COMPARATIVE ANALYSIS FOR PAIR OF DYNAMICAL SYSTEMS, ONE OF WHICH IS LAGRANGIAN.

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Abstract. It is known that some equations of differential geometry are derived from variational principle in form of Euler-Lagrange equations. The equations of geodesic flow in Riemannian geometry is an example. Conversely, having Lagrangian dynamical system in a manifold, one can consider it as geometric equipment of this manifold. Then properties of other dynamical systems can be studied relatively as compared to this Lagrangian one. This gives fruitful analogies for generalization. In present paper theory of normal shift of hypersurfaces is generalized from Riemannian geometry to the geometry determined by Lagrangian dynamical system. Both weak and additional normality equations for this case are derived.

1. Introduction.

Let M be some smooth manifold and let L be Lagrange function of some Lagrangian dynamical system in M. This is smooth scalar function depending on point p of M and on vector $\mathbf{v}$ at this point. In other words, L is a function of point $q=(p,\mathbf{v})$ of tangent bundle TM. We treat L as basic equipment of manifold M (like metric tensor in Riemannian geometry or simplectic structure in simplectic geometry). Therefore we assume L to be satisfying some special requirements. Let $x^1, \ldots, x^n$ be coordinates of point p in some local chart in M, and let $v^1, \ldots, v^n$ be components of vector $\mathbf{v}$ in this local chart:

$$\mathbf{v}=v^1\,\frac{\partial}{\partial x^1}+\ldots+v^n\,\frac{\partial}{\partial x^n}. \tag{1.1}$$

Lagrange function L in local coordinates $x^1, \ldots, x^n$ is represented as the function of $2n$ arguments $L=L(x^1,,\ldots,x^n,v^1,\ldots,v^n)$, while corresponding Lagrangian dynamical system is given by the following ODE's:

$$\dot x^i=v^i, \qquad\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)=\frac{\partial L}{\partial x^i}. \tag{1.2}$$

Due to $\dot x^i=v^i$ in (1.2) vector (1.1) is called *velocity vector*. In addition to (1.2) we consider so called modified Lagrange equations:

$$\dot x^i=\frac{v^i}{\Omega}, \qquad\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right)=\frac{1}{\Omega}\,\frac{\partial L}{\partial x^i}. \tag{1.3}$$

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Denominator Ω in these equations (1.3) is determined by the following formula:

$$\Omega=\sum_{i=1}^{n} v^i\,\frac{\partial L}{\partial v^i}. \tag{1.4}$$

Like original Lagrange equations (1.2), modified equations (1.3) with denominator (1.4) arise in applications. As shown in [1], they describe wave front dynamics for various wave propagation phenomena. Now let's denote by $\mathbf{p}$ the covector at the point $p\in M$ given by its components in local chart:

$$p_i=\frac{\partial L}{\partial v^i}. \tag{1.5}$$

Covector $\mathbf{p}$ is called *momentum covector*. Pair composed by point p and covector $\mathbf{p}$ is a point of cotangent bundle T^*M. Therefore (1.5) defines a map

$$\lambda\colon TM\to T^*M. \tag{1.6}$$

This map is called *Legendre transformation*. It is well known in mechanics (see [2]).

Note that first equations in (1.2) and (1.3) are different. Therefore vector $\mathbf{v}$ cannot be interpreted as velocity vector for modified dynamical system (1.3). For this purpose we introduce vector $\mathbf{u}$ with components

$$u^i=\frac{v^i}{\Omega}. \tag{1.7}$$

Formula (1.7) determines another map, which is similar to (1.6):

$$\mu\colon TM\to TM. \tag{1.8}$$

Definition 1.1. Lagrangian dynamical systems (1.2) and (1.3) and their Lagrange function L are called *regular* if both maps (1.4) and (1.8) are diffeomorphisms and if denominator Ω in modified Lagrange equations (1.3) given by formula (1.4) is positive at all points $q=(p,\mathbf{v})$ of tangent bundle TM, where $|\mathbf{v}|\neq 0$.

In order to get simplest example of regular Lagrangian dynamical system one should assume M to be equipped with Riemannian metric $\mathbf{g}$ and one should take Lagrange function L to be quadratic with respect to velocity vector $\mathbf{v}$:

$$L=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\,v^i\,v^j-U(x^1,\ldots,x^n). \tag{1.9}$$

Though it is simple, this example covers all real mechanical systems if we neglect friction in junctions and other forms of energy pumping and dissipation. First term in (1.9) is kinetic energy, while second term $U(x^1,\ldots,x^n)$ is potential energy of mechanical system. For L of the form (1.9) Legendre transformation (1.6) looks like index lowering procedure in metric $\mathbf{g}$. Indeed, applying (1.5) to (1.9), we find

$$p_i=\sum_{j=1}^{n} g_{ij}\,v^j. \tag{1.10}$$

If L is treated as geometric equipment of manifold M, then for L of the form (1.9) this equipment is equivalent to Riemannian metric in essential. This is the case considered in papers [3] and [4]. In present paper we consider more general case, when geometric equipment of manifold M is determined by some arbitrary regular Lagrange function L, which is not necessarily given by formula (1.9).

Now let's consider another dynamical system. It can be either Lagrangian or not Lagrangian, but we assume it to be second order dynamical system. More precisely, we assume that it is given by differential equations

$$\dot{x}^i = u^i, \qquad\qquad \dot{u}^i = \Phi^i(x^1, \ldots, x^n, u^1, \ldots, u^n). \tag{1.11}$$

We call (1.11) Newtonian dynamical system, since these equation are similar to those expressing Newton's second law for the motion of a particle of unit mass $m = 1$ under the action of force $\mathbf{\Phi}$. Certainly, we cannot touch all problems associated with pairs of dynamical systems of the form (1.2) and (1.11). In present paper we construct theory of normal shift for such pairs of dynamical systems, thus realizing the project claimed in previous paper [1].

Theory of Newtonian dynamical systems *admitting normal shift* was initiated in 1993 in preprint [5]. At first dynamical systems (1.11) in Euclidean space $\mathbb{R}^n$ were considered, then theory was extended for dynamical systems in Riemannian and Finslerian manifolds. This phase, which lasted 7 years from 1993 till 1998, is reflected in theses [6] and [7] (see also appropriate references therein). For recent results in the theory of dynamical systems admitting normal shift see papers [8–12] and papers [1], [3], and [4] already mentioned above. Below in section 2 we give some preliminary information and motivated definitions. Then in further sections of this paper we construct theory of Newtonian dynamical systems (1.11) admitting normal shift for the case of manifolds, geometry of which is not Euclidean, not Riemannian, and even not Finslerian, but is given by Lagrange function L of some regular Lagrangian dynamical system in them.

2. Normal shift of hypersurfaces.

Suppose that S is some arbitrary smooth hypersurface in M. We say that S is equipped with smooth transversal vector field if at each point $p \in S$ some nonzero vector $\mathbf{u}(p)$ transversal to S is fixed. In this case one can consider the following initial data for Newtonian dynamical system (1.11):

$$x^i\Big|_{t=0} = x^i(p), \qquad\qquad u^i\Big|_{t=0} = u^i(p). \tag{2.1}$$

Here $x^i(p)$ are coordinates of point p in some local chart in M and $u^i(p)$ are components of transversal vector $\mathbf{u}(p)$ in this chart. Applying initial data (2.1) to (1.11), we get a family of trajectories of this dynamical system starting at the points of S. In local chart it is represented by functions

$$\begin{cases} x^1 = x^1(t, p), \\ \ldots\ldots\ldots \\ x^n = x^n(t, p). \end{cases} \tag{2.2}$$

These trajectories are transversal to S. If we fix time instant $t \neq 0$ and gather all points of trajectories (2.2) corresponding to this time instant (see Fig. 2.1), we get another hypersurface S_t and diffeomorphism

$$f_t\colon S \to S_t \tag{2.3}$$

binding S_t with initial hypersurface[1]. Diffeomorphism f_t (or, more precisely, the whole set of diffeomorphisms f_t) is called *a shift* of S along trajectories of Newtonian dynamical system (1.11). Note that the shift (2.3) keeps transversality in local. This means that trajectories of shift are transversal not only to initial hypersurface S, but to all hypersurfaces S_t for sufficiently small values of t. Shift f_t is called *normal shift* if it keeps orthogonality of S_t and trajectories in some sense. In previous papers (see [8–12] and earlier) orthogonality was understood in the sense of some metric either Euclidean, Riemannian, or Finslerian. Below we shall understand it in the sense of Lagrange function L as it was suggested in paper [1].

Let $y^1, \ldots, y^{n-1}$ be some local coordinates on initial hypersurface S. Due to diffeomorphisms of shift (2.3) they can be transferred to all hypersurfaces S_t. Functions (2.2) in terms of local coordinates $y^1, \ldots, y^{n-1}$ are written as follows:

$$\begin{cases} x^1 = x^1(t, y^1, \ldots, y^{n-1}), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x^n = x^n(t, y^1, \ldots, y^{n-1}). \end{cases} \tag{2.4}$$

Their time derivatives $u^i = \dot{x}^i$ are components of velocity vector $\mathbf{u}$. It's easy to understand that partial derivatives of these functions with respect to y^i are components of vector tangent to S_t. It is called i-th vector of variation (or, more precisely, i-th vector of variation of trajectories). We denote this vector by $\boldsymbol{\tau}_i$:

$$\boldsymbol{\tau}_i = \tau^1_i\,\frac{\partial}{\partial x^1} + \ldots + \tau^n_i\,\frac{\partial}{\partial x^n},\ \text{ where }\ \tau^s_i = \frac{\partial x^s}{\partial y^i} \tag{2.5}$$

Vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ form a base in tangent space to S_t. For normal shift they should be perpendicular to velocity vector $\mathbf{u}$. According to receipt from paper [1], we define the following deviation functions φ_i:

$$\varphi_i = \langle \mathbf{p} \,|\, \boldsymbol{\tau}_i \rangle = \sum_{s=1}^{n} p_s\,\tau^s_i. \tag{2.6}$$

Here we have no metric, therefore scalar product $\langle \mathbf{p} \,|\, \boldsymbol{\tau}_i \rangle$ is nothing else, but symbolic notation for the sum in right hand side of (2.6).

[1] In general, this is true only for sufficiently small hypersurface S and for time instants t sufficiently close to initial time instant $t = 0$. However, our further consideration is local. Therefore here we shall not discuss the problem of globalization for diffeomorphisms (2.3) referring reader to paper [12], where some aspects of this problem are studied

Definition 2.1. Shift of hypersurface S along trajectories of Newtonian dynamical system (1.11) defined by initial data (2.1) is called *normal shift* if all deviation functions φ_i given by formula (2.6) are identically zero.

Deviation functions $\varphi_1, \ldots, \varphi_{n-1}$ are used as a measure of deviation of shift f_t from being a normal shift. Their vanishing is indicator of normality.

3. Relative form of the equations of Newtonian dynamics.

Note that covector $\mathbf{p}$ in (2.6), according to (1.5), depend on components of vector $\mathbf{v}$. However, $\mathbf{v}$ is not velocity vector for dynamical system (1.11). Vectors $\mathbf{v}$ and $\mathbf{u}$ are bound by the map (1.8), which is expressed by formula (1.7) in local chart. This map is diffeomorphism due to our assumptions (see definition 1.1). Therefore we can transform (1.11) to variables $x^1, \ldots, x^n, v^1, \ldots, v^n$. Here we write

$$\dot{x}^i = \frac{v^i}{\Omega}, \qquad \dot{v}^i = \Psi^i(x^1, \ldots, x^n, v^1, \ldots, v^n). \tag{3.1}$$

One could express $\Psi^1, \ldots, \Psi^n$ through functions $\Phi^1, \ldots, \Phi^n$ in (1.11). However, the latter ones are arbitrary functions. Therefore we can assume $\Psi^1, \ldots, \Psi^n$ in (3.1) to be arbitrary functions as well, with no need to follow their relations to the functions $\Phi^1, \ldots, \Phi^n$ in (1.11).

In the next step we use Legendre transformation (1.6) in order to transform differential equations (3.1) further. Now we write them as

$$\dot{x}^i = \frac{v^i}{\Omega}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{1}{\Omega}\,\frac{\partial L}{\partial x^i} = Q_i, \tag{3.2}$$

where $Q_i = Q_i(x^1, \ldots, x^n, v^1, \ldots, v^n)$. Since λ is diffeomorphism, these equation (3.2) are equivalent to previous ones (3.1). This is relative form of the equations of Newtonian dynamics (1.11). Relative, because in writing them we need another dynamical system (1.3).

4. Extended tensor fields.

Let's consider quantities $Q_1, \ldots, Q_n$ in (3.2). By means of direct calculations one can check up that these quantities are transformed as components of covector under the changes of local charts in M. They define a covector at the point p, where p is a point with local coordinates $x^1, \ldots, x^n$. Let's denote this covector by $\mathbf{Q}$ and note that it depends not only on p, but on components of vector $\mathbf{v}$ as well. This means that $\mathbf{Q}$ fits the definition of extended covector field.

Definition 4.1. Extended covector field $\mathbf{X}$ on a manifold M is a covector-valued function that to each point $q = (p, \mathbf{v})$ of tangent bundle TM puts into correspondence some covector from cotangent space $T^*_p(M)$ at the point $p = \pi(q)$ in M.

Similarly one can define extended tensor fields on the manifold M. First let's consider the following tensor product of tangent and cotangent spaces:

$$T^r_s(p, M) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r \text{ times}} \otimes \underbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}_{s \text{ times}}.$$

Tensor product $T^r_s(p, M)$ is known as a space of (r, s)-tensors at the point p of the manifold M. Pair of integer numbers (r, s) determines the type of tensors. Elements of the space $T^r_s(p, M)$ are called *r-times contravariant and s-times covariant tensors*, or tensors of the type (r, s), or, for brevity, (r, s)-tensors.

Definition 4.2. Extended tensor field $\mathbf{X}$ of the type (r, s) on a manifold M is a tensor-valued function that to each point $q = (p, \mathbf{v})$ of tangent bundle TM puts into correspondence some tensor from tensor space $T^r_s(p, M)$.

As far as I know, extended tensor fields first arose in Finslerian geometry. They was intensively used in theory of Newtonian dynamical systems admitting normal shift (see theses [6], [7] and references therein). Their application to Lagrangian and Hamiltonian dynamical systems is explained in [1], [3], and in [4].

5. Momentum representation for extended tensor fields.

Note that we can replace tangent bundle TM in definition 4.2 by cotangent bundle T^*M. Then we obtain another definition of extended tensor field.

Definition 5.1. Extended tensor field $\mathbf{Y}$ of the type (r, s) on a manifold M is a tensor-valued function that to each point $q = (p, \mathbf{p})$ of cotangent bundle T^*M puts into correspondence some tensor from tensor space $T^r_s(p, M)$.

Extended tensor fields as given by definitions 4.2 and 5.1 are two different objects. However, they can be related to each other by Legendre map (1.6) that links tangent bundle TM and cotangent bundle T^*M. Suppose that

$$\mathbf{X} = \mathbf{Y} \circ \lambda, \qquad\qquad \mathbf{Y} = \mathbf{X} \circ \lambda^{-1}. \tag{5.1}$$

If relationships (5.1) hold, we say that $\mathbf{Y}$ is $\mathbf{p}$-representation (or **momentum** representation) of extended tensor field $\mathbf{X}$. Similarly, in this case we say that $\mathbf{X}$ is $\mathbf{v}$-representation (or **velocity** representation) of extended tensor field $\mathbf{Y}$.

6. Weak normality condition.

Let's consider Newtonian dynamical system written in relative form (3.2) and let's consider some one-parametric family of trajectories $p = p(t, y)$ of this dynamical system. In local chart this family of trajectories is represented by functions

$$\begin{cases} x^1 = x^1(t, y),\\ \dots\dots\dots\dots\\ x^n = x^n(t, y) \end{cases} \tag{6.1}$$

(compare with (2.4)). Time derivatives of (6.1) define vector $\mathbf{v}$ according to first part of the equations (3.2). Components of $\mathbf{v}$ depend on t and y:

$$v^1 = v^1(t, y),\ \dots,\ v^n = v^n(t, y). \tag{6.2}$$

Like in (2.5), we can define variation vector $\boldsymbol{\tau}$. It is given by formula

$$\boldsymbol{\tau} = \tau^1\,\frac{\partial}{\partial x^1} + \ldots + \tau^n\,\frac{\partial}{\partial x^n},\ \text{ where }\ \tau^s = \frac{\partial x^s}{\partial y}. \tag{6.3}$$

Here we have only one variation vector since, besides time variable t, in (6.1) we have only one parameter y. Therefore we have only one deviation function

$$\varphi = \langle \mathbf{p} \,|\, \boldsymbol{\tau} \rangle = \sum_{s=1}^{n} p_s\, \tau^s. \tag{6.4}$$

Let's differentiate (6.2) with respect to parameter y. This yields a series of functions

$$\theta^i = \theta^i(t, y) = \frac{\partial v^i}{\partial y}. \tag{6.5}$$

Double set of functions $\tau^1, \ldots, \tau^n$ and $\theta^1, \ldots, \theta^n$ satisfy a system of linear ordinary differential equations with respect to time variable t. This system is obtained as linearization for (3.2). Differentiating first part of the equations (3.2) with respect to parameter y, we get the following expression for time derivative $\dot{\tau}^i$:

$$\begin{gathered}\dot{\tau}^i = \frac{\theta^i}{\Omega} - \sum_{s=1}^{n} \frac{v^i}{\Omega^2}\, \frac{\partial L}{\partial v^s}\, \theta^s - \\ - \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial^2 L}{\partial v^k\, \partial v^s}\, \frac{v^i\, v^k\, \theta^s}{\Omega^2} - \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial^2 L}{\partial v^k\, \partial x^s}\, \frac{v^i\, v^k\, \tau^s}{\Omega^2}.\end{gathered} \tag{6.6}$$

Differentiating second part of the equations (3.2) with respect to y, we obtain

$$\begin{gathered}\sum_{s=1}^{n} \frac{\partial^2 L}{\partial v^i\, \partial v^s}\, \dot{\theta}^s + \sum_{s=1}^{n} \frac{\partial^2 L}{\partial v^i\, \partial x^s}\, \dot{\tau}^s + \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial^3 L}{\partial v^i\, \partial v^s\, \partial v^k}\, \dot{v}^k\, \theta^s + \\ + \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial^3 L}{\partial v^i\, \partial v^s\, \partial x^k}\, \dot{x}^k\, \theta^s + \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial^3 L}{\partial v^i\, \partial x^s\, \partial v^k}\, \dot{v}^k\, \tau^s + \\ + \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial^3 L}{\partial v^i\, \partial x^s\, \partial x^k}\, \dot{x}^k\, \tau^s - \sum_{s=1}^{n} \frac{\partial^2 L}{\partial x^i\, \partial v^s}\, \frac{\theta^s}{\Omega} - \sum_{s=1}^{n} \frac{\partial^2 L}{\partial x^i\, \partial x^s}\, \frac{\tau^s}{\Omega} + \\ + \sum_{s=1}^{n} \frac{\partial L}{\partial x^i}\, \frac{\partial L}{\partial v^s}\, \frac{\theta^s}{\Omega^2} + \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial L}{\partial x^i}\, \frac{\partial^2 L}{\partial v^k\, \partial v^s}\, \frac{v^k\, \theta^s}{\Omega^2} + \\ + \sum_{k=1}^{n} \sum_{s=1}^{n} \frac{\partial L}{\partial x^i}\, \frac{\partial^2 L}{\partial v^k\, \partial x^s}\, \frac{v^k\, \tau^s}{\Omega^2} = \sum_{s=1}^{n} \frac{\partial Q_i}{\partial x^s}\, \tau^s + \sum_{s=1}^{n} \frac{\partial Q_i}{\partial v^s}\, \theta^s.\end{gathered} \tag{6.7}$$

The equalities (6.7) are not resolved with respect to derivatives $\dot{\theta}^1, \ldots, \dot{\theta}^n$. However, they can be resolved. Indeed, if we denote by $\mathbf{g}$ a matrix with components

$$g_{ij} = \frac{\partial^2 L}{\partial v^i\, \partial v^j}, \tag{6.8}$$

we can see that due to (1.5) it is Jacoby matrix for Legendre transformation (1.6). Matrix (6.8) is non-degenerate since Legendre map λ is diffeomorphism due to our initial assumptions (see definition 1.1 and ending part of section 1 after it).

As we already mentioned above, both (6.6) and (6.7) form a system of homogeneous linear ordinary differential equations with respect to functions $\tau^1, \ldots, \tau^n$ and $\theta^1, \ldots, \theta^n$. Solutions of these equations form linear space of dimension $2n$, We denote it by $\mathfrak{T}$. Looking at (6.4), one can see that φ depends linearly on $\tau^1, \ldots, \tau^n$. Hence all time derivatives of φ (i. e. $\dot{\varphi}$, $\ddot{\varphi}$, and so forth) depend linearly on $\tau^1, \ldots, \tau^n$ and on time derivatives of $\tau^1, \ldots, \tau^n$. Due to equations (6.6) and (6.7) the latter ones can be expressed linearly through $\tau^1, \ldots, \tau^n$ and $\theta^1, \ldots, \theta^n$. This means that for each fixed instance of time t the value of function $\varphi(t)$ itself and values of all time derivatives of this function

$$\varphi^{(k)} = \frac{d^k\varphi}{dt^k}$$

are linear functionals belonging to dual space $\mathfrak{T}^*$. The dimension of $\mathfrak{T}^*$ is finite, it is equal to $2n$. Therefore functions $\varphi,\ \dot{\varphi},\ \ddot{\varphi},\ \ldots,\ \varphi^{(2n)}$ are linearly dependent as elements of $\mathfrak{T}^*$. This means that there are some coefficients $C_0, \ldots, C_{2n}$, which do not vanish simultaneously, such that the following equality holds:

$$\sum_{i=0}^{n} C_i\,\varphi^{(i)} = 0. \tag{6.9}$$

For fixed t coefficients $C_0, \ldots, C_{2n}$ in (6.9) are real numbers. However, if t is not fixed, then $C_0, \ldots, C_{2n}$ depend on t. They also depend on that particular trajectory of dynamical system (3.2), for which functions $\varphi^{(i)}(t)$ are calculated:

$$\sum_{i=0}^{n} C_i(t)\,\varphi^{(i)} = 0. \tag{6.10}$$

Theorem 6.1. *For each trajectory $p = p(t)$ of Newtonian dynamical system* (3.2) *all deviation functions $\varphi = \varphi(t)$ on this trajectory satisfy the same linear homogeneous ordinary differential equation* (6.10) *of the order not greater than $2n$.*

Saying *"all deviation functions"*, in theorem 6.1 we assume that given trajectory $p = p(t)$ can be included into one-parametric family of trajectories by all possible ways. Each such inclusion defines some variation vector $\boldsymbol{\tau} = \boldsymbol{\tau}(t)$ and corresponding deviation function $\varphi = \varphi(t)$ on that trajectory. Functions $C_0, \ldots, C_{2n}$ in (6.10) depend on the trajectory $p = p(t)$, but they do not depend on how this trajectory is included into one parametric family of trajectories.

Theorem 6.1 gives upper estimate for the order of ODE (6.10). For most cases this estimate $2n$ is reached. However, in some special cases real order can be much less than $2n$. We consider one of such special cases.

Definition 6.1. We say that Newtonian dynamical system (3.2) satisfies **weak** normality condition if for each its trajectory $p = p(t)$ there is some second order homogeneous linear ordinary differential equation

$$\ddot{\varphi} = \mathcal{A}(t)\,\dot{\varphi} + \mathcal{B}(t)\,\varphi \tag{6.11}$$

such that all deviation functions on this trajectory satisfy this differential equation.

In other words, weak normality condition is fulfilled when (6.10) reduces to second order differential equation (6.11).

7. Additional normality condition.

Suppose that weak normality condition is fulfilled somehow. Then all deviation functions (2.6) arising in shift construction satisfy second order differential equation of the form (6.11). Coefficients $\mathcal{A}$ and $\mathcal{B}$ depend on shift trajectory (see definition 6.1), therefore here we should write this equation as

$$\ddot{\varphi} = \mathcal{A}(t, y^1, \ldots, y^{n-1})\,\dot{\varphi} + \mathcal{B}(t, y^1, \ldots, y^{n-1})\,\varphi. \tag{7.1}$$

In spite of this minor difference (7.1) is linear homogeneous ODE for φ with respect to time variable t. According to definition 2.1, in order to have normal shift we should provide vanishing of all deviation functions $\varphi_i, \ldots, \varphi_{n-1}$ in (2.6). Due to (7.1) now it is sufficient to provide the following initial data for them:

$$\varphi_i,\Big|_{t=0} = 0, \qquad\qquad \dot{\varphi}_i\Big|_{t=0} = 0. \tag{7.2}$$

The only way to provide these initial conditions is to choose proper initial conditions in (2.1). First part of initial conditions (2.1) says that shift trajectories should start the points of initial hypersurface S. We cannot change this condition. However, we can specify second part of initial conditions (2.1). Note that at each point $p \in S$ we have tangent space $T_p(S)$ embedded into tangent space $T_p(M)$ as a subspace of codimension 1. Denote by $\mathbf{n} = \mathbf{n}(p)$ some nonzero covector that vanishes when contracted with all vectors from $T_p(S)$:

$$\langle \mathbf{n} \,|\, \boldsymbol{\tau} \rangle = \sum_{s=1}^{n} n_s\,\tau^s = 0, \text{ for all } \boldsymbol{\tau} \in T_p(S). \tag{7.3}$$

It is called *normal covector* for S. At each point $p \in S$ normal covector $\mathbf{n}$ is determined uniquely up to a scalar factor. We have no tools for to specify this factor canonically, but, nevertheless, we can glue normal covectors from various points into a smooth covector-valued function $\mathbf{n} = \mathbf{n}(p)$ on S (or locally in small domains covering all points of S). Now let's compare (7.3) with (2.6) and remember that for $t = 0$ variation vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ form a base in tangent spaces to S. Therefore in order to provide first part of initial conditions (7.1) we should direct initial momentum covector $\mathbf{p}$ along normal covector of S. This means that we should specify initial conditions (2.1) as follows:

$$x^i\Big|_{t=0} = x^i(p), \qquad\qquad p_i\Big|_{t=0} = \nu(p)\cdot n_i(p). \tag{7.4}$$

Here $n_i(p)$ are components of normal covector $\mathbf{n}(p)$, while $\nu = \nu(p)$ is some smooth scalar function on S, which is yet undefined.

When applied to Newtonian dynamical system written in relative form (3.2), initial data (7.4) determine initial velocity $\mathbf{v}$ implicitly through Legendre transformation (1.5). However, we can make them explicit initial data if we pass to

$\mathbf{p}$-representation in the equations (3.2). Here these equations look like

$$\dot{x}^i=\frac{1}{\Omega}\,\frac{\partial H}{\partial p_i},\qquad\qquad \dot{p}_i=-\frac{1}{\Omega}\,\frac{\partial H}{\partial x^i}+Q_i. \tag{7.5}$$

Equations (7.5) are the equations of Newtonian dynamical system in the form relative to modified Hamiltonian dynamical system with Hamilton function H. Function H is derived from Lagrange function L as follows:

$$H=h\circ\lambda^{-1},\ \text{ where }\ h=\sum_{i=1}^{n}v^i\,\frac{\partial L}{\partial v^i}-L. \tag{7.6}$$

In other words, extended scalar field H is $\mathbf{p}$-representation of extended scalar field h, where h is given by formula (7.6). Denominator Ω in (7.5) is $\mathbf{p}$-representation of denominator Ω in (3.2). It is expressed through Hamilton function H:

$$\Omega=\sum_{i=1}^{n}p_i\,\frac{\partial H}{\partial p_i}. \tag{7.7}$$

Functions Q_i in (7.5) are components of the same extended covector field $\mathbf{Q}$, as in (3.2), but transformed to $\mathbf{p}$-representation. Their arguments are $x^1,\,\ldots\,,x^n$ and $p_1,\,\ldots\,,p_n$. More details concerning formulas (7.6) and (7.7) and Legendre transformation in whole can be found in book [2] and in papers [1], [3], and [4].

Now let's proceed with initial conditions (7.2). First part of these initial conditions now is fulfilled due to (7.4). We should provide second part of them by proper choice of function $\nu=\nu(p)=\nu(y^1,\ldots,y^{n-1})$ in (7.4). Let's calculate time derivative of deviation function φ_i using differential equations (7.5):

$$\begin{aligned}\dot{\varphi}_i&=\frac{d}{dt}\left(\sum_{s=1}^{n}p_s\,\tau^s_i\right)=\sum_{s=1}^{n}\dot{p}_s\,\tau^s_i+\sum_{s=1}^{n}p_s\,\dot{\tau}^s_i=\\&=-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i+\sum_{s=1}^{n}Q_s\,\tau^s_i+\sum_{s=1}^{n}p_s\,\dot{\tau}^s_i.\end{aligned} \tag{7.8}$$

In order to calculate time derivatives $\dot{\tau}^s_i$ in formula (7.8) we also use differential equations (7.5). As a result we get the following formula for $\dot{\tau}^s_i$:

$$\begin{gathered}\dot{\tau}^s_i=\frac{\partial^2x^s}{\partial t\,\partial y^i}=\frac{\partial}{\partial y^i}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)=\sum_{r=1}^{n}\frac{\partial x^r}{\partial y^i}\,\frac{\partial}{\partial x^r}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)+\\+\sum_{r=1}^{n}\frac{\partial p_r}{\partial y^i}\,\frac{\partial}{\partial p_r}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)=\sum_{r=1}^{n}\tau^r_i\,\frac{\partial}{\partial x^r}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)+\sum_{r=1}^{n}\frac{\partial p_r}{\partial y^i}\,\frac{\partial}{\partial p_r}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right).\end{gathered}$$

Let's substitute this formula into (7.8). Then for time derivative $\dot{\varphi}_i$ we obtain

$$\dot{\varphi}_i=-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i+\sum_{s=1}^{n}Q_s\,\tau^s_i+\sum_{s=1}^{n}\sum_{r=1}^{n}p_s\,\tau^r_i\,\frac{\partial}{\partial x^r}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)+$$

$$+\sum_{s=1}^{n}\sum_{r=1}^{n} p_s\,\frac{\partial p_r}{\partial y^i}\,\frac{\partial}{\partial p_r}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\right)=\sum_{s=1}^{n} Q_s\,\tau^s_i+\sum_{r=1}^{n}\tau^r_i\,\frac{\partial}{\partial x^r}\!\left(\,\sum_{s=1}^{n}\frac{p_s}{\Omega}\,\frac{\partial H}{\partial p_s}\right)+$$
$$+\sum_{r=1}^{n}\frac{\partial p_r}{\partial y^i}\,\frac{\partial}{\partial p_r}\!\left(\,\sum_{s=1}^{n}\frac{p_s}{\Omega}\,\frac{\partial H}{\partial p_s}\right)-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\,\frac{\partial p_s}{\partial y^i}-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i.$$

Note that sums in round brackets are identically equal to unity:

$$\sum_{s=1}^{n}\frac{p_s}{\Omega}\,\frac{\partial H}{\partial p_s}=1. \tag{7.9}$$

This follows from (7.7). Due to (7.9) two terms in the above formula for $\dot\varphi_i$ do vanish. And we get rather simple formula for time derivative $\dot\varphi_i$:

$$\dot\varphi_i=-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\,\frac{\partial p_s}{\partial y^i}-\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i+\sum_{s=1}^{n}Q_s\,\tau^s_i. \tag{7.10}$$

If we recall initial conditions (7.2), then from (7.10) we derive

$$\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}\,\tau^s_i+\sum_{s=1}^{n}\frac{1}{\Omega}\,\frac{\partial H}{\partial p_s}\,\left(\frac{\partial p_s}{\partial y^i}\right)\biggr|_{t=0}=\sum_{s=1}^{n}Q_s\,\tau^s_i. \tag{7.11}$$

Calculating partial derivatives $\partial p_s/\partial y^i$ in the equality (7.11), we should remember that $p_s=\nu\cdot n_s$. This follows from initial data (7.4). Then

$$\left(\frac{\partial p_s}{\partial y^i}\right)\biggr|_{t=0}=\frac{\partial\nu}{\partial y^i}\,n_s+\nu\,\frac{\partial n_s}{\partial y^i}=\frac{1}{\nu}\,\frac{\partial\nu}{\partial y^i}\,p_s+\nu\,\frac{\partial n_s}{\partial y^i}. \tag{7.12}$$

Substituting this expression into (7.11) and using formula (7.7) for Ω, we can transform (7.12) into the partial differential equations for ν:

$$\frac{\partial\nu}{\partial y^i}=-\sum_{s=1}^{n}\frac{\nu^2}{\Omega}\,\frac{\partial n_s}{\partial y^i}\,\frac{\partial H}{\partial p_s}-\sum_{s=1}^{n}\nu\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}-Q_s\right)\tau^s_i. \tag{7.13}$$

Having derived (7.13), we have proved the following lemma.

Lemma 7.1. *Initial conditions* (7.2) *for deviation functions* $\varphi_1,\,\ldots\,,\varphi_{n-1}$ *are equivalent to initial data* (7.4) *for shift trajectories, where scalar function* ν *on* S *satisfies differential equations* (7.13).

In two-dimensional manifold M, when $n=2$, hypersurfaces are curves. In this case we have only one deviation function and only one variable $y=y^1$ as a parameter on S. Therefore (7.13) appears to be ordinary differential equation for the function $\nu=\nu(y)$. We can set initial value problem

$$\nu(y)\Bigr|_{y=0}=\nu_0, \tag{7.14}$$

which is always solvable (at least locally) for all $\nu_0\neq 0$. This means that in two-dimensional case weak normality condition stated in definition 6.1 is sufficient for to arrange normal shift of any predefined hypersurface S in M.

In multidimensional case $n \geqslant 3$ situation changes crucially. Here equations (7.13) form so called *complete system of Pfaff equation* for $\nu = \nu(y^1, \dots, y^{n-1})$. Each separate equation in such system can be treated as ODE. Therefore initial condition like (7.14) is the best way for fixing some particular solution:

$$\nu(p_0) = \nu_0. \tag{7.15}$$

However, initial value problem (7.15) for Pfaff equations (7.13) is not always solvable: some additional conditions should be fulfilled. This is why we had a fork in development of theory of metric normal shift in Riemannian geometry (compare theses [6] and [7]). This fork is present here as well.

Definition 7.1. Complete system of Pfaff equations (7.13) is called *compatible* if initial value problem (7.15) for it is locally solvable for all $\nu_0 \neq 0$.

Let's write Pfaff equations (7.13) formally, denoting by ψ_i their right hand sides:

$$\frac{\partial \nu}{\partial y^i} = \psi_i(\nu, y^1, \dots, y^{n-1}) \tag{7.16}$$

Due to (7.16) we can calculate mixed partial derivatives of ν in two different ways

$$\frac{\partial^2 \nu}{\partial y^i\,\partial y^j} = \frac{\partial \psi_i}{\partial y^j} + \frac{\partial \psi_i}{\partial \nu}\,\psi_j = \vartheta_{ij}(\nu, y^1, \dots, y^{n-1}), \tag{7.17}$$

$$\frac{\partial^2 \nu}{\partial y^j\,\partial y^i} = \frac{\partial \psi_j}{\partial y^i} + \frac{\partial \psi_j}{\partial \nu}\,\psi_i = \vartheta_{ji}(\nu, y^1, \dots, y^{n-1}). \tag{7.18}$$

Equating (7.17) and (7.18), we get compatibility condition for (7.16):

$$\vartheta_{ij}(\nu, y^1, \dots, y^{n-1}) = \vartheta_{ji}(\nu, y^1, \dots, y^{n-1}). \tag{7.19}$$

Lemma 7.2. *Pfaff equations* (7.16) *are compatible in the sense of definition 7.1 if and only if for $\nu \neq 0$ left and right hands sides of* (7.19) *are equal to each other identically as functions of n independent variables y^1, ..., y^{n-1}, and ν.*

Lemma 7.2 is standard result in the theory of Pfaff equations. Proof of this lemma can be found in thesis [6].

Definition 7.2. We say that Newtonian dynamical system (3.2) or, equivalently, dynamical system (7.5) satisfies **additional** normality condition if Pfaff equations (7.13) derived from initial conditions (7.2) are compatible for any hypersurface S in M and for any marked point $p_0 \in S$.

8. Complete and strong normality conditions.

Both **weak** and **additional** normality conditions constitute so called **complete** normality condition in Riemannian geometry (see thesis [6]). We shall keep this terminology saying that Newtonian dynamical system satisfies **complete** normality condition if both conditions from definitions 6.2 and 7.2 are fulfilled[1]. But, apart from this complete normality condition, we shall consider **strong** normality condition given by the following definition.

Definition 8.1. Newtonian dynamical system given by differential equations (3.2) or, equivalently, by differential equations (7.5) satisfies **strong** normality condition if for any hypersurface S in M, for any marked point $p_0 \in S$, and for any real constant $\nu_0 \neq 0$ there is some smaller part S' of S containing marked point p_0 and there is some smooth function $\nu = \nu(p)$ in this smaller part S' normalized by the condition (7.15) and such that initial data (7.4) with this function ν determine normal shift of S' in the sense of definition 2.1.

In simpler words, dynamical systems satisfying strong normality condition are called systems **admitting normal shift** of hypersurfaces. They are able to move normally any predefined hypersurface S in M.

It is easy to note that complete normality condition is sufficient for Newtonian dynamical system to satisfy strong normality condition. Below in section 17 we shall prove that it is not only sufficient, but necessary condition as well.

9. Compatibility equations.

Now suppose that M is a manifold of dimension $n \geqslant 3$. In this case we should study compatibility equations (7.19) for Pfaff system (7.13). For this purpose let's calculate partial derivatives (7.17) and (7.18) explicitly:

$$\begin{aligned}
\frac{\partial^2\nu}{\partial y^i\,\partial y^j} &= -\sum_{s=1}^{n}\frac{\nu^2}{\Omega}\,\frac{\partial H}{\partial p_s}\,\frac{\partial^2 n_s}{\partial y^i\,\partial y^j} - \sum_{s=1}^{n}\nu\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s} - Q_s\right)\frac{\partial^2 x^s}{\partial y^i\,\partial y^j} +\\
&+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{2\,\nu^3}{\Omega^2}\,\frac{\partial H}{\partial p_s}\,\frac{\partial H}{\partial p_r}\,\frac{\partial n_s}{\partial y^i}\,\frac{\partial n_r}{\partial y^j} + \nu\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s} - Q_s\right)\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r} - Q_r\right)\tau^s_i\,\tau^r_j\, +\right.\\
&\left.+\,\frac{2\,\nu^2}{\Omega}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s} - Q_s\right)\frac{\partial H}{\partial p_r}\,\frac{\partial n_r}{\partial y^j}\,\tau^s_i + \frac{\nu^2}{\Omega}\frac{\partial H}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r} - Q_r\right)\frac{\partial n_s}{\partial y^i}\,\tau^r_j\right) -\\
&-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu^2\,\frac{\partial}{\partial p_s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial p_s}{\partial y^i}\,\frac{\partial n_r}{\partial y^j} - \sum_{s=1}^{n}\sum_{r=1}^{n}\nu\,\frac{\partial}{\partial p_s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r} - Q_r\right)\frac{\partial p_s}{\partial y^i}\,\tau^r_j\, -\\
&-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu^2\,\frac{\partial}{\partial x^s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial n_r}{\partial y^j}\,\tau^s_i - \sum_{s=1}^{n}\sum_{r=1}^{n}\nu\,\frac{\partial}{\partial x^s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r} - Q_r\right)\tau^s_i\,\tau^r_j.
\end{aligned}$$

For Pfaff equations (7.13) to be compatible, right hand side of the above equality should be symmetric in indices i and j. Some terms there are obviously symmetric. Below we shall not write such terms explicitly denoting them by dots:

$$\frac{\partial^2\nu}{\partial y^i\,\partial y^j} = \dots + \sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{2\,\nu^2}{\Omega}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s} - Q_s\right)\frac{\partial H}{\partial p_r} - \nu^2\,\frac{\partial}{\partial x^s}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\right)\frac{\partial n_r}{\partial y^j}\,\tau^s_i\, +$$

[1] In two-dimensional case $n = 2$ complete normality condition reduces to weak normality condition since additional normality condition in this case is always fulfilled.

$$
\begin{aligned}
&+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{\nu^2}{\Omega}\,\frac{\partial H}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)-\nu^2\,\frac{\partial}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)\right)\frac{\partial n_s}{\partial y^i}\,\tau^r_j\,-\\
&-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu\,p_s\,\frac{\partial}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial\nu}{\partial y^i}\,\frac{\partial n_r}{\partial y^j}-\sum_{s=1}^{n}\sum_{r=1}^{n}p_s\,\frac{\partial}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)\frac{\partial\nu}{\partial y^i}\,\tau^r_j\,-\\
&-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu^3\,\frac{\partial}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial n_s}{\partial y^i}\,\frac{\partial n_r}{\partial y^j}-\sum_{s=1}^{n}\sum_{r=1}^{n}\nu\,\frac{\partial}{\partial x^s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)\tau^s_i\,\tau^r_j.
\end{aligned}
$$

In the above calculations we used formula (7.12) in order to express partial derivatives $\partial p_s/\partial y^i$ through $\partial n_s/\partial y^i$. As a result we have got partial derivatives $\partial\nu/\partial y^i$ in the above expression. In order to eliminate them now we shall use (7.13):

$$
\begin{aligned}
&\frac{\partial^2\nu}{\partial y^i\,\partial y^j}=\dots+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{2\,\nu^2}{\Omega}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}-Q_s\right)\frac{\partial H}{\partial p_r}-\nu^2\,\frac{\partial}{\partial x^s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)+\right.\\
&\left.+\sum_{q=1}^{n}\nu^2\,p_q\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}-Q_s\right)\right)\frac{\partial n_r}{\partial y^j}\,\tau^s_i+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{\nu^2}{\Omega}\,\frac{\partial H}{\partial p_s}\,\times\right.\\
&\times\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)-\nu^2\,\frac{\partial}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)+\sum_{q=1}^{n}\frac{\nu^2\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)\times\\
&\left.\times\frac{\partial H}{\partial p_s}\right)\frac{\partial n_s}{\partial y^i}\,\tau^r_j+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\sum_{q=1}^{n}\frac{\nu^3\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial H}{\partial p_s}+\frac{\nu^3}{\Omega^2}\,\frac{\partial\Omega}{\partial p_s}\,\frac{\partial H}{\partial p_r}\right)\times\\
&\times\frac{\partial n_s}{\partial y^i}\,\frac{\partial n_r}{\partial y^j}+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\sum_{q=1}^{n}\nu\,p_q\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}-Q_s\right)-\right.\\
&\left.-\,\nu\,\frac{\partial}{\partial x^s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)\right)\tau^s_i\,\tau^r_j.
\end{aligned}
$$

Term with product of partial derivatives $\partial n_s/\partial y^i$ and $\partial n_r/\partial y^j$ in the above expression can be replaced by dots. Indeed, one can easily check up that coefficient of such product of derivatives is symmetric in indices s and r:

$$
\begin{aligned}
&\sum_{q=1}^{n}\frac{\nu^3\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\frac{\partial H}{\partial p_s}+\frac{\nu^3}{\Omega^2}\,\frac{\partial\Omega}{\partial p_s}\,\frac{\partial H}{\partial p_r}=-\sum_{q=1}^{n}\frac{\nu^3\,p_q}{\Omega^3}\,\frac{\partial\Omega}{\partial p_q}\,\frac{\partial H}{\partial p_r}\,\frac{\partial H}{\partial p_s}+\\
&+\sum_{q=1}^{n}\frac{\nu^3\,p_q}{\Omega^2}\,\frac{\partial^2 H}{\partial p_q\,\partial p_r}\,\frac{\partial H}{\partial p_s}+\sum_{q=1}^{n}\frac{\nu^3\,p_q}{\Omega^2}\,\frac{\partial^2 H}{\partial p_q\,\partial p_s}\,\frac{\partial H}{\partial p_r}+\frac{\nu^3}{\Omega^2}\,\frac{\partial H}{\partial p_s}\,\frac{\partial H}{\partial p_r}.
\end{aligned}
$$

Now let's study the term with $\tau^s_i\,\tau^r_j$. For the coefficient in this term we derive

$$
\sum_{q=1}^{n}\nu\,p_q\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}-Q_s\right)-\nu\,\frac{\partial}{\partial x^s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r}-Q_r\right)=
$$

$$= \sum_{q=1}^{n} \nu\, p_q \left(-\frac{1}{\Omega^2}\,\frac{\partial\Omega}{\partial p_q}\,\frac{\partial H}{\partial x^r} + \frac{1}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial x^r} - \frac{\partial Q_r}{\partial p_q}\right)\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s} - Q_s\right) -$$

$$- \nu\left(-\frac{1}{\Omega^2}\,\frac{\partial\Omega}{\partial x^s}\,\frac{\partial H}{\partial x^r} + \frac{1}{\Omega}\,\frac{\partial^2 H}{\partial x^s\,\partial x^r} - \frac{\partial Q_r}{\partial x^s}\right) = \ldots + \sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega^2}\,\frac{\partial^2 H}{\partial p_q\,\partial x^r}\,\frac{\partial H}{\partial x^s} -$$

$$-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial Q_r}{\partial p_q}\,\frac{\partial H}{\partial x^s} + \sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega^2}\,\frac{\partial H}{\partial p_q}\,\frac{\partial H}{\partial x^r}\,Q_s + \sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu\,p_q\,p_k}{\Omega^2}\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial x^r}\,Q_s -$$

$$-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial x^r}\,Q_s + \sum_{q=1}^{n}\nu\,p_q\,\frac{\partial Q_r}{\partial p_q}\,Q_s + \sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega^2}\,\frac{\partial^2 H}{\partial p_q\,\partial x^s}\,\frac{\partial H}{\partial x^r} - \frac{\nu}{\Omega}\,\frac{\partial^2 H}{\partial x^r\,\partial x^s} +$$

$$+\nu\,\frac{\partial Q_r}{\partial x^s} = \cdots - \sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial Q_r}{\partial p_q}\,\frac{\partial H}{\partial x^s} + \sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega^2}\,\frac{\partial H}{\partial p_q}\,\frac{\partial H}{\partial x^r}\,Q_s + \sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu\,p_q\,p_k}{\Omega^2}\,\times$$

$$\times\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial x^r}\,Q_s - \sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial x^r}\,Q_s + \sum_{q=1}^{n}\nu\,p_q\,\frac{\partial Q_r}{\partial p_q}\,Q_s + \nu\,\frac{\partial Q_r}{\partial x^s}.$$

In the above calculations we used formula (7.7) for Ω. As a result for partial derivative (7.17) we have derived the following expression:

$$\frac{\partial^2\nu}{\partial y^i\,\partial y^j} = \cdots + \sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{2\,\nu^2}{\Omega}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s} - Q_s\right)\frac{\partial H}{\partial p_r} - \nu^2\,\frac{\partial}{\partial x^s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right) + \sum_{q=1}^{n}\nu^2\;\times\right.$$

$$\left.\times\,p_q\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s} - Q_s\right)\right)\frac{\partial n_r}{\partial y^j}\,\tau^s_i + \sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{\nu^2}{\Omega}\,\frac{\partial H}{\partial p_s}\,\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r} - Q_r\right) - \right.$$

$$\left.-\,\nu^2\,\frac{\partial}{\partial p_s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r} - Q_r\right) + \sum_{q=1}^{n}\frac{\nu^2\,p_q}{\Omega}\,\frac{\partial}{\partial p_q}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^r} - Q_r\right)\frac{\partial H}{\partial p_s}\right)\frac{\partial n_s}{\partial y^i}\,\tau^r_j +$$

$$+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial Q_r}{\partial p_q}\,\frac{\partial H}{\partial x^s} + \sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega^2}\,\frac{\partial H}{\partial p_q}\,\frac{\partial H}{\partial x^r}\,Q_s + \sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu\,p_q\,p_k}{\Omega^2}\,\times\right.$$

$$\left.\times\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial x^r}\,Q_s - \sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial x^r}\,Q_s + \sum_{q=1}^{n}\nu\,p_q\,\frac{\partial Q_r}{\partial p_q}\,Q_s + \nu\,\frac{\partial Q_r}{\partial x^s}\right)\tau^s_i\,\tau^r_j.$$

Next step in transforming the above expression is based on the following equality:

$$\sum_{s=1}^{n}\sum_{r=1}^{n}\alpha^r_s\,\frac{\partial n_r}{\partial y^j}\,\tau^s_i + \sum_{s=1}^{n}\sum_{r=1}^{n}\beta^s_r\,\frac{\partial n_s}{\partial y^i}\,\tau^r_j = \sum_{s=1}^{n}\sum_{r=1}^{n}(\alpha^r_s - \beta^r_s)\,\frac{\partial n_r}{\partial y^j}\,\tau^s_i + \ldots\;.$$

Here, as we already used above, we denoted by dots terms symmetric in indices i and j. Further for our particular α_{rs} and β_{rs} we derive

$$\alpha^r_s - \beta^r_s = \frac{\nu^2}{\Omega}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s} - Q_s\right)\frac{\partial H}{\partial p_r} - \nu^2\,\frac{\partial}{\partial x^s}\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right) + \sum_{q=1}^{n}\nu^2\,p_q\,\times$$

$$\times\,\frac{\partial}{\partial p_q}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial p_r}\right)\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}-Q_s\right)+\nu^2\,\frac{\partial}{\partial p_r}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}-Q_s\right)-\sum_{q=1}^{n}\frac{\nu^2\,p_q}{\Omega}\,\times$$

$$\times\,\frac{\partial}{\partial p_q}\!\left(\frac{1}{\Omega}\,\frac{\partial H}{\partial x^s}-Q_s\right)\frac{\partial H}{\partial p_r}=\sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu^2}{\Omega^2}\,p_q\,p_k\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial p_r}\,Q_s\,-$$

$$-\sum_{q=1}^{n}\frac{\nu^2}{\Omega}\,p_q\,\frac{\partial^2 H}{\partial p_q\,\partial p_r}\,Q_s-\nu^2\,\frac{\partial Q_s}{\partial p_r}+\sum_{q=1}^{n}\frac{\nu^2}{\Omega}\,p_q\,\frac{\partial H}{\partial p_r}\,\frac{\partial Q_s}{\partial p_q}.$$

Summarizing all above calculations, for partial derivatives (7.17) we obtain

$$\begin{gathered}\frac{\partial^2\nu}{\partial y^i\,\partial y^j}=\cdots+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu^2}{\Omega^2}\,p_q\,p_k\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial p_r}\,Q_s\,-\right.\\ \left.-\sum_{q=1}^{n}\frac{\nu^2}{\Omega}\,p_q\,\frac{\partial^2 H}{\partial p_q\,\partial p_r}\,Q_s-\nu^2\,\frac{\partial Q_s}{\partial p_r}+\sum_{q=1}^{n}\frac{\nu^2}{\Omega}\,p_q\,\frac{\partial H}{\partial p_r}\,\frac{\partial Q_s}{\partial p_q}\right)\frac{\partial n_r}{\partial y^j}\,\tau^s_i\,+\\ +\sum_{s=1}^{n}\sum_{r=1}^{n}\left(-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial Q_r}{\partial p_q}\,\frac{\partial H}{\partial x^s}+\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega^2}\,\frac{\partial H}{\partial p_q}\,\frac{\partial H}{\partial x^r}\,Q_s+\sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu\,p_q\,p_k}{\Omega^2}\,\times\right.\\ \left.\times\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial x^r}\,Q_s-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial x^r}\,Q_s+\sum_{q=1}^{n}\nu\,p_q\,\frac{\partial Q_r}{\partial p_q}\,Q_s+\nu\,\frac{\partial Q_r}{\partial x^s}\right)\tau^s_i\,\tau^r_j.\end{gathered}\tag{9.1}$$

In a similar way for partial derivative (7.18) one can get analogous expression:

$$\begin{gathered}\frac{\partial^2\nu}{\partial y^j\,\partial y^i}=\cdots+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu^2}{\Omega^2}\,p_q\,p_k\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial p_s}\,Q_r\,-\right.\\ \left.-\sum_{q=1}^{n}\frac{\nu^2}{\Omega}\,p_q\,\frac{\partial^2 H}{\partial p_q\,\partial p_s}\,Q_r-\nu^2\,\frac{\partial Q_r}{\partial p_s}+\sum_{q=1}^{n}\frac{\nu^2}{\Omega}\,p_q\,\frac{\partial H}{\partial p_s}\,\frac{\partial Q_r}{\partial p_q}\right)\frac{\partial n_s}{\partial y^i}\,\tau^r_j\,+\\ +\sum_{s=1}^{n}\sum_{r=1}^{n}\left(-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial Q_s}{\partial p_q}\,\frac{\partial H}{\partial x^r}+\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega^2}\,\frac{\partial H}{\partial p_q}\,\frac{\partial H}{\partial x^s}\,Q_r+\sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu\,p_q\,p_k}{\Omega^2}\,\times\right.\\ \left.\times\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial x^s}\,Q_r-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial x^s}\,Q_r+\sum_{q=1}^{n}\nu\,p_q\,\frac{\partial Q_s}{\partial p_q}\,Q_r+\nu\,\frac{\partial Q_s}{\partial x^r}\right)\tau^s_i\,\tau^r_j.\end{gathered}\tag{9.2}$$

Now we are able to write compatibility equations (7.19) in explicit form. It is sufficient to equate partial derivatives (9.1) and (9.2) to each other. However, we shall not do this right now because we need some additional information in order to treat arising compatibility equation properly. We should replace partial derivatives in (9.1) and (9.2) by covariant derivatives, and we should understand some facts from theory of hypersurfaces in a manifold equipped with the only geometric structure given by Lagrange function L.

10. Differentiation of extended tensor fields.

Let's consider smooth extended tensor fields in their **p**-representation as given by definition 5.1. They form a graded ring with respect to standard operations of addition and tensor product. We denote it as follows:

$$\mathbf{T}(M)=\bigoplus_{r=0}^{\infty}\bigoplus_{s=0}^{\infty}T^r_s(M). \tag{10.1}$$

Graded ring (10.1) is equipped with operation of contraction, which is also standard. Moreover, (10.1) possesses the structure of algebra over the ring of smooth functions in T^*M. For this reason it is called *extended algebra of tensor fields*.

In extended algebra of tensor fields (10.1) one can define canonical covariant differentiation, which is called *vertical gradient* or *momentum gradient*:

$$\tilde{\nabla}\colon\ T^r_s(M)\to T^{r+1}_s(M). \tag{10.2}$$

In local chart momentum gradient (10.2) is determined by the following formula:

$$\tilde{\nabla}^q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial p_q}. \tag{10.3}$$

To define another covariant differentiation in $\mathbf{T}(M)$ one need some additional geometric structure in M. This is so called *extended affine connection*. We shall not discuss the nature of this structure (see thesis [6]). Note only that in local chart it is given by its components Γ^k_{ij}, which obey standard transformation rule:

$$\Gamma^k_{ij}=\sum_{m=1}^{n}\sum_{a=1}^{n}\sum_{c=1}^{n}S^k_m\,T^a_i\,T^c_j\,\tilde{\Gamma}^m_{ac}+\sum_{m=1}^{n}S^k_m\,\frac{\partial T^m_i}{\partial x^j}. \tag{10.4}$$

Here S^i_j and T^i_j are components of transition matrices S and T binding coordinates $x^1,\,\ldots\,,\,x^n$ and $\tilde{x}^1,\,\ldots\,,\,\tilde{x}^n$ in two overlapping local charts of M:

$$S^i_j=\frac{\partial x^i}{\partial\tilde{x}^j},\qquad\qquad T^i_j=\frac{\partial\tilde{x}^i}{\partial x^j}. \tag{10.5}$$

Unlike components of standard affine connection, components of extended affine connection Γ in **p**-representation depend not only on $x^1,\,\ldots\,,\,x^n$, but also on components $p_1,\,\ldots\,,\,p_n$ of momentum covector **p**:

$$\Gamma^k_{ij}=\Gamma^k_{ij}(x^1,\ldots,x^n,p_1,\ldots,p_n). \tag{10.6}$$

Thus, if M possesses extended affine connection Γ, one can define *horizontal gradient* or *spatial gradient* ∇ in $\mathbf{T}(M)$. In local chart it is expressed by formula

$$\begin{aligned}\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}&=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^q}+\sum_{a=1}^{n}\sum_{b=1}^{n}p_a\,\Gamma^a_{qb}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial p_b}+\\&+\sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{q\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{q\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned} \tag{10.7}$$

Horizontal gradient ∇ increments by 1 the number of lower indices of tensor field:

$$\nabla\colon\ T^r_s(M)\to T^r_{s+1}(M).\tag{10.8}$$

Using Legendre map, one can transform extended tensor fields to **v**-representation. In **v**-representation vertical gradient $\tilde\nabla$ is called *velocity gradient*. It is a map

$$\tilde\nabla\colon\ T^r_s(M)\to T^r_{s+1}(M)\tag{10.9}$$

similar to (10.8). In local chart this map (10.9) is given by formula

$$\tilde\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^q}.\tag{10.10}$$

Two versions of vertical gradient (10.3) and (10.10) can be related to each other by means of formula using components of matrix (6.8):

$$\tilde\nabla_q\bigl(X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda\bigr)=\sum^n_{k=1}g_{qk}\,\tilde\nabla^k X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda.\tag{10.11}$$

Remember that inverse Legendre map in local chart is given by explicit formula

$$v^i=\frac{\partial H}{\partial p_i}.\tag{10.12}$$

Therefore we can introduce matrix $\mathbf{g}^{-1}$ with the following components:

$$g^{ij}=\frac{\partial^2 H}{\partial p_i\,\partial p_j}.\tag{10.13}$$

Matrices (6.8) and (10.13) are inverse to each other when taken in the same representation, i.e. when both brought to **p**-representation or when both brought to **v**-representation. Matrix (10.13) is used in formula, which is similar to (10.11):

$$\tilde\nabla^q\bigl(X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda^{-1}\bigr)=\sum^n_{k=1}g^{qk}\,\tilde\nabla_k X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda^{-1}.\tag{10.14}$$

Looking at formulas (10.11) and (10.14), we see that matrices (6.8) and (10.13) here do part of work that metric tensor does in Riemannian geometry.

In order to define horizontal gradient for extended tensor fields in **v**-representation we should transform connection components (10.6) by means of Legendre map:

$$\Gamma^k_{ij}\to\Gamma^k_{ij}\circ\lambda=\Gamma^k_{ij}(x^1,\ldots,x^n,v^1,\ldots,v^n).\tag{10.15}$$

Then, using (10.15), we can define horizontal gradient by a formula in local chart:

$$\begin{aligned}\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}&=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^q}-\sum^n_{a=1}\sum^n_{b=1}v^a\,\Gamma^b_{qa}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^b}\,+\\&+\sum^r_{k=1}\sum^n_{a_k=1}\Gamma^{i_k}_{q\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\sum^s_{k=1}\sum^n_{b_k=1}\Gamma^{b_k}_{q\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned}\tag{10.16}$$

In spite of transformation rule (10.15) for connection components, in general, (10.7) and (10.16) are not different representations of the same tensor field. We have the following equality binding these two horizontal gradients:

$$\nabla_q\bigl(X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda\bigr)-\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda=\sum_{s=1}^{n}\nabla_q\tilde{\nabla}_s L\,\cdot\tilde{\nabla}^s X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda. \tag{10.17}$$

However, there is special case, when (10.7) and (10.16) do coincide. This is when

$$\nabla_q\tilde{\nabla}_s L=0. \tag{10.18}$$

In this case (10.17) turns to equality, which is similar to the equality (10.11):

$$\nabla_q\bigl(X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda\bigr)=\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\circ\lambda. \tag{10.19}$$

Definition 10.1. Extended connection Γ is called *concordant* with Lagrange function L if the equality (10.18) holds.

For concordant connections due to (10.19) horizontal gradient can be calculated either in **p** or in **v**-representation yielding the same result, but in different variables. Note that the equality (10.18) can be replaced by equivalent equality for Hamilton function H in **p**-representation. It looks like

$$\nabla_q\tilde{\nabla}^s H=0. \tag{10.20}$$

This equality (10.20) can be derived from (10.18) by direct calculations.

11. Differentiation along lines and hypersurfaces.

Let $p=p(t)$ be a smooth parametric curve in our manifold M. In local chart with coordinates $x^1,\,\ldots\,,\,x^n$ it is represented by functions

$$\begin{cases} x^1=x^1(t),\\ \,.\,.\,.\,.\,.\,.\,.\,.\,.\\ x^n=x^n(t). \end{cases} \tag{11.1}$$

Suppose that at each point p of this curve (6.1) some tensor of the type (r,s) is given, i. e. we have tensor-valued function $\mathbf{X}=\mathbf{X}(t)$. In local chart this tensor function is expressed by its components $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(t)$. If we had standard affine connection in M, we could define covariant derivative of tensor function $\mathbf{X}(t)$ with respect to parameter t along the curve. This is another tensor function $\mathbf{Y}=\nabla_t\mathbf{X}$ with components $Y^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\nabla_t X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ given by the following formula:

$$\begin{aligned}\nabla_t X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}&=\frac{dX^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{dt}+\sum_{k=1}^{r}\sum_{m=1}^{n}\sum_{a_k=1}^{n}\dot{x}^m\,\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\\&-\sum_{k=1}^{s}\sum_{m=1}^{n}\sum_{b_k=1}^{n}\dot{x}^m\,\Gamma^{b_k}_{m\,j_k}X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned} \tag{11.2}$$

However, with extended connection Γ we need come additional data in order to apply formula (11.2) to tensor function $\mathbf{X} = \mathbf{X}(t)$. Indeed, looking at (10.6) and (10.15), we see that in $\mathbf{p}$-representation we need functions

$$p_1(t),\ \dots\ ,\ p_n(t), \tag{11.3}$$

while in $\mathbf{v}$-representation we need another set of functions:

$$v^1(t),\ \dots\ ,\ v^n(t). \tag{11.4}$$

Taken by themselves, functions (11.3) and (11.4) are components of covector-function $\mathbf{p} = \mathbf{p}(t)$ and vector-function $\mathbf{v} = \mathbf{v}(t)$. But combined with functions (11.1), they define lift of initial curve $p = p(t)$ from M to cotangent bundle T^*M and to tangent bundle TM respectively. Once such lift is given, we can use formula (11.2). Covariant derivative given by this formula then is called *covariant derivative with respect to parameter t along the curve $p = p(t)$ due to its lift $q = q(t)$.*

Now let's discuss the question of how tensor function $\mathbf{X} = \mathbf{X}(t)$ on curve could be defined. Surely, it can be given immediately as it is. However, often tensor function $\mathbf{X} = \mathbf{X}(t)$ comes to curve from outer space, i. e. from manifold M. For example, if in M or at least in some neighborhood of our curve some standard (not extended) tensor field $\mathbf{X}$ is given, we can restrict it to the curve $p = p(t)$ thus obtaining tensor function $\mathbf{X} = \mathbf{X}(t)$. In the case of extended tensor field $\mathbf{X}$ we cannot restrict it to the curve $p = p(t)$ immediately. We should first choose a lift of this curve $q = q(t)$ in TM or in T^*M, then we could restrict $\mathbf{X}$ to the lifted curve. Now suppose that tensor function $\mathbf{X} = \mathbf{X}(t)$ on the curve $p = p(t)$ is obtained from extended tensor field $\mathbf{X}$ in this way. Then covariant derivative (11.2) for $\mathbf{X}(t)$ is determined. By direct calculations we can prove that this covariant derivative can be expressed through covariant derivatives (10.7) and (10.3) in $\mathbf{p}$-representation:

$$\nabla_t X^{i_1\dots\,i_r}_{j_1\dots\,j_s} = \sum_{k=1}^{n} \nabla_k X^{i_1\dots\,i_r}_{j_1\dots\,j_s} \cdot \dot{x}^k + \sum_{k=1}^{n} \tilde{\nabla}^k X^{i_1\dots\,i_r}_{j_1\dots\,j_s} \cdot \nabla_t p_k. \tag{11.5}$$

In $\mathbf{v}$-representation we have similar expression

$$\nabla_t X^{i_1\dots\,i_r}_{j_1\dots\,j_s} = \sum_{k=1}^{n} \nabla_k X^{i_1\dots\,i_r}_{j_1\dots\,j_s} \cdot \dot{x}^k + \sum_{k=1}^{n} \tilde{\nabla}_k X^{i_1\dots\,i_r}_{j_1\dots\,j_s} \cdot \nabla_t v^k \tag{11.6}$$

that uses covariant derivatives (10.16) and (10.10). Here $\nabla_t v^k$ and $\nabla_t p_k$ are components of covariant derivatives for vectorial and covectorial functions with components (11.4) and (11.3) that determine lift of curve $p = p(t)$ to TM and T^*M respectively. From general point of view, formulas (11.5) and (11.6) are nothing, but the rule of differentiating composite functions written in terms of covariant derivatives. They are obtained by direct calculations.

Now let S be a smooth hypersurface in M. Denote by $y^1,\ \dots\ ,\ y^{n-1}$ coordinates of point in some local chart of S. Then the following smooth functions

$$\begin{cases} x^1 = x^1(y^1, \dots , y^{n-1}),\\ \dots\dots\dots\dots\dots\dots\\ x^n = x^n(y^1, \dots , y^{n-1}) \end{cases} \tag{11.7}$$

represent hypersurface S in local chart of M. Partial derivatives of functions (11.7) determine tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ to S as it is done in (2.5). Taking one variable y^i in (11.7) and fixing all others, we can consider (11.7) as $n-1$ families of functions of one variable, each corresponding to some family of curves on S. These are coordinate curves forming coordinate network on S (see Fig. 11.1). One can lift coordinate curves to cotangent bundle T^*M by means of normal covector $\mathbf{n} = \mathbf{n}(p)$. However, we shall use another lift given by momentum covector (see initial data (7.4)):

$$\mathbf{p} = \nu(p) \cdot \mathbf{n}(p). \tag{11.8}$$

Then applying Legendre transformation (1.5) to covector (11.8), we obtain vector

$$\mathbf{v} = \mathbf{v}(p) = \tilde{\nabla} L \tag{11.9}$$

at each point $p \in S$. Vector $\mathbf{v}(p)$ is transversal to S as shown on Fig. 11.1. This follows from regularity of Lagrange function (see definition 1.1). Now, using lifts defined by (11.8) and (11.9), we can apply formula (11.2) to coordinate lines of hypersurface S.

Remember that $t = y^i$ is a parameter of i-th coordinate line, while $\boldsymbol{\tau}_i$ is tangent vector to this line corresponding to parameter y^i. Covariant derivative ∇_t with respect to parameter $t = y^i$ along i-th coordinate line by tradition is denoted by $\nabla_{\boldsymbol{\tau}_i}$. This derivative can be applied to any smooth tensorial function defined at the points of hypersurface S. Let's apply it to vector-function $\boldsymbol{\tau}_j$. As a result ge get another vector-function $\nabla_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_j$ on S. Similarly one can consider vector-function $\nabla_{\boldsymbol{\tau}_i}\mathbf{v}$ and covector-function $\nabla_{\boldsymbol{\tau}_i}\mathbf{p}$ on S. The latter one appears to be most important for us. It will be studied in section 13 below.

12. Covariant form of compatibility equations.

In section 7 we have derived Pfaff equations (7.13) for scalar function ν on hypersurface S. Then in section 9 we studied compatibility condition for these Pfaff equations. Explicit form of compatibility condition could be obtained by equating partial derivatives (9.1) and (9.2). But as a result we would obtain huge equality difficult to observe. This means that formulas (9.1) and (9.2) require some preliminary transformations in order to simplify further analysis of compatibility condition they lead to.

First of all we are going to replace partial derivatives $\partial n_r/\partial y^j$ and $\partial n_s/\partial y^i$ by covariant derivatives $\nabla_{\boldsymbol{\tau}_j} p_r$ and $\nabla_{\boldsymbol{\tau}_i} p_s$, assuming that we have some symmetric extended affine connection in M. Let's turn back to the equalities (7.12) and (7.13), let's substitute $\partial\nu/\partial y^i$ from (7.13) into (7.12) and remember formula (10.12):

$$\frac{\partial p_s}{\partial y^i} = \sum_{r=1}^{n} \nu \left(\delta^r_s - \frac{v^r\,p_s}{\Omega}\right) \frac{\partial n_r}{\partial y^i} - \sum_{r=1}^{n} \left(\frac{1}{\Omega}\frac{\partial H}{\partial x^r} - Q_r\right) \tau^r_i\,p_s. \tag{12.1}$$

Denote by P^r_s coefficient in front of partial derivative $\partial n_r/\partial y^i$ in formula (12.1):

$$P^r_s=\delta^r_s-\frac{v^r\,p_s}{\Omega}. \tag{12.2}$$

Formula (12.2) for P^r_s can be rewritten in two equivalent forms:

$$P^r_s=\delta^r_s-\frac{p_s\,\tilde\nabla^r H}{\Omega}=\delta^r_s-\frac{v^r\,\tilde\nabla_s L}{\Omega} \tag{12.3}$$

(compare with formulas (10.12) and (1.5) above). Now we see that (12.3) are components of operator-valued extended tensor field $\mathbf{P}$ either in $\mathbf{p}$ and in $\mathbf{v}$-representations respectively. It is easy to check up the following identity:

$$\mathbf{P}^2=\mathbf{P}\circ\mathbf{P}=\mathbf{P}. \tag{12.4}$$

Formula (12.4) means that $\mathbf{P}$ is a projector-valued extended operator field in M. It projects along velocity vector $\mathbf{v}$ onto a hyperplane defined by momentum covector $\mathbf{p}$. When restricted to hypersurface S due to lift (11.8) in $\mathbf{p}$-representation or due to lift (11.9) in $\mathbf{v}$-representation, it becomes projector field that projects onto tangent hyperplane to S along velocity vector $\mathbf{v}$. In this restricted form components of P appear in formula (12.1). This formula now can be written as

$$\sum^n_{r=1}P^r_s\,\frac{\partial n_r}{\partial y^i}=\frac{1}{\nu}\,\frac{\partial p_s}{\partial y^i}+\sum^n_{r=1}\frac{1}{\nu}\left(\frac{1}{\Omega}\frac{\partial H}{\partial x^r}-Q_r\right)\tau^r_i\,p_s. \tag{12.5}$$

Let's use the following identity that can be checked up immediately:

$$\sum^n_{s=1}P^s_k\,p_s=\sum^n_{s=1}\left(\delta^s_k-\frac{v^s\,p_k}{\Omega}\right)p_s=0. \tag{12.6}$$

Applying (12.4) and (12.6) to the equality (12.5), we find that

$$\sum^n_{r=1}P^r_s\,\frac{\partial n_r}{\partial y^i}=\frac{1}{\nu}\sum^n_{r=1}P^r_s\,\frac{\partial p_r}{\partial y^i}. \tag{12.7}$$

Looking at (12.5), we see that we cannot express partial derivative $\partial n_s/\partial y^i$ in a pure form. However, as appears, formula (12.7) is sufficient for our purposes. Indeed, the term containing partial derivative $\partial n_r/\partial y^j$ in (9.1) can be written as:

$$\ldots-\sum^n_{s=1}\sum^n_{r=1}\sum^n_{q=1}\sum^n_{k=1}\left(\frac{p_q}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,Q_s+\delta^k_q\,\frac{\partial Q_s}{\partial p_q}\right)\nu^2\,P^r_k\,\frac{\partial n_r}{\partial y^j}\,\tau^s_i+\ldots$$

Therefore we can use the above identity (12.7) in order to transform this term:

$$\ldots-\sum^n_{s=1}\sum^n_{r=1}\sum^n_{q=1}\sum^n_{k=1}\left(\frac{p_q}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,Q_s+\delta^k_q\,\frac{\partial Q_s}{\partial p_q}\right)\nu\,P^r_k\,\frac{\partial p_r}{\partial y^j}\,\tau^s_i+\ldots$$

As a result of this transformation for the whole expression (9.1) we obtain

$$
\begin{aligned}
&\frac{\partial^2\nu}{\partial y^i\,\partial y^j}=\dots+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu}{\Omega^2}\,p_q\,p_k\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial p_r}\,Q_s-\right.\\
&\left.-\sum_{q=1}^{n}\frac{\nu}{\Omega}\,p_q\,\frac{\partial^2 H}{\partial p_q\,\partial p_r}\,Q_s-\nu\,\frac{\partial Q_s}{\partial p_r}+\sum_{q=1}^{n}\frac{\nu}{\Omega}\,p_q\,\frac{\partial H}{\partial p_r}\,\frac{\partial Q_s}{\partial p_q}\right)\frac{\partial p_r}{\partial y^j}\,\tau^s_i+\\
&+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial Q_r}{\partial p_q}\,\frac{\partial H}{\partial x^s}+\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega^2}\,\frac{\partial H}{\partial p_q}\,\frac{\partial H}{\partial x^r}\,Q_s+\sum_{q=1}^{n}\sum_{k=1}^{n}\frac{\nu\,p_q\,p_k}{\Omega^2}\,\times\right.\\
&\left.\times\,\frac{\partial^2 H}{\partial p_q\,\partial p_k}\,\frac{\partial H}{\partial x^r}\,Q_s-\sum_{q=1}^{n}\frac{\nu\,p_q}{\Omega}\,\frac{\partial^2 H}{\partial p_q\,\partial x^r}\,Q_s+\sum_{q=1}^{n}\nu\,p_q\,\frac{\partial Q_r}{\partial p_q}\,Q_s+\nu\,\frac{\partial Q_r}{\partial x^s}\right)\tau^s_i\,\tau^r_j.
\end{aligned}
\tag{12.8}
$$

Now we can replace partial derivative $\partial p_r/\partial y^j$ in (12.8) by covariant derivative $\nabla_{\boldsymbol{\tau}_j} p_r$. For this purpose we shall use formula (11.2) with $t = y^j$. This yields

$$
\frac{\partial p_r}{\partial y^j}=\nabla_{\boldsymbol{\tau}_j} p_r+\sum_{\alpha=1}^{n}\sum_{\sigma=1}^{n}\Gamma^{\alpha}_{\sigma r}\,p_\alpha\,\tau^{\sigma}_j. \tag{12.9}
$$

Let's substitute (12.9) into (12.8) and let's use formulas (10.3) and (10.7) in order to express partial derivatives of H and Q_s through covariant derivatives:

$$
\begin{aligned}
&\frac{\partial^2\nu}{\partial y^i\,\partial y^j}=\ldots-\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n}\left(p^q\,\frac{Q_s}{\Omega}+\tilde\nabla^q Q_s\right)\nu\,P^r_q\,\nabla_{\boldsymbol{\tau}_j}p_r\,\tau^s_i+\\
&+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{\nu\,|\mathbf{p}|^2\,\nabla_r H}{\Omega^2}\,Q_s-\sum_{q=1}^{n}p_q\,\frac{\nu\,\nabla_r\tilde\nabla^q H}{\Omega}\,Q_s-\nabla_r Q_s+\right.\\
&\left.+\sum_{q=1}^{n}p_q\,\frac{\nu\,\nabla_r H}{\Omega}\,\tilde\nabla^q Q_s+\frac{\nu\,\nabla_r H}{\Omega}\,Q_s+\sum_{q=1}^{n}\nu\,p_q\,Q_s\,\tilde\nabla^q Q_r\right)\tau^s_i\,\tau^r_j.
\end{aligned}
\tag{12.10}
$$

By transforming (9.2) we can obtain another expression for the same derivative:

$$
\begin{aligned}
&\frac{\partial^2\nu}{\partial y^j\,\partial y^i}=\ldots-\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n}\left(p^q\,\frac{Q_r}{\Omega}+\tilde\nabla^q Q_r\right)\nu\,P^s_q\,\nabla_{\boldsymbol{\tau}_i}p_s\,\tau^r_j+\\
&+\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{\nu\,|\mathbf{p}|^2\,\nabla_s H}{\Omega^2}\,Q_r-\sum_{q=1}^{n}p_q\,\frac{\nu\,\nabla_s\tilde\nabla^q H}{\Omega}\,Q_r-\nabla_s Q_r+\right.\\
&\left.+\sum_{q=1}^{n}p_q\,\frac{\nu\,\nabla_s H}{\Omega}\,\tilde\nabla^q Q_r+\frac{\nu\,\nabla_s H}{\Omega}\,Q_r+\sum_{q=1}^{n}\nu\,p_q\,Q_r\,\tilde\nabla^q Q_s\right)\tau^s_i\,\tau^r_j.
\end{aligned}
\tag{12.11}
$$

Here, as in section 9, we denote by dots terms symmetric in indices r and s. They do not affect ultimate compatibility equation, which will be derived from (12.10) and (12.11). In the equalities (12.10) and (12.11) we used matrix $g^{ij} = \tilde{\nabla}^i\tilde{\nabla}^j H$ from (10.13) as metric tensor and introduced the following notations:

$$p^q = \sum_{k=1}^{n} g^{qk}\,p_q, \qquad\qquad |\mathbf{p}|^2 = \sum_{q=1}^{n}\sum_{k=1}^{n} g^{qk}\,p_q\,p_k. \tag{12.12}$$

In general, matrix (10.13) is not positive, therefore $|\mathbf{p}|^2$ in (12.12) is not necessarily positive number for $\mathbf{p} \neq 0$.

Let's equate right hand sides of (12.10) and (12.11). As a result we obtain compatibility equation (7.19) written in terms of covariant derivatives:

$$\begin{gathered}
\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n}\left(p^q\,\frac{Q_s}{\Omega}+\tilde{\nabla}^q Q_s\right)P^r_q\,\nabla_{\boldsymbol{\tau}_j}p_r\,\tau^s_i\,-\\
-\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n}\left(p^q\,\frac{Q_r}{\Omega}+\tilde{\nabla}^q Q_r\right)P^s_q\,\nabla_{\boldsymbol{\tau}_i}p_s\,\tau^r_j\,=\\
=\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{|\mathbf{p}|^2\,\nabla_r H}{\Omega^2}\,Q_s-\sum_{q=1}^{n}p_q\,\frac{\nabla_r\tilde{\nabla}^q H}{\Omega}\,Q_s-\nabla_r Q_s\,+\right.\\
\left.+\sum_{q=1}^{n}p_q\,\frac{\nabla_r H}{\Omega}\,\tilde{\nabla}^q Q_s+\frac{\nabla_r H}{\Omega}\,Q_s+\sum_{q=1}^{n}p_q\,Q_s\,\tilde{\nabla}^q Q_r\right)\tau^s_i\,\tau^r_j\,-\\
-\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{|\mathbf{p}|^2\,\nabla_s H}{\Omega^2}\,Q_r-\sum_{q=1}^{n}p_q\,\frac{\nabla_s\tilde{\nabla}^q H}{\Omega}\,Q_r-\nabla_s Q_r\,+\right.\\
\left.+\sum_{q=1}^{n}p_q\,\frac{\nabla_s H}{\Omega}\,\tilde{\nabla}^q Q_r+\frac{\nabla_s H}{\Omega}\,Q_r+\sum_{q=1}^{n}p_q\,Q_r\,\tilde{\nabla}^q Q_s\right)\tau^s_i\,\tau^r_j.
\end{gathered}\tag{12.13}$$

The most important point here is that the form of equation (12.13) does not depend on which particular connection is used in covariant derivatives. One can choose any symmetric extended connection Γ in M.

13. Geometry of hypersurfaces.

Though most terms in compatibility equation (12.13) can be treated as components of extended tensor fields in M, the equation in whole is related to hypersurface S. This reveals if we look at partial derivatives $\nabla_{\boldsymbol{\tau}_j}p_r$ and $\nabla_{\boldsymbol{\tau}_i}p_s$. For further treatment of (12.13) we should learn how to treat these partial derivatives.

Let $\boldsymbol{\tau}$ be some arbitrary tangent vector to S. It represented as linear combination of basic tangent vectors: $\boldsymbol{\tau} = \alpha^1\cdot\boldsymbol{\tau}_1 + \ldots + \alpha^{n-1}\cdot\boldsymbol{\tau}_{n-1}$. Let

$$f(\boldsymbol{\tau}) = \nabla_{\boldsymbol{\tau}}\mathbf{p} = \sum_{r=1}^{n}\sum_{j=1}^{n-1}\left(\alpha^j\,\nabla_{\boldsymbol{\tau}_j}p_r\right)\cdot dx^r. \tag{13.1}$$

Then (13.1) defines linear map $f\colon T_p(S) \to T^*_p(M)$. Let's consider composite map

$$\mathbf{b} = -\mathbf{P}^*\circ f\circ\mathbf{P}. \tag{13.2}$$

Projection operator $\mathbf{P}^*$ in (13.2) is a conjugate operator for projector $\mathbf{P}$ with components (12.2). Remember that $\mathbf{P}$ projects onto the subspace $T_p(S)$ in $T_p(M)$ for $p \in S$. Therefore linear map $\mathbf{b}\colon T_p(M) \to T_p^*(M)$ is correctly defined by formula (13.2). Each map from $T_p(M)$ to conjugate space $T_p^*(S)$ is given by some bilinear form. In our case this bilinear form is defined by formula

$$b(\mathbf{X}, \mathbf{Y}) = \langle \mathbf{b}(\mathbf{Y}) \,|\, \mathbf{X} \rangle \,. \tag{13.3}$$

Due to the presence of projection operators $\mathbf{P}$ and $\mathbf{P}^*$ in (13.2) we have

$$b(\mathbf{X}, \mathbf{Y}) = b(\mathbf{P}(\mathbf{X}), \mathbf{Y}) = b(\mathbf{X}, \mathbf{P}(\mathbf{Y})). \tag{13.4}$$

Theorem 13.1. *Bilinear form* (13.3) *defined by linear map* (13.2) *is symmetric.*

Bilinear form (13.3) restricted to tangent space $T_p(S)$ of hypersurface S is known as *second fundamental form* of hypersurface S. Its components

$$\beta_{ij} = b(\boldsymbol{\tau}_i, \boldsymbol{\tau}_j) \tag{13.5}$$

define tensor field in inner geometry of hypersurface S. But for our purposes coordinate representation of bilinear form (13.3) in outer geometry is more preferable:

$$b = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}\, dx^i\, dx^j.$$

In order to prove theorem 13.1 we need some preliminary results. First is given by the following lemma for extended connection components.

Lemma 13.1. *For any symmetric extended connection Γ in M and for any fixed point $q_0 = (p_0, \mathbf{p})$ of cotangent bundle T^*M there is local chart of M such that all connection components $\Gamma^k_{ij}(q) = \Gamma^k_{ij}(x^1, \ldots, x^n, p_1, \ldots, p_n)$ in this chart become zero at the point $q = q_0$.*

Lemma 13.1 is formulated for extended connection in $\mathbf{p}$-representation and for its components (10.6). Similar lemma can be stated for extended connection in $\mathbf{v}$-representation and for its components (10.15).

Lemma 13.2. *For any symmetric extended connection Γ in M and for any fixed point $q_0 = (p_0, \mathbf{v})$ of tangent bundle TM there is local chart of M such that all connection components $\Gamma^k_{ij}(q) = \Gamma^k_{ij}(x^1, \ldots, x^n, v^1, \ldots, v^n)$ in this chart become zero at the point $q = q_0$.*

Here one should emphasize that lemmas 13.1 and 13.2 assert vanishing of Γ^k_{ij} only at single point $q = q_0$. Sometimes they can vanish in whole neighborhood of this point, but this is not the case of general position. Proof of lemmas 13.1 and 13.2 is rather standard. It is based upon formula (10.4). One can find proof of analogous lemma in Chapter V of thesis [6]. With minor changes this proof is applicable to lemmas 13.1 and 13.2.

Now suppose that $x^1, \ldots, x^n$ are coordinates in a chart, existence of which is asserted by lemma 13.1. Then, using transformation formula (10.4), one can check up that linear change variables (i. e. that, for which transition matrices S and T in (10.5) are constant) preserves the property of vanishing of $\Gamma^k_{ij}(q)$ at the point

$q_0 = (p_0, \mathbf{p})$. If we apply this fact to a fixed point $p_0 \in S$, we can choose local coordinates $x^1, \ldots, x^n$ in some neighborhood of p_0 such that

1) fixed point p_0 is the origin in coordinates $x^1, \ldots, x^n$, i.e. $x^i(p_0) = 0$ for the whole range of index $i = 1, \ldots, n$;
2) hypersurface S is given parametrically by the equations
$$\begin{cases} x^1 = y^n, \\ \dots\dots\dots \\ x^{n-1} = y^{n-1}, \\ x^n = z(y^1, \ldots, y^{n-1}), \end{cases} \tag{13.6}$$
where $z(y^1, \ldots, y^{n-1})$ is some smooth function of parameters $y^1, \ldots, y^{n-1}$ vanishing at the origin, i.e. $z(0, \ldots, 0) = 0$, and having extremum there;
3) normal covector $\mathbf{n} = \mathbf{n}(y^1, \ldots, y^{n-1})$ of hypersurface S related with momentum covector $\mathbf{p}$ on S by the equality (11.8) is given by its components:
$$\mathbf{n} = (-z'_1, \ \ldots, -z'_n, 1), \ \text{ where } \ z'_i = \frac{\partial z}{\partial y^i}; \tag{13.7}$$
4) connection components $\Gamma^k_{ij} = \Gamma^k_{ij}(p, \mathbf{p}(p))$, where $\mathbf{p} = \mathbf{p}(p)$ is determined by the equality (11.8), do vanish at the origin $p = p_0$.

Formula (13.7) is derived directly from (13.6). Indeed, if we calculate tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ by using formula (2.5), for τ^s_i we obtain
$$\tau^s_i = \begin{cases} 0 & \text{for } \ s \neq i, n, \\ 1 & \text{for } \ s = i, \\ z'_i & \text{for } \ i = n. \end{cases} \tag{13.8}$$

Now it's easy to see that covector (13.7) is orthogonal to vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ in the sense of the equality (7.3). As we know, normal covector of hypersurface is determined up to a scalar factor. This uncertainty in (13.6) is eliminated by the condition that last component of $\mathbf{n}$ is equal to unity.

Let's consider components of covector $\nabla_{\boldsymbol{\tau}}\mathbf{p}$ in (13.1) for those special coordinates $x^1, \ldots, x^n$ we have chosen above. Applying (11.2) and (12.7), we derive
$$\sum_{r=1}^{n} P^r_s \, \nabla_{\boldsymbol{\tau}_i} p_r = \sum_{r=1}^{n} \nu \, P^r_s \, \nabla_{\boldsymbol{\tau}_i} n_r. \tag{13.9}$$

Applying formula (11.2) again, for covariant derivative $\nabla_i n_r$ we obtain
$$\nabla_{\boldsymbol{\tau}_i} n_r = \frac{\partial n_r}{\partial y^i} - \sum_{\alpha=1}^{n} \sum_{\sigma=1}^{n} \Gamma^\alpha_{\sigma r} \, n_\alpha \, \tau^\sigma_j. \tag{13.10}$$

Remember that in the above formula $\Gamma^\alpha_{\sigma r} = 0$ for $p = p_0$. Therefore, taking into account formula (13.7), from (13.10) we derive the following equality:
$$\nabla_{\boldsymbol{\tau}_i} n_r \Big|_{p=p_0} = \begin{cases} z''_{ir} & \text{for } \ r < n, \\ 0 & \text{for } \ r = n. \end{cases} \tag{13.11}$$

Projector $\mathbf{P}$ projects onto the hyperplane $T_p(S)$ and $\boldsymbol{\tau}_i \in T_p(S)$, hence $\mathbf{P}(\boldsymbol{\tau}_i) = \boldsymbol{\tau}_i$. Then for $\mathbf{b}(\boldsymbol{\tau}_i)$, applying (13.1), (13.2), and (13.9), we derive

$$\mathbf{b}(\boldsymbol{\tau}_i)\Big|_{p=p_0} = -\sum_{s=1}^{n}\sum_{r=1}^{n}(\nu\, P^r_s\ \nabla_{\boldsymbol{\tau}_i} n_r)\cdot dx^s. \tag{13.12}$$

Further, using formulas (13.3), (13.5), (13.11), and (13.12), we obtain

$$\beta_{ij}\Big|_{p=p_0} = -\sum_{s=1}^{n}\sum_{r=1}^{n}\nu\, P^r_s\ \nabla_{\boldsymbol{\tau}_i} n_r\, \tau^s_j = -\sum_{s=1}^{n}\sum_{r=1}^{n}\nu\, P^r_s\, z''_{ir}\, \tau^s_j. \tag{13.13}$$

Now remember formula (12.2) for P^r_s and formula (13.7) for $\mathbf{n}$. Moreover, let's remember that function $z(y^1, \ldots, y^{n-1})$ has extremum at the origin. This means that $z'_i = 0$ for $p = p_0$. Then from (12.2), (13.7), and (13.8) we get

$$P^r_s\Big|_{p=p_0} = \delta^r_s - \frac{v^r\,\delta^n_s}{v^n}, \qquad\qquad \tau^s_i\Big|_{p=p_0} = \delta^s_i. \tag{13.14}$$

Substituting (13.14) into (13.13) we now obtain the following equality:

$$\beta_{ij}\Big|_{p=p_0} = -\nu\, z''_{ij} = -\nu\,\frac{\partial^2 z}{\partial y^i\,\partial y^j}. \tag{13.15}$$

Looking at (13.15), it's easy to see that second fundamental form of hypersurface S is symmetric: $\beta_{ij} = \beta_{ij}$. This is in concordance with classical results for hypersurfaces in Riemannian manifolds. But now we are in quite different geometry.

Second fundamental form $\boldsymbol{\beta}$ with components given by formula (13.5) is a tensor field in S. It is obtained by restricting quadratic form (13.3) to inner geometry of S. Let's calculate components of this form in outer geometry, i. e. in local chart of manifold M at the point $p = p_0$. Due to (13.14) tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ coincide with $n-1$ coordinate tangent vectors at the point p_0:

$$\boldsymbol{\tau}_i\Big|_{p=p_0} = \mathbf{E}_i = \frac{\partial}{\partial x^i} \quad\text{for}\quad i = 1, \ldots, n-1. \tag{13.16}$$

As a consequence of (13.16), (13.3), and (13.5) we obtain

$$b_{ij} = \mathbf{b}(\mathbf{E}_i, \mathbf{E}_j) = \mathbf{b}(\boldsymbol{\tau}_i, \boldsymbol{\tau}_j) = \beta_{ij} \quad\text{for}\quad 1 \leqslant i, j < n. \tag{13.17}$$

Let's take n-th coordinate vector $\mathbf{E}_n = \partial/\partial x^n$. From (13.14) we derive

$$\mathbf{P}(\mathbf{E}_n)\Big|_{p=p_0} = \sum_{i=1}^{n} P^r_n\cdot\mathbf{E}_r = -\sum_{r=1}^{n-1}\frac{v^i}{v_n}\cdot\mathbf{E}_r = -\sum_{i=1}^{n-1}\frac{v^i}{v_n}\cdot\boldsymbol{\tau}_r.$$

Using the above equality and formulas (13.4) and (13.5), for $i < n$ we find

$$b_{in} = b(\mathbf{E}_i, \mathbf{E}_n) = b(\mathbf{E}_i, \mathbf{P}(\mathbf{E}_n)) = -\sum_{r=1}^{n-1}\frac{\beta_{ir}\,v^r}{v^n}. \tag{13.18}$$

In a similar way for component b_{ni} of quadratic form (13.3) we obtain

$$b_{ni} = b(\mathbf{E}_n, \mathbf{E}_i) = b(\mathbf{P}(\mathbf{E}_n), \mathbf{E}_i) = -\sum_{r=1}^{n-1} \frac{\beta_{ri}\, v^r}{v^n}. \tag{13.19}$$

Comparing (13.18) with (13.19) and taking into account symmetry of β_{ij} (see (13.15) above), we find that $b_{in} = b_{ni}$ for $i < n$. Combining this equality with (13.17) and taking into account symmetry of β_{ij} again, we get

$$b_{ij} = b_{ji} \tag{13.20}$$

for all i and j. We have proved the equality (13.20) at one point $p = p_0$ by choosing special local chart in M. But we can choose arbitrary point of M for p_0. Besides, b_{ij} are components of tensor. Therefore, being symmetric in one local chart, they keep symmetry when transformed to another chart. This proves theorem 13.1.

Theorem 13.2. *Let $q_0 = (p_0, \mathbf{p})$ be some fixed point of cotangent bundle T^*M with $\mathbf{p} \neq 0$ and let projector $\mathbf{P}$ be the value of projector-valued extended tensor field* (12.4) *at this point. Then any symmetric quadratic form b in $T_{p_0}(M)$ satisfying the equality* (13.4) *can be determined by some hypersurface S passing through the point p_0 and tangent to null-space of covector $\mathbf{p}$ at this point.*

Proof of theorem 13.2 now is very simple. Indeed, quadratic form b satisfying the equality (13.4) is completely determined by its restriction to null-space of covector $\mathbf{p}$. Then we can choose local chart in M with $\Gamma^k_{ij}(p_0, \mathbf{p}) = 0$ and with first $n-1$ coordinate vectors $\mathbf{E}_1, \ldots, \mathbf{E}_{n-1}$ all being in null-space of covector $\mathbf{p}$ at the point p_0. Let's define matrix β by means of components of quadratic form b:

$$\beta_{ij} = b_{ij} = b(\mathbf{E}_i, \mathbf{E}_j)\Big|_{p=p_0} \quad \text{for} \quad 1 \leqslant i, j < n. \tag{13.21}$$

Now it's sufficient to define hypersurface S by parametric equations (13.6) and take the following function $z = z(y^1, \ldots, y^{n-1})$ in them:

$$z = -\frac{1}{2\,\nu_0} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij}\, y^i\, y^j. \tag{13.22}$$

Here $\nu_0 = \nu(p_0)$ is a constant from (7.15). Due to equalities (13.15), (13.17), and (13.21) it is clear that such hypersurface S reproduces quadratic form b used to define it through (13.21) and (13.22). Thus, theorem 13.2 is proved.

14. Additional normality equations.

Now let's apply theorem 13.2 to the study of the equation (12.13). Let's fix some point p_0 in M and some covector $\mathbf{p}$ at this point. Then, relying upon theorem 13.2, let's take hypersurface S passing through this point tangent to null-space of covector $\mathbf{p}$ and such that its second fundamental form $\boldsymbol{\beta}$ is zero at the point $p = p_0$. This means that quadratic form b in (13.3) is also zero for $p = p_0$. Then due to (13.1) and (13.2) we get $\nabla_{\boldsymbol{\tau}_j} p_r = 0$. Equivalently, $\nabla_{\boldsymbol{\tau}_i} p_s = 0$. Substituting these two equalities into (12.13), we find that first two sums in (12.13) do vanish. Looking at other terms, we see that they are components of an extended tensor field of type

$(0,2)$ contracted with two vectors $\boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_j$ tangent to S. These two vectors depend on the choice of parameters $y^1, \ldots, y^n$, i.e. on the choice of local chart on S. By choosing this local chart properly we can associate $\boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_j$ with two arbitrary vectors in $T_{p_0}(S)$. This means that we can write

$$\boldsymbol{\tau}_i = \mathbf{P}(\mathbf{X}), \qquad\qquad \boldsymbol{\tau}_j = \mathbf{P}(\mathbf{Y}), \tag{14.1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are two arbitrary vectors in $T_{p_0}(M)$. Due to arbitrariness of vectors $\mathbf{X}$ and $\mathbf{Y}$ in (14.1) from rest part of (12.13) we derive

$$\begin{aligned}
&\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{|\mathbf{p}|^2\ \nabla_r H}{\Omega^2}\,Q_s-\sum_{q=1}^{n}p_q\,\frac{\nabla_r\tilde{\nabla}^q H}{\Omega}\,Q_s-\nabla_r Q_s+\right.\\
&\left.+\sum_{q=1}^{n}p_q\,\frac{\nabla_r H}{\Omega}\,\tilde{\nabla}^q Q_s+\frac{\nabla_r H}{\Omega}\,Q_s+\sum_{q=1}^{n}p_q\,Q_s\,\tilde{\nabla}^q Q_r\right)P^s_i\,P^r_j=\\
&=\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\frac{|\mathbf{p}|^2\ \nabla_s H}{\Omega^2}\,Q_r-\sum_{q=1}^{n}p_q\,\frac{\nabla_s\tilde{\nabla}^q H}{\Omega}\,Q_r-\nabla_s Q_r+\right.\\
&\left.+\sum_{q=1}^{n}p_q\,\frac{\nabla_s H}{\Omega}\,\tilde{\nabla}^q Q_r+\frac{\nabla_s H}{\Omega}\,Q_r+\sum_{q=1}^{n}p_q\,Q_r\,\tilde{\nabla}^q Q_s\right)P^s_i\,P^r_j.
\end{aligned}\tag{14.2}$$

Note that the equations (14.2) are partial differential equations for components of extended covector field $\mathbf{Q}$, which is used in the equations of Newtonian dynamics, when they are written in the form (7.5) relative to modified Hamiltonian dynamical. They are written in terms of covariant derivatives (10.3) and (10.7). Though we used some special hypersurface S in order to derive them, in their ultimate form they do not depend on any particular choice of S and, moreover, they can be written in the absence of S at all.

Now remember that $\mathbf{P}(\boldsymbol{\tau}_i)=\boldsymbol{\tau}_i$ and $\mathbf{P}(\boldsymbol{\tau}_j)=\boldsymbol{\tau}_j$. Therefore we can apply (14.2) back to (12.13). As a result we obtain the following equality:

$$\begin{aligned}
&\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n}\left(p^q\,\frac{Q_s}{\Omega}+\tilde{\nabla}^q Q_s\right)P^r_q\,\nabla_{\boldsymbol{\tau}_j}p_r\,\tau^s_i\ =\\
&\qquad=\sum_{s=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n}\left(p^q\,\frac{Q_r}{\Omega}+\tilde{\nabla}^q Q_r\right)P^s_q\,\nabla_{\boldsymbol{\tau}_i}p_s\,\tau^r_j.
\end{aligned}\tag{14.3}$$

This means that the equation (12.13) splits into two parts, first part is (14.3), while second part leads to the equations (14.2).

Both sides of (14.3) do vanish for our special hypersurface S above at its fixed point $p=p_0$. However, for arbitrary hypersurface S they are nonzero, therefore we are to study (14.3) in order to derive other equations for covector field $\mathbf{Q}$. Let's denote by $\mathbf{B}$ extended tensor field with components

$$B^r_s=\sum_{k=1}^{n}\sum_{q=1}^{n}P^r_q\left(p^q\,\frac{Q_k}{\Omega}+\tilde{\nabla}^q Q_k\right)P^k_s. \tag{14.4}$$

It's clear that tensor field $\mathbf{B}$ with components (14.4) is an operator field. Due to the presence of P^k_s and P^r_q in (14.4) we have the following equalities:

$$\mathbf{B} = \mathbf{P} \circ \mathbf{B} = \mathbf{B} \circ \mathbf{P}. \tag{14.5}$$

Relying upon (13.1), (13.2), (13.3) and using (14.4), now we can write (14.3) as

$$b(\boldsymbol{\tau}_i, \mathbf{B}\boldsymbol{\tau}_j) = b(\boldsymbol{\tau}_j, \mathbf{B}\boldsymbol{\tau}_i). \tag{14.6}$$

As we noted above, vectors $\boldsymbol{\tau}_i$ and $\boldsymbol{\tau}_j$ can be replaced by two arbitrary vectors $\mathbf{X}$ and $\mathbf{Y}$, see formulas (14.1). Then (14.6) is transformed to

$$b(\mathbf{X}, \mathbf{B} \circ \mathbf{P}(\mathbf{Y})) = b(\mathbf{Y}, \mathbf{B} \circ \mathbf{P}(\mathbf{X})).$$

Due to (14.5) and theorem 13.1 we can further simplify this relationship:

$$b(\mathbf{X}, \mathbf{B}(\mathbf{Y})) = b(\mathbf{B}(\mathbf{X}), \mathbf{Y}). \tag{14.7}$$

Formula (14.7) means that operator $\mathbf{B}$ is symmetric with respect to bilinear form (13.3). Now we are to utilize this equality.

Let's denote by $W = \operatorname{Im} \mathbf{P}$ the image of projection operator $\mathbf{P} = \mathbf{P}(q_0)$ for some fixed point $q_0 = (p_0, \mathbf{p})$ of cotangent bundle T^*M (see theorem 13.2). Then W is $(n-1)$-dimensional subspace in $T_{p_0}(M)$. It coincides with null-space of covector $\mathbf{p}$. Due to (14.5) subspace W is invariant under the action of operator $\mathbf{B}$. Moreover, operator $\mathbf{B}$ is completely determined by its restriction to W. For instance, if restriction of $\mathbf{B}$ to W is identical operator in W, then $\mathbf{B} = \mathbf{P}$:

$$\mathbf{B}\Big|_W = \mathrm{id}_W \quad \text{implies} \quad \mathbf{B} = \mathbf{P}. \tag{14.8}$$

Bilinear form b is also completely determined by its restriction to subspace W. This follows from relationships (13.4). The equality (14.7) means that restriction of operator B to W is symmetric with respect to restriction of b to W. Now recall theorem 13.2. It means that arbitrary quadratic form in subspace W can be obtained as second fundamental form of some hypersurface S tangent to W. Hence restriction of $\mathbf{B}$ to W is symmetric with respect to all quadratic forms in W. It takes place if and only if the restriction of $\mathbf{B}$ to W is a scalar operator, i. e.

$$\mathbf{B}\Big|_W = \lambda \cdot \mathrm{id}_W\,.$$

This is easy result in linear algebra. Now, applying (14.8) to the above equality, for operator $\mathbf{B}$ itself we derive the following representation:

$$\mathbf{B} = \lambda \cdot \mathbf{P}. \tag{14.9}$$

Here λ is some scalar factor. It can be expressed explicitly through trace of $\mathbf{B}$:

$$\lambda = \frac{\operatorname{tr} \mathbf{B}}{n-1}. \tag{14.10}$$

Formula (14.9) is important result. Combining it with (14.4), we obtain

$$\sum_{k=1}^{n}\sum_{q=1}^{n} P^r_q \left(p^q\,\frac{Q_k}{\Omega}+\tilde\nabla^q Q_k\right)P^k_s=\lambda\,P^r_s. \tag{14.11}$$

Substituting (14.10) for scalar factor λ in (14.11), we get the following equality:

$$\begin{aligned}&\sum_{k=1}^{n}\sum_{q=1}^{n} P^r_q \left(p^q\,\frac{Q_k}{\Omega}+\tilde\nabla^q Q_k\right)P^k_s=\\ &\qquad=\sum_{k=1}^{n}\sum_{q=1}^{n}\left(p^q\,\frac{Q_k}{\Omega}+\tilde\nabla^q Q_k\right)\frac{P^k_q\,P^r_s}{n-1}.\end{aligned} \tag{14.12}$$

This is another additional normality equation. Both (14.2) and (14.12) form a system of partial differential equations for components of extended convector field $\mathbf{Q}$ that defines Newtonian dynamical system in form (7.5) relative to Hamiltonian dynamical system with Hamilton function H. Having derived these equations, we proved the following theorem.

Theorem 14.1. *Additional normality condition stated in definition 7.2 for Newtonian dynamical system* (7.5) *in multidimensional case $n\geqslant 3$ is equivalent to the system of additional normality equations* (14.2) *and* (14.12) *that should be fulfilled at all points $q=(p,\mathbf{p})$ of cotangent bundle T^*M, where $\mathbf{p}\neq 0$.*

15. Connection invariance.

Deriving additional normality equations (14.2) and (14.12) above we used some symmetric extended connection Γ. Components of this connection are present in (14.2) due to covariant derivatives ∇_r and ∇_s. However, above we did not specify which particular symmetric connection Γ is used. This means that additional normality equations (14.2) and (14.12) should be invariant under transformations

$$\Gamma^k_{ij}\to\Gamma^k_{ij}+T^k_{ij}, \tag{15.1}$$

where T^k_{ij} are components of symmetric extended tensor field of type $(1,2)$. Differential equations (14.12) are obviously invariant under transformations (15.1) since momentum gradient $\tilde\nabla$ is defined without use of connection components (see formula (10.3)). In general, differential equations (14.2) are not invariant under these transformations. However, if component of covector field $\mathbf{Q}$ satisfy differential equations (14.12), then equations (14.2) become invariant under transformations (15.1). In other words, this means that differential equations (14.2) are invariant under transformations (15.1) modulo differential equations (14.12). This fact can be checked up by direct calculations.

16. Weak normality equations.

Let's fix some point $q_0=(p_0,\mathbf{p}_0)$ of cotangent bundle T^*M with $\mathbf{p}_0\neq 0$. It yields initial data for Newtonian dynamical system written in form (7.5) and defines a trajectory $p=p(t)$ of this dynamical system passing through the point $p=p_0$

at time instant $t = 0$. Null-space of covector $\mathbf{p}_0$ is a hyperplane in tangent space $T_{p_0}(M)$. Let's denote it by α. One can draw various hypersurfaces passing through the point p_0 and tangent to hyperplane α at this point. Suppose that S is one of such hypersurfaces and suppose that $\mathbf{n} = \mathbf{n}(p)$ is smooth normal covector of S in some neighborhood of the point p_0. At the very point $p = p_0$ we have the equality

$$\mathbf{p}_0 = \nu_0 \cdot \mathbf{n}(p_0), \ \text{ where } \ \nu_0 \neq 0. \tag{16.1}$$

Now let's take some smooth function on S normalized by the condition (7.15) and set up initial data (7.4) for Newtonian dynamical system (7.5). Solving Cauchy problem with these initial data, we get a family of trajectories for dynamical system (7.5), which includes our initial trajectory passing through the point $p = p_0$. This is easily seen if we compare (16.1) and (7.15).

In local coordinates the family of trajectories constructed just above is represented by functions (2.4). Using them, we define variation vectors (2.5) and deviation functions (2.6). Now suppose that Newtonian dynamical system (7.5) satisfy strong normality condition (see definition 8.1). This means that by proper choice of function $\nu(p)$ we can make all deviation functions $\varphi_1, \ldots, \varphi_{n-1}$ to be identically zero. Hence initial conditions (7.2) are fulfilled. From section 7 we know that initial data (7.2) are equivalent to Pfaff equations (7.13) for ν. We can vary constant $\nu_0 \neq 0$ in normalizing condition (7.15), and for each value of this constant due to strong normality condition we would have some function $\nu(p)$ on S satisfying Pfaff equations (7.13). Due to lemma 7.2 then Pfaff equations (7.13) are compatible in the sense of definition 7.1. Thus we have proved the following theorem.

Theorem 16.1. *Strong normality condition implies additional normality condition for Newtonian dynamical system* (7.5)*.*

Additional normality condition is formulated in definition 7.2. In section 14 we have shown that additional normality condition is equivalent to additional normality equations (14.2) and (14.12). Thus strong normality condition leads to the equations (14.2) and (14.12) for components of covector field $\mathbf{Q}$. However, it can yield much more. Indeed, it implies initial condition

$$\ddot{\varphi}_i\Big|_{t=0} = 0 \tag{16.2}$$

in addition to initial conditions (7.5). Let's calculate second derivatives $\ddot{\varphi}_i$ for deviation functions $\varphi_1, \ldots, \varphi_{n-1}$ by differentiating formula (7.10). First of all note that formula (7.10) itself can be written in terms of covariant derivatives

$$\dot{\varphi}_i = -\sum_{s=1}^{n} \frac{\tilde{\nabla}^s H}{\Omega}\, \nabla_{\boldsymbol{\tau}_i} p_s - \sum_{s=1}^{n} \left(\frac{\nabla_s H}{\Omega} - Q_s \right) \tau_i^s. \tag{16.3}$$

Note that the equations of Newtonian dynamics in form (7.5) also admit covariant derivatives instead of partial derivatives in them:

$$\dot{x}^s = \frac{\tilde{\nabla}^s H}{\Omega}, \qquad\qquad \nabla_t p_s = -\frac{\nabla_s H}{\Omega} + Q_s. \tag{16.4}$$

Applying covariant derivative $\nabla_{\boldsymbol{\tau}_i}$ to (16.4) and denoting $\nabla_{\boldsymbol{\tau}_i}\mathbf{p} = \boldsymbol{\xi}_i$, we obtain the following differential equations for components of vector $\boldsymbol{\tau}_i$ and covector $\boldsymbol{\xi}_i$:

$$\nabla_t\tau^s_i = \sum_{r=1}^{n}\tilde{\nabla}^r\!\left(\frac{\tilde{\nabla}^s H}{\Omega}\right)\xi_{ri} + \sum_{r=1}^{n}\nabla_r\!\left(\frac{\tilde{\nabla}^s H}{\Omega}\right)\tau^r_i, \tag{16.5}$$

$$\begin{aligned}
&\nabla_t\xi_{si} + \sum_{r=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n} D^{mq}_{rs}\,p_m\left(\frac{\nabla_q H}{\Omega} - Q_q\right)\tau^r_i + \\
&\quad + \sum_{q=1}^{n}\sum_{m=1}^{n}\frac{\tilde{\nabla}^q H\,p_m}{\Omega}\left(\sum_{r=1}^{n}\tilde{R}^m_{sqr}\;\tau^r_i + \sum_{r=1}^{n} D^{mr}_{qs}\,\xi_{ri}\right) = \\
&\quad = -\sum_{r=1}^{n}\tilde{\nabla}^r\!\left(\frac{\nabla_s H}{\Omega} - Q_s\right)\xi_{ri} - \sum_{r=1}^{n}\nabla_r\!\left(\frac{\nabla_s H}{\Omega} - Q_s\right)\tau^r_i,
\end{aligned} \tag{16.6}$$

Here D^{mq}_{rs} and D^{mr}_{qs} are components of dynamic curvature tensor $\mathbf{D}$ given by formula

$$D^{kr}_{ij} = -\frac{\partial\Gamma^k_{ij}}{\partial p_r}. \tag{16.7}$$

Tensor $\mathbf{D}$ has no analogs in Riemannian geometry since its components (16.7) do vanish for non-extended connections. In (16.6) we have quantities $\tilde{R}^m_{sqr}$ which are components of another curvature tensor $\tilde{\mathbf{R}}$ given by formula

$$\begin{aligned}
\tilde{R}^k_{rij} &= \frac{\partial\Gamma^k_{jr}}{\partial x^i} - \frac{\partial\Gamma^k_{ir}}{\partial x^j} + \sum_{m=1}^{n}\Gamma^k_{im}\,\Gamma^m_{jr} - \sum_{m=1}^{n}\Gamma^k_{jm}\,\Gamma^m_{ir} + \\
&+ \sum_{m=1}^{n}\sum_{\alpha=1}^{n} p_\alpha\,\Gamma^\alpha_{mi}\,\frac{\partial\Gamma^k_{jr}}{\partial p^m} - \sum_{m=1}^{n}\sum_{\alpha=1}^{n} p_\alpha\,\Gamma^\alpha_{mj}\,\frac{\partial\Gamma^k_{ir}}{\partial p^m}.
\end{aligned} \tag{16.8}$$

In Riemannian geometry (16.8) reduces to standard formula for components of Riemann curvature tensor.

Now let's differentiate (16.3) with respect to time variable t. It is equivalent to applying covariant derivative ∇_t to to this equality:

$$\begin{aligned}
\ddot{\varphi}_i &= -\sum_{s=1}^{n}\nabla_t\left(\frac{\tilde{\nabla}^s H}{\Omega}\right)\xi_{si} - \sum_{s=1}^{n}\frac{\tilde{\nabla}^s H}{\Omega}\,\nabla_t\xi_{si} - \\
&-\sum_{s=1}^{n}\nabla_t\left(\frac{\nabla_s H}{\Omega} - Q_s\right)\tau^s_i - \sum_{s=1}^{n}\left(\frac{\nabla_s H}{\Omega} - Q_s\right)\nabla_t\tau^s_i.
\end{aligned} \tag{16.9}$$

Substituting (16.5) and (16.6) into (16.9), we obtain the following equality for $\ddot{\varphi}_i$:

$$\ddot{\varphi}_i = \sum_{r=1}^{n}\left(\sum_{s=1}^{n}\frac{\nabla_s\Omega}{\Omega^2}\,\frac{\tilde{\nabla}^s H}{\Omega} + \sum_{s=1}^{n}\frac{\tilde{\nabla}^s\Omega}{\Omega^2}\left(-\frac{\nabla_s H}{\Omega} + Q_s\right)\right)\tilde{\nabla}^r H\;\xi_{ri} -$$

$$
-\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{\tilde\nabla^s H}{\Omega}\left(\tilde\nabla^r Q_s+\frac{\tilde\nabla^r\Omega}{\Omega}\,Q_s\right)\xi_{ri}+\sum_{r=1}^{n}\left(\sum_{s=1}^{n}\frac{\nabla_s\Omega}{\Omega^2}\,\frac{\tilde\nabla^s H}{\Omega}+\right.
$$

$$
\left.+\sum_{s=1}^{n}\frac{\tilde\nabla^s\Omega}{\Omega^2}\left(-\frac{\nabla_s H}{\Omega}+Q_s\right)\right)\nabla_r H\ \tau^r_i+\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\nabla_s Q_r-\nabla_r Q_s-\right.
$$

$$
\left.-\frac{\nabla_r\Omega}{\Omega}\,Q_s\right)\frac{\tilde\nabla^s H}{\Omega}\,\tau^r_i+\sum_{r=1}^{n}\sum_{s=1}^{n}\left(-\frac{\nabla_s H}{\Omega}+Q_s\right)\tilde\nabla^s Q_r\,\tau^r_i
$$

Terms with curvature tensors are canceled due to the following identities:

$$
[\nabla_i,\,\nabla_j]H=-\sum_{k=1}^{n}\sum_{s=1}^{n}p_k\,\tilde R^k_{sij}\,\tilde\nabla^s H, \tag{16.10}
$$

$$
[\nabla_i,\,\tilde\nabla^j]H=\sum_{k=1}^{n}\sum_{s=1}^{n}p_k\,D^{kj}_{is}\,\tilde\nabla^s H. \tag{16.11}
$$

The identities similar to (16.10) and (16.11) in $\mathbf{v}$-representation were derived in Chapter III of thesis [6].

Thus, formula for $\ddot\varphi_i$ is derived (see above). It is rather huge. In order to simplify this formula we introduce two extended fields $\boldsymbol\alpha$ and $\boldsymbol\beta$ with components

$$
\begin{aligned}
\alpha^r=&\left(\sum_{s=1}^{n}\frac{\nabla_s\Omega}{\Omega^2}\,\frac{\tilde\nabla^s H}{\Omega}+\sum_{s=1}^{n}\frac{\tilde\nabla^s\Omega}{\Omega^2}\left(-\frac{\nabla_s H}{\Omega}+Q_s\right)\right)\tilde\nabla^r H-\\
&-\sum_{s=1}^{n}\frac{\tilde\nabla^s H}{\Omega}\left(\tilde\nabla^r Q_s+\frac{\tilde\nabla^r\Omega}{\Omega}\,Q_s\right)
\end{aligned}\tag{16.12}
$$

$$
\begin{aligned}
\beta_r=&\left(\sum_{s=1}^{n}\frac{\nabla_s\Omega}{\Omega^2}\,\frac{\tilde\nabla^s H}{\Omega}+\sum_{s=1}^{n}\frac{\tilde\nabla^s\Omega}{\Omega^2}\left(-\frac{\nabla_s H}{\Omega}+Q_s\right)\right)\nabla_r H+\\
&+\sum_{s=1}^{n}\left(\nabla_s Q_r-\nabla_r Q_s-\frac{\nabla_r\Omega}{\Omega}\,Q_s\right)\frac{\tilde\nabla^s H}{\Omega}-\sum_{s=1}^{n}\left(\frac{\nabla_s H}{\Omega}-Q_s\right)\tilde\nabla^s Q_r.
\end{aligned}\tag{16.13}
$$

Using notations (16.12) and (16.13), we can write formula for $\ddot\varphi_i$ as follows:

$$
\ddot\varphi_i=\sum_{s=1}^{n}\alpha^s\,\nabla_{\boldsymbol\tau_i}p_s+\sum_{s=1}^{n}\beta_s\,\tau^s_i. \tag{16.14}
$$

Then, using projector $\mathbf{P}$ with components (12.3), we can transform this expression:

$$
\begin{aligned}
\ddot\varphi_i=&\sum_{r=1}^{n}\sum_{s=1}^{n}\alpha^r\,P^s_r\,\nabla_{\boldsymbol\tau_i}p_s+\sum_{r=1}^{n}\sum_{s=1}^{n}\beta_r\,P^r_s\,\tau^s_i+\\
&+\sum_{r=1}^{n}\sum_{s=1}^{n}\alpha^r\,p_r\,\frac{\tilde\nabla^s H}{\Omega}\,\nabla_{\boldsymbol\tau_i}p_s+\sum_{r=1}^{n}\sum_{s=1}^{n}\beta_r\,\frac{\tilde\nabla^r H}{\Omega}\,p_s\,\tau^s_i.
\end{aligned}\tag{16.15}
$$

Comparing (16.15) with formula (16.3) for $\dot{\varphi}_i$, we can write formula (16.15) as

$$\begin{gathered}\ddot{\varphi}_i+\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle\,\dot{\varphi}_i=\sum_{r=1}^{n}\sum_{s=1}^{n}\alpha^r\,P^s_r\,\nabla_{\boldsymbol{\tau}_i}p_s+\sum_{r=1}^{n}\sum_{s=1}^{n}\beta_r\,P^r_s\,\tau^s_i-\\-\sum_{s=1}^{n}\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle\left(\frac{\nabla_sH}{\Omega}-Q_s\right)\tau^s_i+\sum_{r=1}^{n}\sum_{s=1}^{n}\beta_r\,\frac{\tilde{\nabla}^rH}{\Omega}\,p_s\,\tau^s_i.\end{gathered}\tag{16.16}$$

Now let's introduce other two extended fields σ and $\boldsymbol{\eta}$ with components

$$\eta_r=\beta_r-\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle\left(\frac{\nabla_rH}{\Omega}-Q_r\right),\qquad\qquad\sigma=\sum_{r=1}^{n}\frac{\tilde{\nabla}^rH}{\Omega}\,\eta_r.\tag{16.17}$$

Then, taking into account formula (2.6) for φ_i, we can write (16.16) as follows:

$$\ddot{\varphi}_i+\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle\,\dot{\varphi}_i-\sigma\,\varphi_i=\sum_{r=1}^{n}\sum_{s=1}^{n}\alpha^r\,P^s_r\,\nabla_{\boldsymbol{\tau}_i}p_s+\sum_{r=1}^{n}\sum_{s=1}^{n}\eta_r\,P^r_s\,\tau^s_i.\tag{16.18}$$

Combining (16.18) with (7.2) and (16.2), we obtain the equality

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\alpha^r\,P^s_r\,\nabla_{\boldsymbol{\tau}_i}p_s+\sum_{r=1}^{n}\sum_{s=1}^{n}\eta_r\,P^r_s\,\tau^s_i=0.\tag{16.19}$$

If we remember operator $\mathbf{b}$ determined by (13.1) and (13.2) and if we use formulas (13.3) and (13.4) for bilinear form b, then (16.19) is written as

$$-\langle\mathbf{b}(\boldsymbol{\tau}_i)\,|\,\mathbf{P}\boldsymbol{\alpha}\rangle+\langle\boldsymbol{\eta}\,|\,\mathbf{P}\boldsymbol{\tau}_i\rangle=0.\tag{16.20}$$

Now recall that in the beginning of this section we have taken a point $q_0=(p_0,\mathbf{p}_0)$ and considered a set of hypersurfaces in M passing through the point p_0 tangent to null-space of covector $\mathbf{p}_0\neq 0$. Therefore, when equality (16.20) is written for the point p_0, we can replace $\mathbf{P}\boldsymbol{\tau}_i$ by $\mathbf{PX}$, where $\mathbf{X}$ is an arbitrary vector of tangent space $T_{p_0}(M)$. In a similar way, due to theorem 13.2, covector $\mathbf{b}(\boldsymbol{\tau}_i)$ in (16.20) can be replaced by arbitrary covector $\mathbf{y}\in T^*_{p_0}(M)$. Hence (16.20) breaks into two parts

$$\langle\mathbf{y}\,|\,\mathbf{P}\boldsymbol{\alpha}\rangle=0,\qquad\qquad\langle\boldsymbol{\eta}\,|\,\mathbf{PX}\rangle=0\tag{16.21}$$

with arbitrary vector $\mathbf{X}$ and arbitrary covector $\mathbf{y}$. In coordinate form these two equalities (16.21) are equivalent to the following ones:

$$\sum_{s=1}^{n}P^r_s\,\alpha^s=0,\qquad\qquad\sum_{s=1}^{n}P^s_r\,\eta_s=0.\tag{16.22}$$

Looking at (16.12), (16.13), and (16.17), we see that the equalities (16.22) form a system of partial differential equations for components of covector $\mathbf{Q}$ written in terms of covariant derivatives ∇ and $\tilde{\nabla}$. They are called *weak normality equations*. Let's write them explicitly. For the first equation (16.22) we have

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{\tilde{\nabla}^s H}{\Omega}\left(\tilde{\nabla}^r Q_s+\frac{\tilde{\nabla}^r\Omega}{\Omega}\,Q_s\right)P^q_r=0. \tag{16.23}$$

Second equality (16.22) leads to more huge equations

$$\begin{aligned}\sum_{r=1}^{n}\sum_{s=1}^{n}\Biggl(\biggl(\nabla_s Q_r+\frac{\nabla_s\Omega}{\Omega}\,Q_r-\nabla_r Q_s+\frac{\nabla_r\Omega}{\Omega}\,Q_s\biggr)\frac{\tilde{\nabla}^s H}{\Omega}\,+\\ +\sum_{m=1}^{n}\left(\frac{\nabla_r H}{\Omega}-Q_r\right)\left(\tilde{\nabla}^m Q_s+\frac{\tilde{\nabla}^m\Omega}{\Omega}\,Q_s\right)\frac{\tilde{\nabla}^s H}{\Omega}\,p_m\,-\\ -\left(\frac{\nabla_s H}{\Omega}-Q_s\right)\left(\tilde{\nabla}^s Q_r+\frac{\tilde{\nabla}^s\Omega}{\Omega}\,Q_r\right)\Biggr)P^r_q=0.\end{aligned} \tag{16.24}$$

Theorem 16.2. *Strong normality condition for Newtonian dynamical system* (16.3) *implies weak normality equations* (16.23) *and* (16.24) *to be fulfilled at all points* $q=(p,\mathbf{p})$ *of cotangent bundle* T^*M*, where* $\mathbf{p}\neq 0$.

Note that theorem 16.2 is stated for Newtonian dynamical system written in form (16.4). However, weak normality equations (16.23) and (16.24) are invariant under the transformations (15.1) (this can be proved by direct calculations). Therefore theorem 16.2 is equally applicable to Newtonian dynamical system written in form of the equations (7.5).

17. Equivalence of strong and complete normality conditions.

In section 16 we have derived weak normality equations (16.23) and (16.24) from strong normality condition for Newtonian dynamical system (see definition 8.1). Here we reveal their relation to weak normality condition considered in section 6. As in section 6, let's consider one-parametric family of trajectories $p=p(t,y)$ of Newtonian dynamical system (16.4). In local coordinates it is represented by functions (6.1). Differentiating them with respect to parameter y, we obtain variation vector $\boldsymbol{\tau}$, see formula (6.3). Then we can define deviation function (6.4). Unlike section 6, in the above calculations in section 16 we used $\mathbf{p}$-representation rather than $\mathbf{v}$-representation. Indeed, the equations of Newtonian dynamics (16.4) and all normality equations (14.2), (14.12), (16.23), and (16.24) are written in terms of Hamilton function H and in terms of covariant derivatives (10.3) and (10.7), which require momentum representation of extended tensor fields. For this reason, instead of functions $\theta^i(t,y)$ in (6.5), we consider components of vector $\boldsymbol{\xi}=\nabla_{\boldsymbol{\tau}}\mathbf{p}$:

$$\xi_s=\nabla_{\boldsymbol{\tau}}p_s=\frac{\partial p_s}{\partial y}-\sum_{k=1}^{n}\sum_{q=1}^{n}\Gamma^k_{sq}\,p_k\,\tau^q$$

(compare with formula (12.9) above). Functions $\tau^1,\,\ldots\,,\,\tau^n$, $\xi^1,\,\ldots\,,\,\xi^n$ considered as functions of time variable t for fixed y satisfy a system ordinary differential equations which are quite the same as the equations (16.5) and (16.6) above, except for the absence of index i now:

$$\nabla_t\tau^s=\sum_{r=1}^{n}\tilde{\nabla}^r\!\left(\frac{\tilde{\nabla}^sH}{\Omega}\right)\xi_{ri}+\sum_{r=1}^{n}\nabla_r\!\left(\frac{\tilde{\nabla}^sH}{\Omega}\right)\tau^r,\tag{17.1}$$

$$\begin{gathered}\nabla_t\xi_s+\sum_{r=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n}D^{mq}_{rs}\,p_m\left(\frac{\nabla_qH}{\Omega}-Q_q\right)\tau^r+\\ +\sum_{q=1}^{n}\sum_{m=1}^{n}\frac{\tilde{\nabla}^qH\,p_m}{\Omega}\left(\sum_{r=1}^{n}\tilde{R}^m_{sqr}\,\tau^r+\sum_{r=1}^{n}D^{mr}_{qs}\,\xi_r\right)=\\ =-\sum_{r=1}^{n}\tilde{\nabla}^r\!\left(\frac{\nabla_sH}{\Omega}-Q_s\right)\xi_r-\sum_{r=1}^{n}\nabla_r\!\left(\frac{\nabla_sH}{\Omega}-Q_s\right)\tau^r.\end{gathered}\tag{17.2}$$

These equations (17.1) and (17.2) can be understood as linearizations of the equations of Newtonian dynamics (16.4). They are linear equations with respect to functions $\tau^1,\ldots,\tau^n,\xi^1,\ldots,\xi^n$, but with non-constant coefficients. Their coefficients are functions of time variable t determined by a trajectory $p=p(t)$ of Newtonian dynamical system (16.4). For a fixed trajectory $p=p(t)$, i.e. when parameter y in $p=p(t,y)$ is fixed, solutions of the equations (17.1) and (17.2) form n-dimensional linear space. We denote this space by $\mathfrak{T}$. In essential, it is the same space $\mathfrak{T}$ as in section 6.

Let's consider deviation function φ determined by formula (6.4) and its time derivatives. Now φ and functions $\tau^1,\ldots,\tau^n,\xi^1,\ldots,\xi^n$ are not related to any hypersurface S. Nevertheless, repeating the same steps as in deriving formula (16.18), one can derive the following equality for deviation function φ:

$$\ddot{\varphi}+\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle\,\dot{\varphi}-\sigma\,\varphi=\sum_{r=1}^{n}\sum_{s=1}^{n}\alpha^r\,P^s_r\,\xi_s+\sum_{r=1}^{n}\sum_{s=1}^{n}\eta_r\,P^r_s\,\tau^s.\tag{17.3}$$

Remember that weak normality equations (16.23) and (16.24) are expanded form of the equations (16.22). Therefore, if weak normality equations are fulfilled, then (17.3) reduces to second order ordinary differential equation for φ:

$$\ddot{\varphi}+\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle\,\dot{\varphi}-\sigma\,\varphi=0.\tag{17.4}$$

Comparing (17.4) with (6.11), we find that Newtonian dynamical system fits the definition 6.1 under the condition that weak normality equations (16.23) and (16.24) for extended covector field $\mathbf{Q}$ are fulfilled.

Converse result is also valid, i.e. weak normality condition stated in definition 6.1 implies weak normality equations (16.23) and (16.24) to be fulfilled. Let's prove it. Suppose that we take some trajectory of $p=p(t)$ of Newtonian dynamical system (16.4). This determines coefficients in linear differential equations (17.1) and (17.2) for components of $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$ and fixes linear space $\mathfrak{T}$. Then, according to definition 6.1, for each solution of the equations (17.1) and (17.2) corresponding deviation function φ should satisfy differential equation (6.11). For time derivatives $\dot{\varphi}$ and $\ddot{\varphi}$ we have the following equalities:

$$\dot{\varphi} = -\sum_{s=1}^{n} \frac{\tilde{\nabla}^s H}{\Omega}\,\xi^s - \sum_{s=1}^{n}\left(\frac{\nabla_s H}{\Omega} - Q_s\right)\tau^s, \tag{17.5}$$

$$\ddot{\varphi} = \sum_{s=1}^{n} \alpha^s\,\xi^s + \sum_{s=1}^{n} \beta_s\,\tau^s. \tag{17.6}$$

(compare with (16.3) and (16.14) above). For the function φ itself we have formula (6.4). Due to differential equation (6.11) from (17.5) and (17.6) we derive

$$\sum_{s=1}^{n}\left(\alpha^s + \mathcal{A}\,\frac{\tilde{\nabla}^s H}{\Omega}\right)\xi^s + \sum_{s=1}^{n}\left(\beta_s + \mathcal{A}\left(\frac{\nabla_s H}{\Omega} - Q_s\right) - \mathcal{B}\,p_s\right)\tau^s = 0.$$

According to definition 6.1, this equality should be fulfilled for all solutions of differential equations (17.1) and (17.2). Therefore

$$\alpha^s + \mathcal{A}\,\frac{\tilde{\nabla}^s H}{\Omega} = 0, \tag{17.7}$$

$$\beta_s + \mathcal{A}\left(\frac{\nabla_s H}{\Omega} - Q_s\right) - \mathcal{B}\,p_s = 0. \tag{17.8}$$

Multiplying (17.7) by p_s and summing over s running from 1 to n, we obtain

$$\mathcal{A} = -\sum_{s=1}^{n} \alpha^s\,p_s = -\,\langle \mathbf{p}\,|\,\boldsymbol{\alpha}\rangle\,. \tag{17.9}$$

In a similar way, multiplying (17.8) by $\tilde{\nabla}^s H$ and summing over s running from 1 to n, we derive formula for coefficient $\mathcal{B}$ in (6.11):

$$\mathcal{B} = \sum_{s=1}^{n} \frac{\tilde{\nabla}^s H}{\Omega}\left(\beta_s + \mathcal{A}\left(\frac{\nabla_s H}{\Omega} - Q_s\right)\right) = \sigma. \tag{17.10}$$

Now, let's substitute (17.10) and (17.9) back into the equations (17.7) and (17.8). Then let's multiply (17.7) and (17.8) by P^q_s and P^s_q respectively and sum them over index s. This yields the equalities (16.22) which are equivalent to weak normality equations (16.23) and (16.24). Thus we have proved the following theorem.

Theorem 17.1. *Weak normality condition stated in definition 6.1, when applied to Newtonian dynamical system written as* (7.5)*, is equivalent to the system of weak normality equations* (16.23) *and* (16.24) *that should be fulfilled at all points* $q = (p, \mathbf{p})$ *of cotangent bundle* T^*M*, where* $\mathbf{p} \neq 0$.

Combining theorems 16.1, 16.2 and 17.1, we obtain another theorem.

Theorem 17.2. *Strong and complete normality conditions are equivalent to each other either for* $n = 2$ *and in multidimensional case for* $n \geqslant 3$.

18. Summary and conclusions.

Primary goal of present paper is to generalize theory of Newtonian dynamical systems admitting normal shift from Riemannian geometry to the geometry determined by some Lagrangian or, equivalently, by some Hamiltonian dynamical system. In the above sections this goal is reached in essential:

- we have found proper statement for the concept of normal shift in Lagrangian geometry and have defined class of Newtonian dynamical systems admitting normal shift of hypersurfaces;
- we have derived complete system of normality equations thus obtaining effective tool for studying this class of dynamical systems.

However, some details of constructed theory appear to be different from those one could expect in the beginning. Indeed, being generalization of Riemannian geometry, geometry of Lagrangian dynamical system could have some connection canonically associated with Lagrange function. But even if such connection does exist, theory of normal shift does not reveal it. All normality equations (14.2), (14.12), (16.23), and (16.24) are invariant under transformations (15.1). Therefore they are of connection-free nature. There is a problem of writing them in coordinate covariant tensorial form without use of connection at all. This problem will be considered in separate paper.

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