of this equation as described just below (see also [2] and [3]).

Let's take some smooth surface σ in $\mathbb{R}^3$. We assume that σ is level surface with $\varphi = 0$ for the solution $\varphi(x^1, x^2, x^3)$ of the equation (1.9) that we are going to construct. Denote by y^1 and y^2 inner curvilinear coordinates of points on σ. Then we can write the equations determining points of σ in parametric form:

$$\begin{cases} x^1 = x^1(y^1, y^2), \\ x^2 = x^2(y^1, y^2), \\ x^3 = x^3(y^1, y^2). \end{cases} \tag{1.18}$$

At each point of σ we have unit normal vector $\mathbf{n}$. Let's denote it by $\mathbf{n} = \mathbf{n}(y^1, y^2)$. Assuming σ to be orientable, we can take $\mathbf{n}(y^1, y^2)$ to be smooth function of y^1 and y^2. Under these assumptions we define vector function

$$\mathbf{p} = \nu \cdot \mathbf{n} \tag{1.19}$$

on σ, getting scalar factor $\nu = \nu(y^1, y^2)$ from the following equality:

$$H(\nu\, n_1,\, \nu\, n_2,\, \nu\, n_3, x^1, x^2, x^3) = 0. \tag{1.20}$$

Then we use vector function (1.19) in order to set up Cauchy problem

$$x^i\Big|_{t=0} = x^i(y^1, y^2), \qquad\qquad p_i\Big|_{t=0} = p_i(y^1, y^2) \tag{1.21}$$

for Hamilton equations (1.15). Solving this Cauchy problem (1.21), we obtain two-parametric family of characteristic lines given by functions (1.11) that extend initial functions (1.18). They possess property 1), since we determine ν by (1.20). They also possess property 2), since we determine $\mathbf{p}$ by (1.19) (at least for initial surface σ). These characteristic lines fill some neighborhood of initial surface σ. Therefore we can treat t, y^1, y^2 as curvilinear coordinates in $\mathbb{R}^3$ and consider (1.11) as transition functions to these curvilinear coordinates. Then integral

$$\varphi = \int\limits_0^t \Omega\, dt, \tag{1.22}$$

where Ω is given by right hand side of (1.17), yields a solution of partial differential equation (1.9) expressed in curvilinear coordinates t, y^1, and y^2. This solution (1.22) satisfies zero boundary-value condition on σ:

$$\varphi\Big|_{\sigma} = 0. \tag{1.23}$$

In other words, (1.23) means that σ is zero level surface for the function φ.

Note that in curvilinear coordinates t, y^1, y^2 initial surface σ is given by the equation $t = 0$. However, other level surfaces of the function φ are not given by the equations $t = \text{const}$. In order to change this situation we should choose another set of curvilinear coordinates s, y^1, y^2, where $s = \varphi(t, y^1, y^2)$. This means, that we change parametrization of characteristic lines (1.11) without changing them as geometric sets of points. In new parameter s characteristic lines of the equation (1.9) are given by modified Hamilton equations

$$\dot{x}^i = \frac{1}{\Omega}\,\frac{\partial H}{\partial p_i}, \qquad\qquad \dot{p}_i = -\frac{1}{\Omega}\,\frac{\partial H}{\partial x^i}, \tag{1.24}$$

where denominator Ω is determined by right hand side of (1.17). In our particular case, when function H is given by formula (1.10), we have $\Omega = (p_1)^2 + (p_2)^2 + (p_3)^2$. Hence $\Omega \neq 0$ for $\mathbf{p} \neq 0$.

Let's fix new curvilinear coordinates s, y^1, y^2. Here $\varphi(s, y^1, y^2) = s$ by definition. Now let's return to initial wave equation (1.1) and to formula (1.5) for the function S in asymptotical power expansion (1.3). It's important to note that t in (1.5) do not coincide with t in (1.11) and in Hamilton equations (1.15), where t was used as a parameter on characteristic lines of the equation (1.6). Therefore now in the expression for S we have both s and t (and t is time variable again):

$$S = S(t, s, y^1, y^2) = t - s.$$

For exponential factor $e^{i\lambda S}$ in (1.3), taking into account that $\lambda = \omega$, we get:

$$e^{i\lambda S} = e^{i\omega(t-s)}. \tag{1.25}$$

Right hand side of (1.25) corresponds to plane wave propagating in the direction of s-axis. In original Cartesian coordinates x^1, x^2, x^3 this looks like non-plain wave propagating along characteristic lines of the equation (1.9). Level surfaces of the function φ are the surfaces of constant phase in such wave. They are called *wave fronts*. The equation $t - s = \text{const}$, when transformed to Cartesian coordinates x^1, x^2, x^3, describes moving surface, that gradually passes positions of level surfaces of the function φ. This process is called wave front dynamics. It's very important that this process can be understood as a **motion of separate points of wave front**, each obeying modified Hamilton equations (1.24). For this reason these equations are called **the equations of wave front dynamics**.

Another important point concerning wave front dynamics, that we noted above, is that level surfaces of the function φ are perpendicular to characteristic lines (1.11). Therefore wave front dynamics is a normal displacement (or **normal shift**) of initial surface σ along trajectories of modified Hamiltonian dynamical system.

2. More complicated example.

Let M be some Riemannian manifold. Denote by ∇ standard covariant differentiation determined by metric connection Γ in M. The following differential operator is called Laplace-Beltrami operator in the manifold M:

$$\triangle = \sum_{i=1}^{n}\sum_{j=1}^{n} g^{ij}\,\nabla_i\,\nabla_j. \tag{2.1}$$

Here g^{ij} are components of metric tensor in local coordinates $x^1, \ldots, x^n$, while ∇_i and ∇_j are symbols of covariant derivatives in these local coordinates. Differential operator H is called *fiberwise spherically symmetric* if it is represented as a polynomial of Laplace-Beltrami operator (2.1):

$$H(p, D) = \sum_{k=0}^{m} a_k(p) \,\triangle^k. \tag{2.2}$$

Here p is a point of M and D is a formal symbol for differentiation. Coefficients $a_0, \ldots, a_m$ in (2.2) are arbitrary smooth functions of $p \in M$. Note that H is scalar operator. Operator (2.2) can be applied either to scalar field or tensorial field in M, yielding the field of the same type as that it was applied to.

Now let's add differentiation in time variable $\partial_t = \partial/\partial t$ and let's introduce large parameter λ to (2.2). As a result we get differential operator

$$H(p, \lambda^{-1} D) = \sum_{s=0}^{m} \sum_{k=0}^{m} \frac{a_{sk}(p)}{(i\,\lambda)^{s+2k}} \,{\partial_t}^s \,\triangle^k.$$

The following differential equation in M is an analog of wave equation (1.1):

$$H(p, \lambda^{-1} D)\psi = 0. \tag{2.3}$$

Short wave asymptotics $\lambda \to \infty$ for this equations is described by the same asymptotical expansion (1.3) as in case of standard wave equation. Coefficients $a_{sk}(p)$ in (2.3) do not depend on t. Therefore we can choose S to be linear function of t:

$$S = t - \varphi(p), \tag{2.4}$$

just like it was in (1.5). Substituting (2.4) into (1.3) and substituting (1.3) into (2.3), we derive differential equation for phase function $\varphi(p)$ in (2.4):

$$\sum_{k=0}^{m} \left(\sum_{s=0}^{m} a_{sk}(p) \right) \cdot |\nabla\varphi|^{2k} = \sum_{k=0}^{m} b_k(p) \cdot |\nabla\varphi|^{2k} = 0. \tag{2.5}$$

Here $|\nabla\varphi|$ is modulus of covector field $\nabla\varphi$ measured in Riemannian metric $\mathbf{g}$:

$$|\nabla\varphi|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} g^{ij} \,\nabla_i\varphi \,\nabla_j\varphi.$$

Let's denote $\nabla\varphi$ by $\mathbf{p}$ as it was done above in section 1 (see formula (1.8)):

$$\mathbf{p} = \nabla\varphi = \begin{Vmatrix} \partial\varphi/\partial x^1 \\ \vdots \\ \partial\varphi/\partial x^n \end{Vmatrix}.$$

Now we can write (2.5) as polynomial equation with respect to components of $\mathbf{p}$:

$$\sum_{k=0}^{m} b_k(p) \cdot |\mathbf{p}|^{2k} = 0. \tag{2.6}$$

Let's denote left hand side of (2.6) by $H = H(p, \mathbf{p})$. The equation (2.5), written as

$$H(p, \nabla\varphi) = 0, \tag{2.7}$$

is exact analog of the equation (1.9) from section 1. Further steps in solving this equation are quite similar to those in section 1 (they are described in details in paper [4]). Below we shall not discuss them. However, we shall point out most important features of wave front dynamics in the limit of short waves $\lambda = \omega \to \infty$ for generalized wave equation (2.3). They are the following ones:

— wave fronts are level hypersurfaces $\sigma_t = \{p \in M : \ \varphi(p) = t\}$ for the function $\varphi(p)$, where $\varphi(p)$ is a solution of differential equation (2.7);
— time evolution of wave fronts σ_t in M can be described in terms of motion of their points obeying modified Hamilton equations

$$\dot{x}^i = \frac{1}{\Omega}\,\frac{\partial H}{\partial p_i}, \qquad\qquad \dot{p}_i = -\frac{1}{\Omega}\,\frac{\partial H}{\partial x^i}, \tag{2.8}$$

where Hamilton function H is determined by left hand side of (2.6)

$$H(p, \mathbf{p}) = \sum_{k=0}^{m} b_k(p) \cdot |\mathbf{p}|^{2k}, \tag{2.9}$$

and denominator Ω in (2.8) is determined by formula

$$\Omega = \sum_{i=1}^{n} p_i\,\frac{\partial H}{\partial p_i};$$

— wave front dynamics for wave equation (2.3) in short wave limit $\lambda \to \infty$ is a normal shift of of initial wave front hypersurface σ along trajectories of modified Hamiltonian dynamical system (2.8), this means that orthogonality of wave fronts σ_t and trajectories of shift is preserved in time;
— normal shift of hypersurface σ is initiated by Cauchy problem data

$$x^i\,\Bigr|_{t=0} = x^i(p), \qquad\qquad p_i\,\Bigr|_{t=0} = \nu(p) \cdot n_i(p) \tag{2.10}$$

for the equations (2.8), where $n_i(p)$ are covariant components of normal vector $\mathbf{n}(p)$ at the point $p \in \sigma$, and $\nu(p)$ is a scalar factor determined by the equation

$$H(p, \nu \cdot \mathbf{n}(p)) = 0. \tag{2.11}$$

For us the most important feature of wave front dynamics, among those listed above, is the phenomenon of normal shift. It was revealed in simplest case considered in section 1. It is also present in more complicated case related to some Riemannian metric. Our aim below is to reveal this phenomenon for the case of general Hamilton function H, which is not restricted by formula (2.9). In order to do this we need to introduce geometrical technique, which is not new, but nevertheless, is not commonly known. It seems to me, that this technique first appeared in Finslerian geometry (see [5] and [6]). We used this technique in [7–22], where theory of Newtonian dynamical systems admitting normal shift was developed (see also theses [23], [24], and recent papers [25–29]).

3. Extended tensor fields.

Let's consider Hamilton function (2.9). It depends on two arguments p and $\mathbf{p}$, where p is a point of manifold M, while $\mathbf{p}$ is cotangent vector at the point p, i.e. $\mathbf{p}$ is an element of cotangent space $T^*_p(M)$. Both p and $\mathbf{p}$, taken together, form a pair $q = (p, \mathbf{p})$ which is a point of cotangent bundle T^*M. This means that H is a scalar field in cotangent bundle T^*M. But we shall treat it as *extended scalar field* in M as defined below. Let's consider the following tensor product:

$$T^r_s(p, M) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r \text{ times}} \otimes \underbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}_{s \text{ times}}$$

Tensor product $T^r_s(p, M)$ is known as a space of (r, s)-tensors at the point $p \in M$. Pair of integer numbers (r, s) determines type of tensors. Elements of $T^r_s(p, M)$ are called *r-times contravariant and s-times covariant tensors* or simply (r, s)-tensors.

Definition 3.1. *Extended* tensor field $\mathbf{X}$ of type (r, s) in M is a tensor-valued function that maps each point $q = (p, \mathbf{p})$ of some domain $G \subseteq T^*M$ to a tensor of the space $T^r_s(p, M)$. If $G = T^*M$, then $\mathbf{X}$ is called *global* extended tensor field.

Note a trick: arguments of extended tensor fields belong to cotangent bundle T^*M, while their values are tensors related to base manifold M. If we replace T^*M by tangent bundle TM, we can state another definition of extended tensor field.

Definition 3.2. *Extended* tensor field $\mathbf{X}$ of type (r, s) in M is a tensor-valued function that maps each point $q = (p, \mathbf{p})$ of some domain $G \subseteq TM$ to a tensor of the space $T^r_s(p, M)$. If $G = TM$, then $\mathbf{X}$ is called *global* extended tensor field.

In the case of arbitrary smooth manifold M definitions 3.1 and 3.2 lead to different theories. But for Riemannian manifold M tangent bundle TM and cotangent bundle T^*M are bound with each other by duality maps:

$$\mathbf{g}\colon TM \to T^*M, \qquad\qquad \mathbf{g}^{-1}\colon T^*M \to TM. \tag{3.1}$$

In local coordinates duality maps (3.1) are represented as index lowering and index raising procedures in arguments of extended tensor field $\mathbf{X}$:

$$p_i = \sum_{j=1}^{n} g_{ij}\, p^j, \qquad\qquad p^i = \sum_{j=1}^{n} g^{ij}\, p_j.$$

Due to duality maps (3.1) two objects introduced by definitions 3.1 and 3.2 are the same in essential. We call them *covariant* and *contravariant* representations of extended tensor field $\mathbf{X}$.

In local coordinates $x^1, \ldots, x^n$ extended tensor field $\mathbf{X}$ is represented by its components $X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}(x^1, \ldots, x^n, p_1, \ldots, p_n)$ or $X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}(x^1, \ldots, x^n, p^1, \ldots, p^n)$, depending on which representation (covariant or contravariant) is used. Extended tensor field $\mathbf{X}$ is called *smooth* if its components are smooth functions.

Smooth extended tensor fields form a ring, we denote it by $\mathfrak{F} = \mathfrak{F}(T^*M)$ in the case of covariant representation, and by $\mathfrak{F} = \mathfrak{F}(TM)$ in the case of contravariant representation. The whole set of smooth extended tensor fields in M is equipped with operations of 1) summation, 2) multiplications by scalars, 3) tensor product, 4) contraction. It forms bi-graded algebra over the ring $\mathfrak{F}$. We denote this algebra by $\mathbf{T}(M)$ and call it *an algebra of extended tensor fields* in M:

$$\mathbf{T}(M)=\bigoplus_{r=0}^{\infty}\bigoplus_{s=0}^{\infty}T^r_s(M). \tag{3.2}$$

Definition 3.3. A map $D\colon\mathbf{T}(M)\to\mathbf{T}(M)$ is called a *differentiation* of extended algebra of tensor fields, if the following conditions are fulfilled:

(1) concordance with grading: $D(T^r_s(M))\subset T^r_s(M)$;
(2) $\mathbb{R}$-linearity: $D(\mathbf{X}+\mathbf{Y})=D(\mathbf{X})+D(\mathbf{Y})$ and $D(\lambda\mathbf{X})=\lambda D(\mathbf{X})$ for $\lambda\in\mathbb{R}$;
(3) commutation with contractions: $D(C(\mathbf{X}))=C(D(\mathbf{X}))$;
(4) Leibniz rule: $D(\mathbf{X}\otimes\mathbf{Y})=D(\mathbf{X})\otimes\mathbf{Y}+\mathbf{X}\otimes D(\mathbf{Y})$.

Theory of differentiations in extended algebra of tensor fields (3.2) is considered in Chapters II, III, and IV of thesis [23]. In this section below we shall mention some facts from this theory needed for further use.

Suppose that $\mathbf{T}(M)$ is extended algebra of tensor fields in M taken in contravariant representation. Then the set of its differentiations $\mathfrak{D}(M)$ possesses the structure of module over the ring $\mathfrak{F}(TM)$. The set of extended vector fields (i. e. summand $T^1_0(M)$ in direct sum (3.2)) also possesses the structure of $\mathfrak{F}(TM)$-module. Therefore the following definition is consistent.

Definition 2.1. *Covariant differentiation* ∇ in the algebra of extended tensor fields $\mathbf{T}(M)$ is a homomorphism of $\mathfrak{F}(TM)$-modules $\nabla: T^1_0(M)\to\mathfrak{D}(M)$. Image of vector field $\mathbf{Y}$ under such homomorphism denoted by $\nabla_{\mathbf{Y}}$ is called *covariant differentiation along vector field Y.*

For each covariant differentiation the expression $\nabla_{\mathbf{Y}}\mathbf{X}$ is $\mathfrak{F}(TM)$-linear with respect to $\mathbf{Y}$. Therefore ∇ can be treated as a map $\nabla: T^r_s(M)\to T^r_{s+1}(M)$. Each smooth manifold M possesses exactly one canonical covariant differentiation $\tilde\nabla$ which is called *vertical gradient.* In local coordinates it is expressed by formula

$$\tilde\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial p^q}. \tag{3.3}$$

In order to define other covariant differentiations one need some additional geometric structures in M. Thus, if M possesses affine connection Γ, one can define *horizontal gradient* ∇. In local coordinates it is expressed by formula

$$\begin{aligned}\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}&=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^q}-\sum_{a=1}^{n}\sum_{b=1}^{n}p^a\,\Gamma^b_{qa}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial p^b}+\\&+\sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{q\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{q\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned}\tag{3.4}$$

If we take covariant representation of the algebra of extended tensor fields $\mathbf{T}(M)$, then formulas for vertical and horizontal gradients are transformed as follows:

$$\tilde{\nabla}^q X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial p_q}. \tag{3.5}$$

$$\begin{gathered}\nabla_q X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial x^q} + \sum_{a=1}^{n}\sum_{b=1}^{n} p_a\,\Gamma^a_{qb}\,\frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial p_b} +\\ + \sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{q\,a_k}\,X^{i_1\ldots\, a_k\ldots\, i_r}_{j_1\ldots\ldots\ldots\, j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{q\,j_k}\,X^{i_1\ldots\ldots\ldots\, i_r}_{j_1\ldots\, b_k\ldots\, j_s}.\end{gathered} \tag{3.6}$$

In the case of arbitrary smooth manifold M gradients defined by formulas (3.3) and (3.4) are not related to those defined by formulas (3.5) and (3.6). However, if M is Riemannian manifold, then $\tilde{\nabla}$ and ∇ defined by these two ways appear to be the same differentiations[1] in different representations of algebra $\mathbf{T}(M)$. This fact is expressed by the following commutation relationships:

$$\nabla(\mathbf{X}\circ\mathbf{g}) = (\nabla\mathbf{X})\circ\mathbf{g}, \qquad\qquad \tilde{\nabla}(\mathbf{X}\circ\mathbf{g}) = (\tilde{\nabla}\mathbf{X})\circ\mathbf{g}.$$

Here $\mathbf{g}$ is duality map (3.1) defined by metric tensor of Riemannian manifold.

4. Legendre transformation.

Legendre transformation is usually used to relate Lagrangian and Hamiltonian dynamical systems. Suppose that M is smooth manifold and let $L(p,\mathbf{v})$ be smooth extended scalar field in M taken in contravariant representation. Then dynamical system in tangent bundle TM described by differential equations

$$\dot{x}^i = v^i, \qquad\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial x^i} \tag{4.1}$$

is called Lagrangian dynamical system. Let's apply covariant derivative (3.5) to L. As a result we get covector field $\mathbf{p} = \mathbf{p}(p,\mathbf{v})$ with components

$$p_i = \tilde{\nabla}_i L = \frac{\partial L}{\partial v^i}. \tag{4.2}$$

If pair $(p,\mathbf{v})$ is a point of tangent bundle TM, then pair $(p,\mathbf{p})$ is a point of cotangent bundle T^*M. This means that derivatives (4.2) determine a map

$$\lambda\colon TM \to T^*M. \tag{4.3}$$

This map is known as Legendre transformation (see [30]). Below we assume Legendre transformation (4.3) to be invertible. Moreover we assume inverse map

$$\lambda^{-1}\colon T^*M \to TM \tag{4.4}$$

to be smooth. Under these assumptions we can treat direct and inverse Legendre transformations (4.3) and (4.4) as nonlinear analogs of duality maps (3.1). Lagrange

[1] One should only lower index q in (3.5).

function $L(p, \mathbf{v})$ and Lagrange equations (4.1) are associated with tangent bundle TM. Vector $\mathbf{v}$ in arguments of Lagrange function is called *velocity vector*, while covector $\mathbf{p}$ with components (4.2) is called *momentum covector*. This gives rise to the following terminology. If $\mathbf{X}$ is an extended tensor field in contravariant representation and if $\mathbf{Y} = \mathbf{X} \circ \lambda^{-1}$, then we say that $\mathbf{Y}$ is $\mathbf{p}$-representation or *momentum representation* for $\mathbf{X}$, while $\mathbf{X}$ is called $\mathbf{v}$-representation or *velocity representation* for $\mathbf{Y}$. For the case of general smooth manifold M (without Riemannian metric) direct and inverse Legendre transformations (4.3) and (4.4) bind the following two representations of extended tensor fields:

covariant **p**-representation	contravariant **v**-representation

If M is Riemannian manifold, we have four representations per each extended field:

covariant **p**-representation	contravariant **v**-representation
contravariant **p**-representation	covariant **v**-representation

Now let's consider the following two extended scalar fields h and H:

$$h = \sum_{i=1}^{n} v^i\,\tilde{\nabla}_i L - L, \qquad\qquad H = h \circ \lambda^{-1}. \tag{4.5}$$

Scalar field H is known as Hamilton function, while h is its $\mathbf{v}$-representation. Lagrange function L and its $\mathbf{p}$-representation $l = L \circ \lambda^{-1}$ can be expressed through Hamilton function H by formulas similar to (4.5):

$$l = \sum_{i=1}^{n} p_i\,\tilde{\nabla}^i H - H, \qquad\qquad L = l \circ \lambda. \tag{4.6}$$

Applying Legendre transformation to Lagrangian dynamical system (4.1), we can transform them to Hamiltonian dynamical system in cotangent bundle T^*M:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}.$$

This fact is well known (see [30]), as well as above formulas (4.5) and (4.6).

5. Modified Hamiltonian and Lagrangian dynamical systems in Riemannian manifolds.

Now suppose that manifold M is equipped with Riemannian metric $\mathbf{g}$. All results we discussed in section 2 were obtained under this assumption. Remember that wave front dynamics is described by modified Hamilton equations (2.8). Using (3.5) and (3.6), we can replace partial derivatives in them by covariant derivatives:

$$\dot{x}^i = \frac{\tilde{\nabla}^i H}{\Omega}, \qquad\qquad \nabla_t p_i = -\frac{\nabla_i H}{\Omega}, \tag{5.1}$$

Time derivatives $\dot{x}^1, \ldots, \dot{x}^n$ in (5.1) are components of tangent vector to trajectory $p = p(t)$, while ∇_t is standard covariant derivative with respect to parameter t along this curve. Similar to original Hamilton equations (4.5), modified Hamilton equations are associated with cotangent bundle T^*M. Using inverse Legendre map (4.4), one can transform them to **v**-representation. This was done in paper [4]. As a result modified Lagrange equations were obtained:

$$\dot{x}^i = \frac{v^i}{\Omega}, \qquad\qquad \nabla_t(\tilde{\nabla}_i L) = \frac{\nabla_i L}{\Omega}. \tag{5.2}$$

Denominator Ω in original **p**-representation is given by formula

$$\Omega = \sum_{i=1}^{n} p_i\, \tilde{\nabla}^i H, \tag{5.3}$$

Upon passing to **v**-representation in (5.2) formula (5.3) transforms to

$$\Omega = \sum_{i=1}^{n} v^i\, \tilde{\nabla}_i L. \tag{5.4}$$

Formula (5.4) means that we can express Ω in **v**-representation explicitly through Lagrange function L.

6. Newtonian dynamical systems admitting normal shift in Riemannian manifolds.

One can see that vector **v** for modified Lagrangian dynamics (5.2) do not coincide with actual velocity vector. If we denote actual velocity vector by **u**, then we derive

$$u^i = \frac{v^i}{\Omega}. \tag{6.1}$$

Formula (6.1) defines nonlinear map similar to Legendre map λ:

$$\mu\colon TM \to TM \tag{6.2}$$

If this map (6.2) is invertible and if inverse map μ^{-1} is smooth, then one can transform modified Lagrange equations (5.2) to the following form:

$$\dot{x}^i = u^i, \qquad\qquad \nabla_t u^i = F^i(x^1, \ldots, x^n, u^1, \ldots, u^n). \tag{6.3}$$

Differential equations (6.3) determine Newtonian dynamical system in M. Extended vector field **F** with components $F^1, \ldots, F^n$ in (6.3) is called *force field* of this Newtonian dynamical system.

Note that nonlinear map (6.2) is more complicated than Legendre map. Almost each modified Lagrangian dynamical system can be transformed to Newtonian form (at least locally). However, converse is not true. Moreover, even if it is known that Newtonian dynamical system (6.3) is derived from modified Lagrangian dynamical system (5.2), there is no explicit formula for **F**. In paper [4] one can find explicit formula for $\mathbf{F}$, but in very special case, when Lagrange function L is *fiberwise spherically symmetric* with respect to Riemannian metric $\mathbf{g}$:

$$F_k = -|\mathbf{u}| \cdot \sum_{i=1}^{n} \frac{\nabla_i W}{W'} \cdot \left(2\, N^i\, N_k - \delta^i_k\right). \tag{6.4}$$

Here $W = W(p, |\mathbf{u}|) = h \circ \mu^{-1}$ is $\mathbf{u}$-representation for Hamilton function H and W' is partial derivative of W with respect to its $(n+1)$-th argument $u = |\mathbf{u}|$, N^i and N_k are components of unit vector $\mathbf{N} = \mathbf{u}/|\mathbf{u}|$.

Remember that in the example considered in section 2 Hamilton function H is given by formula (2.9). It is fiberwise spherically symmetric, i. e. $H = H(p, |\mathbf{p}|)$. Applying inverse Legendre map we get fiberwise spherically symmetric function $h = h(p, |\mathbf{v}|) = H \circ \lambda$. Its $\mathbf{u}$ representation then is fiberwise spherically symmetric function W in formula (6.4). Thus we have the following theorem.

Theorem 6.1. *For generalized wave equation* (2.3) *in Riemannian manifold M wave front dynamics in the limit of short waves is described by Newtonian dynamical system* (6.3) *with force field* (6.4).

Theorem 6.1 is the main result of paper [4]. It establishes a link between wave propagation phenomena and the *theory of dynamical systems admitting normal shift* (see papers [7–22]). Below we give brief introduction to this theory.

Let M be Riemannian manifold and let σ be some smooth hypersurface in M. Suppose that p is a point of σ and $\mathbf{n}(p)$ is a unit normal vector to σ at the point p. Under these assumptions we can consider initial data

$$x^i\Big|_{t=0} = x^i(p), \qquad\qquad u^i\Big|_{t=0} = \nu(p) \cdot n^i(p) \tag{6.5}$$

for Newtonian dynamical system (6.3). Similar to initial data (2.10) for modified Hamiltonian dynamical system, here initial data (6.5) define *a shift* of hypersurface σ along trajectories of Newtonian dynamical system (6.3). This shift is called *normal shift* if hypersurfaces σ_t, which are obtained from σ by shift, keep orthogonality to shift trajectories in time.

Definition 6.1. Newtonian dynamical system (6.3) is called a system *admitting normal shift* if for any hypersurface σ there is a smooth function $\nu = \nu(p)$ on σ such that initial data (6.5) with this function ν define normal shift of σ along trajectories of dynamical system (6.3).

Suppose that p_0 is some fixed point of hypersurface σ and let ν_0 be some fixed constant. Let's normalize $\nu(p)$ by the following condition:

$$\nu(p_0) = \nu_0. \tag{6.6}$$

Definition 6.2. Say that Newtonian dynamical system (6.3) satisfies *strong normality condition* if for any hypersurface σ, for any point $p_0 \in \sigma$, and for any constant $\nu_0 \neq 0$ there is a smooth function $\nu = \nu(p)$ on σ normalized by the condition (6.6) and such that initial data (6.5) with this function ν define normal shift of σ along trajectories of dynamical system (6.3).

Strong normality condition, in contrast to *the normality condition* from definition 6.1, is less obvious. But it is more convenient for to study by mathematical methods. In papers [12] and [13] the following two systems of differential equations for the force field $\mathbf{F}$ of Newtonian dynamical system (6.3) were derived:

$$\begin{cases} \displaystyle\sum_{i=1}^{n}\left(v^{-1}\,F_i+\sum_{j=1}^{n}\tilde{\nabla}_i\left(N^j\,F_j\right)\right)P^i_k=0,\\ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j+\nabla_jF_i-2\,v^{-2}\,F_i\,F_j\right)N^j\,P^i_k+\\ \displaystyle+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde{\nabla}_jF_i}{v}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde{\nabla}_jF_r}{v}\,F_i\right)P^i_k=0, \end{cases} \tag{6.7}$$

$$\begin{cases} \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_i\,\tilde{\nabla}_mF_j}{v}-\nabla_iF_j\right)=\\ \displaystyle\qquad=\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_j\,\tilde{\nabla}_mF_i}{v}-\nabla_jF_i\right),\\ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}P^j_\sigma\,\tilde{\nabla}_jF^i\,P^\varepsilon_i=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}\frac{P^j_m\,\tilde{\nabla}_jF^i\,P^m_i}{n-1}\,P^\varepsilon_\sigma. \end{cases} \tag{6.8}$$

The equations (6.7) were called *weak normality equations*, while other equations (6.8) were called *additional normality equations*. In Chapter V of thesis [23] the following theorem was proved.

Theorem 6.2. *Newtonian dynamical system* (6.3) *satisfies strong normality condition if and only if its force field* $\mathbf{F}$ *satisfies complete system of normality equations consisting of weak normality equations* (6.7) *and including additional normality equations* (6.8) *in the case of higher dimensions* $n\geqslant 3$*.*

Weak normality equations (6.7) are related to *weak normality condition.* In order to formulate this condition let's consider one-parametric family of trajectories of Newtonian dynamical system (6.3). Denote it as follows:

$$p=p(t,y). \tag{6.9}$$

Here t is time variable and y is a parameter. In local coordinates this one-parametric family of trajectories (6.9) is expressed by functions

$$\begin{cases} x^1=x^1(t,y),\\ \ldots\ldots\ldots\ldots\\ x^n=x^n(t,y). \end{cases} \tag{6.10}$$

Differentiating (6.10) with respect to parameter y, we get vector $\boldsymbol{\tau}$ with components

$$\tau^i=\frac{\partial x^i}{\partial y}. \tag{6.11}$$

Vector $\boldsymbol{\tau}$ is called *vector of variation* of trajectories. Then from (6.3) we derive

$$\begin{aligned}\nabla_{tt}\tau^k=-\sum_{q=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}R^k_{qij}\,\tau^i\,u^j\,u^q\,+\\+\sum_{q=1}^{n}\nabla_t\tau^q\,\tilde{\nabla}_qF^k+\sum_{q=1}^{n}\tau^q\,\nabla_qF^k.\end{aligned}\tag{6.12}$$

Here R^k_{qij} components of curvature tensor for metric $\mathbf{g}$, while u^j and u^q are components of velocity vector $\mathbf{u}$. Function φ defined as scalar product

$$\varphi=(\mathbf{u}\,|\,\boldsymbol{\tau})=\sum_{i=1}^{n}\sum_{j=1}^{n}g_{ij}\,u^i\,\tau^j\tag{6.13}$$

is called *function of deviation*. From (6.12) one can derive the following ordinary differential equation for the function of deviation (6.13):

$$\sum_{i=0}^{2n}C_i(t)\,\varphi^{(i)}=0.\tag{6.14}$$

For general Newtonian dynamical system (6.3) this is homogeneous ordinary differential equation of the order $2n$ (see details in Chapter V of thesis [23]). However, in special cases the equation (6.14) can reduce to lower order differential equation. Weak normality condition below specifies one of such cases. Indeed, let's consider some trajectory $p=p(t)$ of Newtonian dynamical system (6.3). It can be included into one-parametric family of trajectories (6.9) by various ways. This defines various variation vectors $\boldsymbol{\tau}$ with components satisfying differential equations (6.12) and various deviation functions (6.13) on the trajectory $p=p(t)$.

Definition 6.3. Say that Newtonian dynamical system (6.3) satisfies *weak normality condition* if for each its trajectory $p=p(t)$ and for any vector of variation $\boldsymbol{\tau}$ on this trajectory corresponding function of deviation $\varphi(t)$ satisfies homogeneous second order ordinary differential equation

$$\ddot{\varphi}=\mathcal{A}(t)\,\dot{\varphi}+\mathcal{B}(t)\,\varphi\tag{6.15}$$

with coefficients depending only on choice of trajectory $p=p(t)$.

As it was shown in paper [12], weak normality condition is equivalent to weak normality equations (6.7) for the force field of Newtonian dynamical system (6.3).

Now let's proceed with additional normality condition. In order to formulate this condition let's consider some smooth hypersurface σ in M and let's fix some point p_0 on σ. Denote by $y^1,\ldots,y^{n-1}$ local coordinates on σ in some neighborhood of fixed point p_0. Setting up initial data (6.5), we can define a family of trajectories $p=p(t,y^1,\ldots,y^n)$ of Newtonian dynamical system (6.3) starting at the points of σ. Now this is $(n-1)$-parametric family of trajectories expressed by functions

$$\begin{cases}x^1=x^1(t,y^1,\ldots,y^{n-1}),\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ x^n=x^n(t,y^1,\ldots,y^{n-1}).\end{cases}\tag{6.16}$$

in local coordinates $x^1, \ldots, x^n$ in M. Differentiating these functions (6.16) with respect to parameters $y^1, \ldots, y^{n-1}$, as it is done in (6.11), we get $n-1$ variation vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$. It's easy to note that variation vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ form a frame of tangent vectors to hypersurfaces σ_t obtained by shifting initial hypersurface σ along trajectories (6.16) of dynamical system (6.3). Therefore corresponding deviation functions $\varphi_1, \ldots, \varphi_{n-1}$ serve as measure of orthogonality of σ_t and shift trajectories (6.16). They should be identically zero in order to provide orthogonality of shift: $\varphi_i(t, y^1, \ldots, y^{n-1}) = 0$. If we consider initial data

$$\varphi_k\Bigr|_{t=0} = 0, \qquad\qquad \dot\varphi_k\Bigr|_{t=0} = 0, \tag{6.17}$$

then we can see that first part of initial conditions (16.17) is fulfilled due to initial data (6.5). Second part of these conditions can be transformed to differential equations for the function $\nu = \nu(p) = \nu(y^1, \ldots, y^{n-1})$ in (6.5):

$$\frac{\partial \nu}{\partial y^i} = -\nu^{-1}\,(\mathbf{F}\,|\,\boldsymbol{\tau}_i). \tag{6.18}$$

If $\dim M = n \geqslant 3$, then the equations (6.18) form complete system of Pfaff equations for scalar function ν. The condition of its compatibility is known as additional normality condition.

Definition 6.4. Say that Newtonian dynamical system (6.3) satisfies *additional normality condition* if for any smooth hypersurface σ in M and for any local coordinates $y^1, \ldots, y^{n-1}$ on σ corresponding Pfaff equations (6.18) are compatible.

In paper [13] it was shown that for $n \geqslant 3$ additional normality condition is equivalent to additional normality equations (6.8) for the force field of Newtonian dynamical system (6.3). In two-dimensional case $n = 2$ situation is quite different. Here we have only one parameter $y = y^1$ and (6.18) turns to unique ordinary differential equation, which is compatible with itself in anyway. Therefore in two-dimensional case additional normality condition is always fulfilled. This special case is studied in thesis [24].

In higher dimensional case $n \geqslant 3$ complete system of normality equations includes both (6.7) and (6.8). For this case in Chapter VII of thesis [23] explicit formula for general solution of complete system of normality equations was derived:

$$F_k = \frac{h(W)\,N_k}{W'} - |\mathbf{u}| \cdot \sum_{i=1}^{n} \frac{\nabla_i W}{W'} \cdot \left(2\,N^i\,N_k - \delta^i_k\right). \tag{6.19}$$

Comparing formulas (6.4) and (6.19) we see that they are quite similar. They differ only by first term in (6.19), where $h = h(w)$ is arbitrary function of one variable. This fact indicates that modified Lagrangian dynamical systems (5.2) describing wave front dynamics and Newtonian dynamical systems (6.3) admitting normal shift of hypersurfaces in Riemannian manifolds are closely related with each other. In further sections we are going to reveal this relation in more general case, when manifold M is not equipped with Riemannian metric. Problem of interpreting first term in (6.19) should be considered in separate paper.

7. Weak normality phenomenon for modified Lagrangian dynamical systems.

Let M be a smooth manifold which is not equipped with Riemannian metric, but which is equipped with modified Lagrangian dynamical system. Let $L = L(p, \mathbf{v})$ be Lagrange function for this system. This is extended scalar field in contravariant $\mathbf{v}$-representation. In the absence of Riemannian metric we cannot use spatial gradient (3.4). Therefore we write modified Lagrange equations as

$$\dot{x}^i = \frac{v^i}{\Omega}, \qquad\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{1}{\Omega}\,\frac{\partial L}{\partial x^i}. \tag{7.1}$$

Denominator Ω in (7.1) is determined by formula (5.4):

$$\Omega = \sum_{i=1}^{n} v^i\,\tilde{\nabla}_i L = \sum_{i=1}^{n} v^i\,\frac{\partial L}{\partial v^i}. \tag{7.2}$$

Remember that formula (4.2) defines Legendre map (4.3). Below we assume this map λ to be invertible, and moreover, we assume inverse map λ^{-1} to be smooth. Local invertibility of λ means that matrix $\mathbf{g}$ with components

$$\mu_{ij} = \frac{\tilde{\nabla}_i\tilde{\nabla}_j L}{2} = \frac{1}{2}\,\frac{\partial^2 L}{\partial v^i\,\partial v^j} \tag{7.3}$$

is non-degenerate. We shall assume this matrix to be positive:

$$\boldsymbol{\mu} > 0. \tag{7.4}$$

For real mechanical systems this condition $\boldsymbol{\mu} > 0$ is fulfilled since kinetic energy K of such systems is positive quadratic function of velocity vector. For such systems components of matrix $\boldsymbol{\mu}$ do not depend on $\mathbf{v}$, hence one can choose $\boldsymbol{\mu}$ to be metric tensor for Riemannian metric in M. However, we shall consider more general case, when $\boldsymbol{\mu}$ is extended tensor field with components depending on $\mathbf{v}$.

In addition to inequality (7.4) we shall assume that denominator Ω in modified Lagrange equations (7.1) (which is determined by (7.2)) is positive function:

$$\Omega > 0, \quad \text{for} \quad \mathbf{v} \neq 0. \tag{7.5}$$

This assumption is consistent since for real mechanical systems $\Omega = 2\,K$.

Now let's consider one-parametric family of trajectories $p = p(t, y)$ of modified Lagrangian dynamical system (7.1). In local coordinates these curves are expressed by functions (6.10). Formula (6.11) then defines vector of variation $\boldsymbol{\tau}$. In order to define function of deviation φ we could use formula (6.13) with matrix (7.3) as metric. However, we choose another formula for φ:

$$\varphi = \langle \mathbf{p}\,|\,\boldsymbol{\tau}\rangle = \sum_{i=1}^{n} \tilde{\nabla}_i L\ \tau^i. \tag{7.6}$$

Here $\mathbf{p}$ is momentum covector defined by formula (4.2), while angular brackets denote contraction of vector and covector[1]. Function φ in (7.1) can be treated as

[1] Such notations are often used in quantum mechanics. See [31].

scalar product of vectors $\mathbf{v}$ and $\boldsymbol{\tau}$. This scalar product is linear with respect to vector $\boldsymbol{\tau}$, but it is nonlinear respect to vector $\mathbf{v}$. Such scalar products usually arise in Finslerian geometry (see Chapter VIII of thesis [23]).

Let $\mathbf{v} = \mathbf{v}(t, y)$ be velocity vector for one-parametric family of trajectories $p = p(t, y)$ of modified Lagrangian dynamical system (7.1). In local coordinates this vector-function is expressed by the following scalar functions:

$$\begin{cases} v^1 = v^1(t, y), \\ \dots\dots\dots\dots \\ v^n = v^n(t, y). \end{cases} \tag{7.7}$$

Differentiating (7.7) with respect to parameter y, we get series of functions

$$\theta^i = \frac{\partial v^i}{\partial y} = \dot{\tau}^i. \tag{7.8}$$

In contrast to τ^i in (6.11), these functions $\theta^1, \dots, \theta^n$ are not interpreted as components of vector. We shall use them in order to simplify further calculations. Differentiating first equation (7.1) with respect to y, we obtain

$$\begin{gathered} \dot{\tau}^i = \frac{\theta^i}{\Omega} - \sum_{s=1}^{n} \frac{v^i}{\Omega^2}\,\frac{\partial L}{\partial v^s}\,\theta^s - \\ - \sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial^2 L}{\partial v^k\,\partial v^s}\,\frac{v^i\,v^k\,\theta^s}{\Omega^2} - \sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial^2 L}{\partial v^k\,\partial x^s}\,\frac{v^i\,v^k\,\tau^s}{\Omega^2}. \end{gathered} \tag{7.9}$$

Differentiating second equation (7.1) with respect to parameter y, we get

$$\begin{gathered} \sum_{s=1}^{n} \frac{\partial^2 L}{\partial v^i\,\partial v^s}\,\dot{\theta}^s + \sum_{s=1}^{n} \frac{\partial^2 L}{\partial v^i\,\partial x^s}\,\dot{\tau}^s + \sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial^3 L}{\partial v^i\,\partial v^s\,\partial v^k}\,\dot{v}^k\,\theta^s + \\ + \sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial^3 L}{\partial v^i\,\partial v^s\,\partial x^k}\,\dot{x}^k\,\theta^s + \sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial^3 L}{\partial v^i\,\partial x^s\,\partial v^k}\,\dot{v}^k\,\tau^s + \\ + \sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial^3 L}{\partial v^i\,\partial x^s\,\partial x^k}\,\dot{x}^k\,\tau^s = \sum_{s=1}^{n} \frac{1}{\Omega}\,\frac{\partial^2 L}{\partial x^i\,\partial v^s}\,\theta^s + \\ + \sum_{s=1}^{n} \frac{1}{\Omega}\,\frac{\partial^2 L}{\partial x^i\,\partial x^s}\,\tau^s - \sum_{s=1}^{n} \frac{\partial L}{\partial x^i}\,\frac{\partial L}{\partial v^s}\,\frac{\theta^s}{\Omega^2} - \\ - \sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial L}{\partial x^i}\,\frac{\partial^2 L}{\partial v^k\,\partial v^s}\,\frac{v^k\,\theta^s}{\Omega^2} - \sum_{k=1}^{n}\sum_{s=1}^{n} \frac{\partial L}{\partial x^i}\,\frac{\partial^2 L}{\partial v^k\,\partial x^s}\,\frac{v^k\,\tau^s}{\Omega^2}. \end{gathered} \tag{7.10}$$

Both (7.9) and (7.10) form a system of homogeneous linear ordinary differential equations with respect to functions $\tau^1, \dots, \tau^n$ and $\theta^1, \dots, \theta^n$. This system of equations is an analog of equations (6.12) considered above.

Function of deviation φ defined by formula (7.6) depends linearly on components of vector $\boldsymbol{\tau}$. Let's calculate time derivatives of this function. For $\dot{\varphi}$ we get

$$\begin{gathered}
\dot{\varphi}=\sum_{s=1}^{n}\frac{\partial L}{\partial v^s}\,\dot{\tau}^s+\sum_{k=1}^{n}\frac{d}{dt}\!\left(\frac{\partial L}{\partial v^s}\right)\tau^s=\sum_{s=1}^{n}\frac{\partial L}{\partial v^s}\,\dot{\tau}^s+\\
+\sum_{s=1}^{n}\frac{\partial L}{\partial x^s}\,\frac{\tau^s}{\Omega}=\sum_{s=1}^{n}\frac{\partial L}{\partial v^s}\,\frac{\theta^s}{\Omega}-\sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial L}{\partial v^s}\,\frac{v^s}{\Omega^2}\,\frac{\partial L}{\partial v^r}\,\theta^r-\\
-\sum_{s=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\frac{\partial L}{\partial v^s}\,\frac{\partial^2 L}{\partial v^k\,\partial v^r}\,\frac{v^s\,v^k\,\theta^r}{\Omega^2}-\sum_{s=1}^{n}\sum_{k=1}^{n}\sum_{r=1}^{n}\frac{\partial L}{\partial v^s}\,\frac{\partial^2 L}{\partial v^k\,\partial x^r}\,\frac{v^s\,v^k\,\tau^r}{\Omega^2}+\\
+\sum_{s=1}^{n}\frac{\partial L}{\partial x^s}\,\frac{\tau^s}{\Omega}=\sum_{s=1}^{n}\frac{\partial L}{\partial x^s}\,\frac{\tau^s}{\Omega}-\sum_{k=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 L}{\partial v^k\,\partial v^r}\,\frac{v^k\,\theta^r}{\Omega}-\sum_{k=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 L}{\partial v^k\,\partial x^r}\,\frac{v^k\,\tau^r}{\Omega}.
\end{gathered}$$

In the above calculations we used second equation (7.1), used formula (7.2) for denominator Ω, and used differential equations (7.9). As a result we obtained

$$\dot{\varphi}=\sum_{s=1}^{n}\frac{\partial L}{\partial x^s}\,\frac{\tau^s}{\Omega}-\sum_{k=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 L}{\partial v^k\,\partial v^r}\,\frac{v^k\,\theta^r}{\Omega}-\sum_{k=1}^{n}\sum_{r=1}^{n}\frac{\partial^2 L}{\partial v^k\,\partial x^r}\,\frac{v^k\,\tau^r}{\Omega}.\tag{7.11}$$

Now let's differentiate (7.11) once more. This yields

$$\begin{gathered}
\ddot{\varphi}=-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^i\,\partial v^s}\,\frac{v^i\,\dot{\theta}^s}{\Omega}-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^i\,\partial x^s}\,\frac{v^i\,\dot{\tau}^s}{\Omega}+\sum_{s=1}^{n}\frac{\partial L}{\partial x^s}\,\frac{\dot{\tau}^s}{\Omega}-\\
-\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^3 L}{\partial v^k\,\partial v^i\,\partial v^s}\,\frac{\dot{v}^k\,v^i\,\theta^s}{\Omega}-\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^k\,\partial v^i\,\partial v^s}\,\frac{\dot{x}^k\,v^i\,\theta^s}{\Omega}-\\
-\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^3 L}{\partial v^k\,\partial v^i\,\partial x^s}\,\frac{\dot{v}^k\,v^i\,\tau^s}{\Omega}-\sum_{k=1}^{n}\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^3 L}{\partial x^k\,\partial v^i\,\partial x^s}\,\frac{\dot{x}^k\,v^i\,\tau^s}{\Omega}-\\
-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^i\,\partial v^s}\,\frac{\dot{v}^i\,\theta^s}{\Omega}-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^i\,\partial x^s}\,\frac{\dot{v}^i\,\tau^s}{\Omega}+\\
+\sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^k\,\partial x^s}\,\frac{\dot{v}^k\,\tau^s}{\Omega}+\sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^k\,\partial x^s}\,\frac{\dot{x}^k\,\tau^s}{\Omega}-\frac{\dot{\Omega}}{\Omega}\,\dot{\varphi}.
\end{gathered}$$

Using equations (7.10), we can eliminate all entries of derivatives $\dot{\theta}^s$ from the above expression for $\ddot{\varphi}$. As a result we get reduced formula

$$\ddot{\varphi}=-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^i\,\partial v^s}\,\frac{v^i\,\theta^s}{\Omega^2}-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^i\,\partial x^s}\,\frac{v^i\,\tau^s}{\Omega^2}+$$

$$+\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial L}{\partial x^i}\,\frac{\partial L}{\partial v^s}\,\frac{v^i\,\theta^s}{\Omega^3}+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial L}{\partial x^i}\,\frac{\partial^2 L}{\partial v^k\,\partial v^s}\,\frac{v^i\,v^k\,\theta^s}{\Omega^3}+$$
$$+\sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial L}{\partial x^i}\,\frac{\partial^2 L}{\partial v^k\,\partial x^s}\,\frac{v^i\,v^k\,\tau^s}{\Omega^3}+\sum_{s=1}^{n}\frac{\partial L}{\partial x^s}\,\frac{\dot\tau^s}{\Omega}-$$
$$-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^i\,\partial v^s}\,\frac{\dot v^i\,\theta^s}{\Omega}+\sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^k\,\partial x^s}\,\frac{\dot x^k\,\tau^s}{\Omega}-\frac{\dot\Omega}{\Omega}\,\dot\varphi.$$

Now we eliminate all entries of $\dot\tau^s$, using equations (7.9) for this purpose:

$$\begin{gathered}\ddot\varphi=-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^i\,\partial v^s}\,\frac{v^i\,\theta^s}{\Omega^2}-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^i\,\partial x^s}\,\frac{v^i\,\tau^s}{\Omega^2}+\\ +\sum_{s=1}^{n}\frac{\partial L}{\partial x^s}\,\frac{\theta^s}{\Omega^2}-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^i\,\partial v^s}\,\frac{\dot v^i\,\theta^s}{\Omega}+\sum_{k=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^k\,\partial x^s}\,\frac{\dot x^k\,\tau^s}{\Omega}-\frac{\dot\Omega}{\Omega}\,\dot\varphi.\end{gathered}\tag{7.12}$$

Then, using first equation in (7.1), we express time derivative $\dot x^k$ through v^k. As a result formula (7.12) for $\ddot\varphi$ reduces to the following one:

$$\begin{gathered}\ddot\varphi=-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial x^i\,\partial v^s}\,\frac{v^i\,\theta^s}{\Omega^2}+\sum_{s=1}^{n}\frac{\partial L}{\partial x^s}\,\frac{\theta^s}{\Omega^2}-\\ -\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\partial^2 L}{\partial v^i\,\partial v^s}\,\frac{\dot v^i\,\theta^s}{\Omega}-\frac{\dot\Omega}{\Omega}\,\dot\varphi.\end{gathered}\tag{7.13}$$

Now, if one take into account second equation (7.1) written in expanded form, then three terms in right hand side of (7.13) can be canceled. This yields

$$\ddot\varphi+\frac{\dot\Omega}{\Omega}\,\dot\varphi=0.\tag{7.14}$$

Thus, in the end of huge calculations we get very simple relationship (7.14), which is homogeneous second order linear ordinary differential equation. It is even more simple than analogous equation (6.15) considered in previous section. Now, if we formulate definition 6.3 respective to modified Lagrangian dynamical system (7.1) and if we use formula (7.6) for φ, then from (7.14) we derive the following theorem.

Theorem 7.1. *Each modified Lagrangian dynamical system* (7.1) *satisfies* **weak normality condition** *with respect to deviation functions* (7.6) *determined by its own Lagrange function L.*

8. Additional normality phenomenon.

In order to reproduce results of section 6 in present more complicated geometric environment we should consider some hypersurface σ in M, and we should arrange a shift of σ by means of modified Lagrangian dynamical system (7.1). Fortunately we should not invent something absolutely new for this purpose. Wave front dynamics considered in section 2 suggests a way of how to do this. In the absence of

Riemannian metric we cannot choose unit normal vector on σ. However, we can take normal covector $\mathbf{n} = \mathbf{n}(p)$, which is unique up to a scalar factor. Then we can set up Cauchy problem with the following initial data:

$$x^i\Bigr|_{t=0} = x^i(p), \qquad\qquad p_i\Bigr|_{t=0} = \nu(p) \cdot n_i(p) \tag{8.1}$$

(compare with (2.10) above). Here p is a point of σ and p_i are components of momentum covector $\mathbf{p}$ defined by formula (4.2):

$$p_i = \tilde{\nabla}_i L = \frac{\partial L}{\partial v^i}.$$

Initial data (8.1) determine initial velocity $\mathbf{v}$ implicitly through initial momentum covector $\mathbf{p}$ due to invertibility of Legendre map λ. Applying initial data (8.1) to modified Lagrangian dynamical system (7.1), we obtain a family of trajectories $p = p(t, y^1, \ldots, y^n)$ starting at the points of hypersurface σ. Similar to (6.16), in local coordinates these trajectories are expressed by the following functions:

$$\begin{cases} x^1 = x^1(t, y^1, \ldots, y^{n-1}),\\ \dots\dots\dots\dots\dots\dots\\ x^n = x^n(t, y^1, \ldots, y^{n-1}). \end{cases} \tag{8.2}$$

These functions (8.2) determine variation vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ with components

$$\tau^i_j = \frac{\partial x^i}{\partial y^j}$$

(compare with (6.11) and (7.8)). Each variation vector determines corresponding deviation function according to the formula (7.6). We denote these deviation functions by $\varphi_1, \ldots, \varphi_{n-1}$ as in section 6.

Definition 8.1. Shift of initial hypersurface σ determined by modified Lagrangian dynamical system (7.1) and by initial data (8.1) for it is called *normal shift* in inner geometry of dynamical system (7.1) if all deviation functions $\varphi_1, \ldots, \varphi_{n-1}$ are identically zero.

Due to differential equation (7.14) for deviation functions in order to arrange a normal shift of σ it is sufficient to provide initial conditions

$$\varphi_i,\Bigr|_{t=0} = 0, \qquad\qquad \dot{\varphi}_i\Bigr|_{t=0} = 0 \tag{8.3}$$

just the same as in (6.17). First part of initial conditions (8.3) is fulfilled due to initial data (8.1). Second part of these conditions should be transformed to differential equations for the function $\nu = \nu(p) = \nu(y^1, \ldots, y^{n-1})$ in (8.1). For this purpose we could use formula (7.11) derived in section 7. However, initial data (8.1) explicitly relate function $\nu = \nu(p)$ with initial value of momentum covector $\mathbf{p}$, while relation to velocity vector $\mathbf{v}$ is implicit. Therefore it is easier to transform formula (7.6) to $\mathbf{p}$-representation. This yields

$$\varphi_i = \langle \mathbf{p} \,|\, \boldsymbol{\tau}_i \rangle = \sum_{s=1}^{n} p_s\, \tau^s_i. \tag{8.4}$$

Remember that modified Lagrange equations, when transformed to **p**-representation, look like modified Hamilton equations (2.8):

$$\dot{x}^i=\frac{1}{\Omega}\,\frac{\partial H}{\partial p_i},\qquad\qquad \dot{p}_i=-\frac{1}{\Omega}\,\frac{\partial H}{\partial x^i}. \tag{8.5}$$

Here Hamilton function H is determined by formula (4.5) and Ω is given by formula (5.3). Now, differentiating formula (8.4), we obtain

$$\dot{\varphi}_i=\sum_{s=1}^{n}\dot{\overline{p_s\,\tau^s_i}}=\sum_{s=1}^{n}\dot{p}_s\,\tau^s_i+\sum_{s=1}^{n}p_s\,\dot{\tau}^s_i=-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i+\sum_{s=1}^{n}p_s\,\dot{\tau}^s_i. \tag{8.6}$$

In order to calculate time derivatives $\dot{\tau}^s_i$ in formula (8.6) we use first part of modified Hamilton equations (8.5). A a result for $\dot{\tau}^s_i$ we get

$$\begin{aligned}\dot{\tau}^s_i&=\frac{\partial^2x^s}{\partial t\,\partial y^i}=\frac{\partial}{\partial y^i}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)=\sum_{r=1}^{n}\frac{\partial x^r}{\partial y^i}\,\frac{\partial}{\partial x^r}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)+\\&+\sum_{r=1}^{n}\frac{\partial p_r}{\partial y^i}\,\frac{\partial}{\partial p_r}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)=\sum_{r=1}^{n}\tau^r_i\,\frac{\partial}{\partial x^r}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)+\sum_{r=1}^{n}\frac{\partial p_r}{\partial y^i}\,\frac{\partial}{\partial p_r}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right).\end{aligned}$$

Let's substitute this expression into (8.6). This yields

$$\begin{aligned}\dot{\varphi}_i&=-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i+\sum_{s=1}^{n}\sum_{r=1}^{n}p_s\,\tau^r_i\,\frac{\partial}{\partial x^r}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)+\\&+\sum_{s=1}^{n}\sum_{r=1}^{n}p_s\,\frac{\partial p_r}{\partial y^i}\,\frac{\partial}{\partial p_r}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)=\sum_{r=1}^{n}\tau^r_i\,\frac{\partial}{\partial x^r}\left(\sum_{s=1}^{n}\frac{p_s}{\Omega}\,\frac{\partial H}{\partial p_s}\right)+\\&+\sum_{r=1}^{n}\frac{\partial p_r}{\partial y^i}\,\frac{\partial}{\partial p_r}\left(\sum_{s=1}^{n}\frac{p_s}{\Omega}\,\frac{\partial H}{\partial p_s}\right)-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\,\frac{\partial p_s}{\partial y^i}-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i.\end{aligned}$$

First two terms in right hand side of the above equality are identically zero. This follows from formula (5.3) for Ω. Thus for $\dot{\varphi}_i$ we have

$$\dot{\varphi}_i=-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\,\frac{\partial p_s}{\partial y^i}-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i. \tag{8.7}$$

If we recall initial conditions (8.3), then from (8.7) we derive

$$\sum_{s=1}^{n}\frac{\partial H}{\partial x^s}\,\tau^s_i+\sum_{s=1}^{n}\frac{\partial H}{\partial p_s}\,\left.\left(\frac{\partial p_s}{\partial y^i}\right)\right|_{t=0}=0. \tag{8.8}$$

Calculating partial derivatives $\partial p_s/\partial y^i$ in (8.8), we should remember (8.1). Then

$$\left.\left(\frac{\partial p_s}{\partial y^i}\right)\right|_{t=0}=\frac{\partial\nu}{\partial y^i}\,n_s+\nu\,\frac{\partial n_s}{\partial y^i}=\frac{1}{\nu}\,\frac{\partial\nu}{\partial y^i}\,p_s+\nu\,\frac{\partial n_s}{\partial y^i}. \tag{8.9}$$

Substituting this expression into (8.8) and using formula (5.3) for Ω, we can transform (8.8) to the partial differential equations for ν:

$$\frac{1}{\nu}\,\frac{\partial\nu}{\partial y^i}=-\sum_{s=1}^{n}\frac{\nu}{\Omega}\,\frac{\partial n_s}{\partial y^i}\,\frac{\partial H}{\partial p_s}-\sum_{s=1}^{n}\frac{\partial H}{\partial x^s}\,\frac{\tau^s_i}{\Omega}. \tag{8.10}$$

Differential equations (8.10) are analogs of the equations (6.18). If $n\geqslant 3$, then they form complete system of Pfaff equations for the function $\nu=\nu(y^1,\ldots,y^{n-1})$. Therefore we can formulate *additional normality condition* for modified Lagrangian dynamical system (7.1) as compatibility condition for Pfaff equations (8.10).

Suppose that $n\geqslant 3$. Let's examine if differential equations (8.10) are compatible. For this purpose let's calculate second order partial derivatives of ν using (8.10):

$$\begin{aligned}
\frac{\partial^2\nu}{\partial y^i\,\partial y^j}&=-\sum_{s=1}^{n}\frac{\nu^2}{\Omega}\,\frac{\partial H}{\partial p_s}\,\frac{\partial^2 n_s}{\partial y^i\,\partial y^j}-\sum_{s=1}^{n}\frac{\partial H}{\partial x^s}\,\frac{\nu}{\Omega}\,\frac{\partial^2 x^s}{\partial y^i\,\partial y^j}+\\
&+\frac{2\,\nu^3}{\Omega^2}\sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial H}{\partial p_s}\,\frac{\partial H}{\partial p_r}\,\frac{\partial n_s}{\partial y^i}\,\frac{\partial n_r}{\partial y^j}+\frac{\nu}{\Omega^2}\sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial H}{\partial x^s}\,\frac{\partial H}{\partial x^r}\,\tau^s_i\,\tau^r_j+\\
&+\frac{2\,\nu^2}{\Omega^2}\sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial H}{\partial x^s}\,\frac{\partial H}{\partial p_r}\,\frac{\partial n_r}{\partial y^j}\,\tau^s_i+\frac{\nu^2}{\Omega^2}\sum_{s=1}^{n}\sum_{r=1}^{n}\frac{\partial H}{\partial p_s}\,\frac{\partial H}{\partial x^r}\,\frac{\partial n_s}{\partial y^i}\,\tau^r_j-\\
&-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu^2\,\frac{\partial}{\partial p_s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial p_s}{\partial y^i}\,\frac{\partial n_r}{\partial y^j}-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu\,\frac{\partial}{\partial p_s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)\frac{\partial p_s}{\partial y^i}\,\tau^r_j-\\
&-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu^2\,\frac{\partial}{\partial x^s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial n_r}{\partial y^j}\,\tau^s_i-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu\,\frac{\partial}{\partial x^s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)\tau^s_i\,\tau^r_j.
\end{aligned}$$

For Pfaff equations (8.10) to be compatible, right hand side of the above equality should be symmetric in indices i and j. First four terms there are obviously symmetric. Below we shall not write such terms explicitly denoting them by dots:

$$\begin{aligned}
\frac{\partial^2\nu}{\partial y^i\,\partial y^j}&=\cdots+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{2\,\nu^2}{\Omega^2}\,\frac{\partial H}{\partial x^s}\,\frac{\partial H}{\partial p_r}-\nu^2\,\frac{\partial}{\partial x^s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\right)\frac{\partial n_r}{\partial y^j}\,\tau^s_i+\\
&+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{\nu^2}{\Omega^2}\,\frac{\partial H}{\partial p_s}\,\frac{\partial H}{\partial x^r}-\nu^2\,\frac{\partial}{\partial p_s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)\right)\frac{\partial n_s}{\partial y^i}\,\tau^r_j-\\
&-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu\,p_s\,\frac{\partial}{\partial p_s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial\nu}{\partial y^i}\,\frac{\partial n_r}{\partial y^j}-\sum_{s=1}^{n}\sum_{r=1}^{n}p_s\,\frac{\partial}{\partial p_s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)\frac{\partial\nu}{\partial y^i}\,\tau^r_j-\\
&-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu^3\,\frac{\partial}{\partial p_s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial n_s}{\partial y^i}\,\frac{\partial n_r}{\partial y^j}-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu\,\frac{\partial}{\partial x^s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)\tau^s_i\,\tau^r_j.
\end{aligned}$$

In the above calculations we used formula (8.9) for partial derivatives $\partial p_s/\partial y^i$.

Below we use the equations (8.10) for to express partial derivatives $\partial\nu/\partial y^i$:

$$
\begin{aligned}
\frac{\partial^2\nu}{\partial y^i\,\partial y^j}&=\dots+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{2\,\nu^2}{\Omega^2}\,\frac{\partial H}{\partial x^s}\,\frac{\partial H}{\partial p_r}-\nu^2\,\frac{\partial}{\partial x^s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)+\right.\\
&\left.+\sum_{q=1}^{n}\frac{\nu^2\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial H}{\partial x^s}\right)\frac{\partial n_r}{\partial y^j}\,\tau^s_i+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{\nu^2}{\Omega^2}\,\frac{\partial H}{\partial p_s}\,\frac{\partial H}{\partial x^r}\,-\right.\\
&\left.-\nu^2\,\frac{\partial}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)+\sum_{q=1}^{n}\frac{\nu^2\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)\frac{\partial H}{\partial p_s}\right)\frac{\partial n_s}{\partial y^i}\,\tau^r_j+\\
&+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\sum_{q=1}^{n}\frac{\nu^3\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial H}{\partial p_s}+\frac{\nu^3}{\Omega^2}\,\frac{\partial\Omega}{\partial p_s}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial n_s}{\partial y^i}\,\frac{\partial n_r}{\partial y^j}+\\
&+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)\frac{\partial H}{\partial x^s}+\frac{\nu}{\Omega^2}\,\frac{\partial\Omega}{\partial x^s}\,\frac{\partial H}{\partial x^r}\right)\tau^s_i\,\tau^r_j.
\end{aligned}
$$

Now we are able to write compatibility condition for Pfaff equations (8.10). It breaks into three separate parts. These are the following equalities:

$$
\begin{aligned}
&\frac{2\,\nu^2}{\Omega^2}\,\frac{\partial H}{\partial x^s}\,\frac{\partial H}{\partial p_r}-\nu^2\,\frac{\partial}{\partial x^s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)+\sum_{q=1}^{n}\frac{\nu^2\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial H}{\partial x^s}=\\
&=\frac{\nu^2}{\Omega^2}\,\frac{\partial H}{\partial p_r}\,\frac{\partial H}{\partial x^s}-\nu^2\,\frac{\partial}{\partial p_r}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\right)+\sum_{q=1}^{n}\frac{\nu^2\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\right)\frac{\partial H}{\partial p_r},
\end{aligned}
\tag{8.11}
$$

$$
\begin{aligned}
&\sum_{q=1}^{n}\frac{\nu^3\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial H}{\partial p_s}+\frac{\nu^3}{\Omega^2}\,\frac{\partial\Omega}{\partial p_s}\,\frac{\partial H}{\partial p_r}=\\
&\qquad=\sum_{q=1}^{n}\frac{\nu^3\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)\frac{\partial H}{\partial p_r}+\frac{\nu^3}{\Omega^2}\,\frac{\partial\Omega}{\partial p_r}\,\frac{\partial H}{\partial p_s},
\end{aligned}
\tag{8.12}
$$

$$
\begin{aligned}
&\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}\right)\frac{\partial H}{\partial x^s}+\frac{\nu}{\Omega^2}\,\frac{\partial\Omega}{\partial x^s}\,\frac{\partial H}{\partial x^r}=\\
&\qquad=\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\right)\frac{\partial H}{\partial x^r}+\frac{\nu}{\Omega^2}\,\frac{\partial\Omega}{\partial x^r}\,\frac{\partial H}{\partial x^s}.
\end{aligned}
\tag{8.13}
$$

It is easy to check that all these three equalities (8.11), (8.12), and (8.13) turn to identities if we substitute (5.3) for Ω. Therefore we can formulate the following main result of this section.

Theorem 8.1. *Each modified Lagrangian dynamical system* (7.1) *satisfies* **additional normality condition** *with respect to deviation functions* (7.6) *determined by its own Lagrange function* L.

One should note here that Pfaff equations (8.10) are not only compatible, but they are also explicitly integrable in form of equality $H(p, \nu \cdot \mathbf{n}(p)) = \text{const}$, which is similar to the equality (2.11).

9. Summary and conclusions.

Theorems 7.1 and 8.1 form main result of present paper. Now we are to understand this result. Thus, we have arbitrary smooth manifold M without Riemannian metric, but equipped with Lagrange function $L = L(p, \mathbf{v})$ defining invertible Legendre transformation λ and satisfying two conditions (7.4) and (7.5). It's clear that these conditions are rather non-restrictive. Despite to the absence of Riemannian metric, under the above assumptions

1) one can define concept of *normal shift* with respect to geometric structures determined only by Lagrange function L;
2) one can formulate *weak and additional normality conditions*;
3) one can prove that modified Lagrangian dynamical system (7.1) satisfies both normality conditions with respect to geometric structures determined by its own Lagrange function L;

The results listed above generalize a part of *theory of dynamical systems admitting normal shift* from Riemannian geometry to the geometry of Lagrangian dynamics. However, this is not complete generalization. Indeed, in Riemannian case geometry was determined by metric tensor $\mathbf{g}$, while dynamics was determined by force field $\mathbf{F}$. Theory is based on the interplay of these two structures. Here both geometry and dynamics are determined by Lagrange function L yet. In further generalizations one should introduce another dynamical system in M (either Lagrangian or not Lagrangian), and then one should measure its capability to implement normal shift of hypersurfaces in geometry determined by L.

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[1]Papers [7–22] are arranged here in the order they were written. However, the order of publication not always coincides with the order of writing.

[2]For the convenience of reader we give direct reference to archive file. This is the following URL address: **http://arXiv.org/eprint/math.DG/0002202**.

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