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MINIMAL TORI IN FIVE-DIMENSIONAL SPHERE IN $\mathbb{C}^3$.

Abstract. Special class of surfaces in five-dimensional sphere in $\mathbb{C}^3$ is considered. Immersion equations for minimal tori of that class are shown to be reducible to the equation $u_{z\bar{z}} = e^u - e^{-2u}$ which is integrable by means of inverse scattering method. Finite-gap minimal tori are constructed.

1. Introduction.

Minimal surfaces in multidimensional spaces naturally arise as classical trajectories of relativistic strings with Lagrangians suggested by Nambu (see [1]) and by Polyakov (see [2]). From geometric point of view these are surfaces of zero mean curvature. Their immersion into environment space often is given by the equations integrable by means of inverse scattering method (see [3–5]). Minimal surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ are described by Liouville equation $u_{z\bar{z}} = e^u$. This equation is nonlinear, but can be linearized by Backlund transformation (see [6, 7]). In higher dimensional spaces and in non-flat spaces immersion equations for minimal surfaces are not linearizable. But they possess Lax pairs (see [5]), therefore one can rather effectively study their solutions.

In this paper minimal tori in five-dimensional sphere $S^5 \subset \mathbb{C}^3$ are considered whose immersion is described by Bullough-Dodd-Jiber-Shabat equation[1]

$$u_{z\bar{z}} = e^u - e^{-2u}. \tag{1.1}$$

On a base of construction of finite-gap solutions for this equation (see [8]) finite-gap minimal tori which are complexly normal in S^5 are constructed. Situation here is similar to that considered in [9] and [10]. There substantial advances in describing tori of constant mean curvature in $\mathbb{R}^3$, in S^3, and in H^3 on a base of finite gap solutions of Sine-Gordon equation $u_{z\bar{z}} = \sin u$ were achieved. In the framework of affine geometry the equation (1.1) were considered in [11].

2. Complexly normal surfaces in Hermitian sphere in $\mathbb{C}^3$ and their scalar invariants.

Let's consider complex space $\mathbb{C}^3$ with standard Hermitian scalar product

$$\langle \mathbf{A} \,|\, \mathbf{B} \rangle = \sum_{i=1}^{3} \bar{A}_i \, B_i \tag{2.1}$$

and with associated Euclidean scalar product

$$(\mathbf{A} \,|\, \mathbf{B}) = \mathrm{Re}\big(\langle \mathbf{A} \,|\, \mathbf{B} \rangle\big). \tag{2.2}$$

[1]After this paper had been published S. P. Tsarev discovered that the equation (1.1) was first introduced by Tzitzeica in [12]. Now it is called Tzitzeica equation. See also [13] for more details.

Let $\mathbf{r}(x^1, x^2)$ be vector-valued function with values in $\mathbb{C}^3$ defining immersion of real two-dimensional surface T into the sphere S_R of radius R in $\mathbb{C}^3$. Denote by $\mathbf{E}_1$ and $\mathbf{E}_2$ basic tangent vectors of this two-dimensional surface:

$$\mathbf{E}_1 = \partial_1 \mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^1}, \qquad \mathbf{E}_2 = \partial_2 \mathbf{r} = \frac{\partial \mathbf{r}}{\partial x^2}.$$

Scalar product (2.1) induces Hermitian metric on T:

$$h_{ij} = \langle \mathbf{E}_i \,|\, \mathbf{E}_j \rangle = g_{ij} + i\,\omega_{ij}. \tag{2.3}$$

Its real part is a Riemannian metric induced by scalar product (2.2), while imaginary part of Hermitian metric (2.3) is skew-symmetric tensor

$$\omega_{ij} = (i \cdot \mathbf{E}_i \,|\, \mathbf{E}_j) \tag{2.4}$$

defining closed 1-form $\boldsymbol{\omega}$ on T. Operator-valued tensor field

$$\Omega^i_j = \sum_{k=1}^{2} g^{ik}\,\omega_{kj}$$

has zero trace, while $\det \Omega$ is a scalar invariant of metric (2.3) on T.

Definition. Immersion $\mathbf{r}\colon T \to M \subset \mathbb{C}^3$ of real two-dimensional surface T into real submanifold M of codimension 1 in $\mathbb{C}^3$ is called *complexly normal* immersion if at each point of T Euclidean unit normal vector $\mathbf{N}$ of M is orthogonal to tangent plane to T in Hermitian metric, i. e. $\langle \mathbf{E}_i \,|\, \mathbf{N} \rangle = 0$.

For complexly normal surface T we define vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ which are orthogonal to $\mathbf{N}$ in Hermitian metric and orthogonal to vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ in Euclidean metric. We define them by the relationship

$$\mathbf{F}_i = i \cdot \mathbf{E}_i + \sum_{s=1}^{2} \Omega^s_i \cdot E_s. \tag{2.5}$$

Vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ define one more tensor field related to g_{ij} and ω_{ij}:

$$f_{ij} = g_{ij} + \sum_{k=1}^{2} \sum_{s=1}^{2} \omega_{ik}\, g^{ks}\, \omega_{sj}.$$

For associated tensor field F^i_j we derive

$$F^i_j = \sum_{k=1}^{2} g^{ik}\, f_{kj} = \delta^i_j + \sum_{k=1}^{2} \sum_{r=1}^{2} \sum_{s=1}^{2} g^{ik}\, \omega_{kr}\, g^{rs}\, \omega_{sj} = \delta^i_j + \sum_{r=1}^{2} \Omega^i_r\, \Omega^r_j.$$

Scalar invariants of this field can be expressed through invariant $\det \Omega$.

Vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ form a moving frame in tangent space to surface T, while vectors $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{N}$, and $i \cdot \mathbf{N}$ form complementary moving frame in normal space. Dynamics of first frame is given by the equations

$$\partial_i \mathbf{E}_j = \sum_{k=1}^{2} \Gamma^k_{ij} \cdot \mathbf{E}_k + \sum_{k=1}^{2} T^k_{ij} \cdot \mathbf{F}_k + (b_{ij} + i\, d_{ij}) \cdot \mathbf{N}, \tag{2.6}$$

where Γ^k_{ij} are components of metric connection of T given by well-known formula

$$\Gamma^k_{ij} = \sum_{s=1}^{2} \frac{g^{ks}}{2} \left(\partial_i g_{sj} + \partial_j g_{is} - \partial_s g_{ij} \right). \tag{2.7}$$

Dynamics of vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ is completely determined by dynamics of $\mathbf{E}_1$ and $\mathbf{E}_2$ due to the relationship (2.5). Dynamics of unit normal vector $\mathbf{N}$ of submanifold M along T is given by the following equation:

$$\partial_i \mathbf{N} = \sum_{k=1}^{2} L^k_i \cdot \mathbf{E}_k + \sum_{k=1}^{2} M^k_i \cdot \mathbf{F}_k + i\, S_i \cdot \mathbf{N}. \tag{2.8}$$

Tensor fields b_{ij}, d_{ij}, L^k_i, and M^k_i, which appear as coefficients in the equations (2.6) and (2.8), are related to each other by a number of relationships that follow from our specific choice of frames. Due to orthogonality of $\mathbf{N}$ and $\mathbf{E}_j$ we get

$$b_{ij} = -\sum_{k=1}^{2} L^k_i\, g_{kj}, \tag{2.9}$$

while orthogonality of vectors $i \cdot \mathbf{N}$ and $\mathbf{E}_j$ yields

$$d_{ij} = \sum_{k=1}^{2} M^k_i\, f_{kj} - \sum_{k=1}^{2} L^k_i\, \omega_{kj}. \tag{2.10}$$

Differentiating (2.4) and using the relationship (2.6), we derive the equality

$$\nabla_s \omega_{ij} = \sum_{k=1}^{2} T^k_{sj}\, f_{ki} - \sum_{k=1}^{2} T^k_{si}\, f_{kj}, \tag{2.11}$$

which completely determines skew-symmetric in indices i and j part of tensor

$$t_{isj} = \sum_{k=1}^{2} T^k_{is}\, f_{kj}.$$

In the case, when manifold M is a sphere S_R of radius R, the above relationships (2.8)–(2.11) simplify substantially. In this case radius-vector $\mathbf{r}(x^1, x^2)$ is collinear to normal vector: $\mathbf{r} = R \cdot \mathbf{N}$. Therefore

$$\mathbf{E}_i = \partial_i \mathbf{r} = R \cdot \partial_i \mathbf{N}.$$

Comparing this equality with (2.8), we get

$$L^k_i = \frac{1}{R}\, \delta^k_i, \qquad M^k_i = 0, \qquad S_i = 0.$$

Further from (2.9) and (2.10) for matrices of second fundamental forms we derive

$$b_{ij} = -\frac{1}{R}\, g_{ij}, \qquad d_{ij} = -\frac{1}{R}\, \omega_{ij}.$$

Matrix d_{ij} is symmetric, while matrix ω_{ij} is skew-symmetric. Therefore both matrices are zero: $d_{ij} = \omega_{ij} = 0$. Thus, for $M = S_R$ and for complexly normal embedding of surface T vectors $\mathbf{F}_1$ and $\mathbf{F}_2$ coincide with vectors $i \cdot \mathbf{E}_1$ and $i \cdot \mathbf{E}_2$ respectively, while relationships (2.6) and (2.8) are written as

$$\nabla_i \mathbf{E}_j = \sum_{k=1}^{2} T^k_{ij} \cdot \mathbf{F}_k - \frac{1}{R}\, g_{ij} \cdot \mathbf{N}, \qquad\qquad \partial_i \mathbf{N} = \frac{1}{R}\, \mathbf{E}_i. \tag{2.12}$$

Due to the equality (2.11) tensor

$$T_{kij} = \sum_{s=1}^{2} T^s_{ij}\, g_{sk}$$

is symmetric with respect to all its indices. Gauss equation, Peterson-Coddazi equation and Ricci equation are obtained as compatibility conditions for the equations (2.12). Here is Gauss equation

$$R^s_{kij} = \sum_{r=1}^{2} T^r_{jk}\, T^s_{ir} - \sum_{r=1}^{2} T^r_{ik}\, T^s_{jr} + \frac{g_{jk}\, \delta^s_i - g_{ik}\, \delta^s_j}{R^2}, \tag{2.13}$$

where R^s_{kij} is Riemann curvature tensor determined by metric connection (2.7) according to standard formula

$$R^s_{kij} = \partial_i \Gamma^s_{kj} - \partial_j \Gamma^s_{ki} - \sum_{r=1}^{2} \Gamma^r_{ki}\, \Gamma^s_{rj} + \sum_{r=1}^{2} \Gamma^r_{kj}\, \Gamma^s_{ri}. \tag{2.14}$$

Peterson-Coddazi and Ricci equations in this case are united into one equation

$$\nabla_i T_{jsk} - \nabla_j T_{isk} = 0. \tag{2.15}$$

Symmetric tensor T_{ijk} has two scalar invariants of second order

$$H^2 = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{s=1}^{2} T^{is}_i\, T^j_{js}, \qquad\qquad k = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{s=1}^{2} T^{ijs}\, T_{ijs} \tag{2.16}$$

and an invariant of fourth order determined by the relationship

$$q = \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \sum_{r=1}^{2} \sum_{p=1}^{2} \sum_{s=1}^{2} T^i_{jk}\, T^{jk}_s\, T^s_{rp}\, T^{rp}_i. \tag{2.17}$$

Due to specific features of two-dimensional case ($\dim T = 2$) invariants (2.16) and (2.17) form maximal set of functionally independent invariants of symmetric tensor T_{ijk}. Moreover, in two-dimensional case due to symmetry of Riemann curvature tensor R^s_{kij} tensorial equation (2.13) is equivalent to one scalar equation that binds Gaussian curvature K of the surface T with curvatures H and k of tensor T_{ijk}:

$$2\, K = \sum_{j=1}^{2} g^{kj}\, R^s_{ksj} = H^2 - k + 2\, R^{-2}. \tag{2.18}$$

Invariant H in (2.16) coincides with the length of averaged normal vector of T:

$$H \cdot \mathbf{n} = \sum_{i=1}^{2} T^{ik}_i \cdot \mathbf{F}_k. \tag{2.19}$$

Here H is mean curvature of the surface T embedded into S, while unit vector $\mathbf{n}$ tangent to sphere S_R in (2.19) is a unit vector of averaged normal of T.

3. Complexly normal tori of zero mean curvature.

The condition of vanishing of mean curvature is very restricting condition for the class of surfaces in question. Indeed, from the condition $H = 0$ due to the equality (2.19) we have

$$\sum_{i=1}^{2} T_i^{ik} = 0.$$

If we take into account symmetry of tensor T_{ijk}, the above equality means that in this tensor we have only two independent components. In order to use this circumstance let's choose isothermal coordinates on the surface T, i. e. coordinates $x = x^1 = \operatorname{Re} z$ and $y = x^2 = \operatorname{Im} z$ for which metric g_{ij} is conformally Euclidean: $\mathbf{g} = 2\,R^2\,e^u\,dz\,d\bar{z}$. In this case components of tensor T_{ij}^k are expressed through two independent quantities A and B:

$$\begin{aligned} &T_{11}^1 = A, && T_{12}^2 = T_{21}^2 = T_{22}^1 = -A,\\ &T_{22}^2 = B, && T_{12}^1 = T_{21}^1 = T_{11}^2 = -B. \end{aligned} \tag{3.1}$$

Let's calculate components of metric connection Γ_{ij}^k by using formula (2.7):

$$\begin{aligned} &\Gamma_{11}^1 = \frac{u_x}{2}, && \Gamma_{11}^2 = -\frac{u_y}{2}, && \Gamma_{12}^1 = \Gamma_{21}^1 = \frac{u_y}{2},\\ &\Gamma_{22}^2 = \frac{u_y}{2}, && \Gamma_{22}^1 = -\frac{u_x}{2}, && \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{u_x}{2}. \end{aligned} \tag{3.2}$$

Then let's substitute (3.1) into Peterson-Coddazi-Ricci equation (2.15). Upon completing calculations and taking into account (3.2) we find

$$\partial_x\left(e^u\,A\right) = \partial_y\left(e^u\,B\right), \qquad \partial_y\left(e^u\,A\right) = -\partial_x\left(e^u\,B\right). \tag{3.3}$$

It's easy to see that relationships (3.3) do coincide with Cauchy-Riemann equations for holomorphic function $G(z) = e^u\,A + i\,e^u\,B$.

If $G(z)$ is identically zero, then we have trivial case. In this case due to (2.12) the subspace defined as a span of vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{N}$ contains all derivatives of these vectors. Hence this subspace do not change when we vary x and y. This means that vectors $\mathbf{E}_1$, $\mathbf{E}_2$, and $\mathbf{N}$ belong to some fixed three-dimensional real subspace of $\mathbb{C}^3$, while T is a central section of the sphere S_R within this three-dimensional subspace, i. e. T is two-dimensional sphere of radius R or its part.

Let's consider the case $G(z) \not\equiv 0$. In this case we shall assume that T is compact closed surface of toric topology. Taking z for uniformizing parameter inherited from universal cover $\mathbb{C} \to T$, we get $G(z) = \mathrm{const} \neq 0$ since $G(z)$ then is holomorphic function on complex torus T. At the expense of simultaneous change of scale along axes x and y by the same factor we can satisfy additional condition $|G(z)| = 1$.

Then we write $G(z) = \cos\vartheta + i\,\sin\vartheta$. Now (3.1) is written as

$$\begin{aligned} T^1_{11} &= e^{-u}\,\cos\vartheta, & T^2_{22} &= e^{-u}\,\sin\vartheta,\\ T^2_{11} &= -e^{-u}\,\sin\vartheta, & T^1_{22} &= -e^{-u}\,\cos\vartheta,\\ T^1_{12} &= T^1_{21} = -e^{-u}\,\sin\vartheta, & T^2_{12} &= T^2_{21} = e^{-u}\,\cos\vartheta. \end{aligned} \tag{3.4}$$

Using tensor T^k_{ij} of the form (3.4), we can calculate invariants k and q determined by formulas (2.16) and (2.17):

$$k = \frac{2\,e^{-3u}}{R^2}, \qquad\qquad q = \frac{2\,e^{-6u}}{R^4}. \tag{3.5}$$

If we calculate curvature tensor R^s_{kij} for connection components (3.2) using formula (2.14) and if we substitute it into Gauss equation (2.18), we obtain the equation for the function $u(x,y)$:

$$u_{xx} + u_{yy} = 4\,e^{-2u} - 4\,e^{u},$$

This equation does coincide with (1.1). Its solution corresponding to the embedding of two-dimensional torus into $S_R \subset \mathbb{C}^3$ is double-periodic with some grid of periods in the plane of variables x and y. Below we consider class of finite-gap minimal surfaces in sphere S_R including compact minimal tori which are complexly normal in this sphere.

Finite gap solutions of the equation $u_{z\bar z} = e^u - e^{-2u}$ and associated orthonormal frame in $\mathbb{C}^3$.

Let's consider Riemannian surface Γ of even genus g with two distinguished points P_0 and P_∞ such that

1) there is a meromorphic function $\lambda(P)$ on Γ with divisor of zeros and poles $3\,P_0 - 3\,P_\infty$;
2) there is a holomorphic involution σ such that $\lambda(\sigma P) = -\lambda(P)$;
2) there is an antiholomorphic involution τ of separating type such that

$$\lambda(\tau P)\overline{\lambda(P)} = 1. \tag{4.1}$$

Points of Γ which are stable under the action of τ (i.e. $\tau P = P$) form a closed curve or a set of several closed curves. These curves are called invariant cycles of τ. For antiholomorphic involution of separating type τ its invariant cycles break Γ into two domains: Γ_0 containing point P_0 and Γ_∞ containing point P_∞. Due to (4.1) all invariant cycles of τ are projected onto unit circle on complex λ-plane. The number of these cycles is less or equal to three. This number determines the number of real tori in Jacobian $\mathrm{Jac}(\Gamma)$. Each such torus is composed by classes of divisors D of degree g such that

$$D + \tau D - P_0 - P_\infty = C,$$

where C is divisor of canonic class on Γ. Due to (4.2) each real divisor D (e.e. divisor from real torus in $\mathrm{Jac}(\Gamma)$) determines some Abelian differential of the third kind $\omega(P)$ with zeros at the points of divisor $D + \tau D$ and with residues

$$\operatorname*{Res}_{P=P_0}\omega(P) = +i, \qquad\qquad \operatorname*{Res}_{P=P_\infty}\omega(P) = -i,$$

at the points P_0 and P_∞, where $\omega(P)$ has simple poles. Under the action of τ differential $\omega(P)$ is transformed as follows:

$$\omega(\tau P) = \overline{\omega(P)}. \tag{4.3}$$

Therefore it is real valued on invariant cycles of τ. Real torus T_0 in $\mathrm{Jac}(\Gamma)$ is distinguished among other real tori by the following property: for divisor D from this torus differential $\omega(P)$ is positive on all invariant cycles of τ with respect to natural orientation of boundary $\partial\Gamma_\infty$.

Upon fixing torus T_0 in $\mathrm{Jac}(\Gamma)$ let's consider its subset consisting of divisors invariant under the action of composite map $\tau \circ \sigma$:

$$\tau D = \sigma D. \tag{4.4}$$

This subset is not empty. It is real torus T_0 in Prym variety $\mathrm{Prym}(\Gamma)$. For divisors of this torus we can complete (4.3) by another relationship

$$\omega(\sigma P) = \omega(P),$$

which follows from (4.4) and from invariance of points P_0 and P_∞ under the action of involution σ.

Let's fix local parameters $k^{-1}(P)$ and $q^{-1}(P)$ in the neighborhood of distinguished points P_0 and P_∞ by the conditions

$$k^3(P) = \lambda(P), \qquad \overline{k(\tau P)} = q(P). \tag{4.5}$$

Now, having fixed some positive divisor $D \in T_0 \subset \mathrm{Prym}(\Gamma)$ of degree g, we construct vectorial Baker-Achiezer function $\psi(z, P)$ with values in $\mathbb{C}^3$ such that

$$\begin{aligned}
\psi_1(P) &= e^{i\,k(P)\,z}\left(k^{-1}(P) + \dots\right), \\
\psi_2(P) &= e^{i\,k(P)\,z}\left(k^{-2}(P) + \dots\right), \\
\psi_1(P) &= e^{i\,k(P)\,z}\left(k^{-3}(P) + \dots\right)
\end{aligned} \tag{4.6}$$

in the neighborhood of distinguished point P_∞ and such that

$$\begin{aligned}
\psi_1(P) &= e^{i\,q(P)\,\bar z}\left(q^{1}(P) + \dots\right) e^{-u}, \\
\psi_2(P) &= e^{i\,q(P)\,\bar z}\left(q^{2}(P) + \dots\right) e^{u}, \\
\psi_1(P) &= e^{i\,q(P)\,\bar z}\left(q^{3}(P) + \dots\right)
\end{aligned} \tag{4.7}$$

in the neighborhood of another distinguished point P_0. Functions ψ_1, ψ_2, and ψ_3 are uniquely determined by divisor D and by conditions (4.5) and (4.7) (see [8]). They satisfy the following differential equations:

$$\begin{aligned}
\partial_z \psi_1 &= -u_z\,\psi_1 + i\,\lambda\,\psi_3, & \partial_{\bar z}\psi_1 &= i\,e^{-2u}\,\psi_2, \\
\partial_z \psi_2 &= u_z\,\psi_2 + i\,\psi_1, & \partial_{\bar z}\psi_2 &= i\,e^{u}\,\psi_3, \\
\partial_z \psi_3 &= i\,\psi_2, & \partial_{\bar z}\psi_2 &= i\,\lambda^{-1}\,e^{u}\,\psi_1.
\end{aligned} \tag{4.8}$$

Compatibility condition of these equations is equivalent to the equation (1.1) for the function $u = u(z,\bar z)$ in (4.8). Condition $D \in T_0 \subset \mathrm{Prym}(\Gamma)$ provides that u is real-valued smooth function. There is an explicit formula for u in terms of Prym theta-functions (see [8]). Respective to ψ_1, ψ_2, and ψ_3 the same condition expressed by (4.4) and (4.5) yields

$$\begin{aligned}
\psi_1(\sigma P) &= -\lambda^{-1}(P)\, e^{-u}\, \overline{\psi_2(\tau P)},\\
\psi_2(\sigma P) &= \lambda^{-1}(P)\, e^{u}\, \overline{\psi_1(\tau P)},\\
\psi_3(\sigma P) &= -\lambda^{-2}(P)\, \overline{\psi_2(\tau P)}.
\end{aligned} \tag{4.9}$$

Remarkable feature of spectral problems associated with integrable nonlinear equations is the presence of bilinear forms (pairings or generalized Wronskians), which are in concordance with Lax operators and which in finite-gap case possess some "resonant" properties. The latter property can be used in constructing soliton-like solutions on finite-gap background for these equations and in Cauchy kernels on Riemann surfaces. For spectral problem (4.8) such pairing is given by formula

$$\begin{aligned}
\Omega(P,Q) = \{\psi(P)\,|\,\psi(\sigma Q)\} &= \psi_1(P)\,\psi_2(\sigma Q)\,\lambda(P) -\\
&- \psi_2(P)\,\psi_1(\sigma Q)\,\lambda(P) - \psi_3(P)\,\psi_3(\sigma Q)\,\lambda^2(P).
\end{aligned} \tag{4.10}$$

Let's differentiate (4.10) with respect to z and $\bar z$. Taking into account differential equations (4.8), we obtain the equalities

$$\begin{aligned}
&\partial_z \Omega(P,Q) = i\,(\lambda(Q) - \lambda(P))\,\lambda(P)\,\psi_2(P)\,\psi_3(\sigma Q),\\
&\partial_{\bar z} \Omega(P,Q) = i\, e^{u}\, \big(\lambda(P)\,\lambda^{-1}(Q) - 1\big)\,\lambda(P)\,\psi_3(P)\,\psi_1(\sigma Q)
\end{aligned}$$

Looking at these equalities, we see that for $Q = P$ function (4.10) does not depend on z and $\bar z$. Moreover, function $W(P) = \Omega(P,P)$ is meromorphic on Γ, it can be calculated explicitly in the following form:

$$W(P) = \frac{i\, d\lambda(P)}{\lambda(P)\,\omega(P)}. \tag{4.11}$$

Note that $\lambda : \Gamma \to \mathbb{C}$ is three-sheeted covering. Therefore each value λ of the function $\lambda(P)$ has multiplicity 3, i. e. in general case there are three distinct points P_1, P_2, and P_3 such that $\lambda(P_1) = \lambda(P_2) = \lambda(P_3) = \lambda$. Resonant property of $\Omega(P,Q)$ then is expressed by the following equality:

$$\Omega(P_i,P_j) = \begin{cases} W(P_i) & \text{for } \ P_i = P_j,\\ \ \ 0 & \text{for } \ P_i \neq P_j. \end{cases} \tag{4.12}$$

For each value of λ such that $|\lambda| = 1$ points P_1, P_2, and P_3 are on invariant cycles of antiholomorphic involution τ. They are stable under the action of τ. Using them, we can compose the matrix $U = U(\lambda, z, \bar z)$ of the following form:

$$U = \left\|\begin{matrix}
\dfrac{e^{u/2}\,\psi_1(P_1)}{\sqrt{W(P_1)}} & \dfrac{e^{-u/2}\,\psi_2(P_1)}{\sqrt{W(P_1)}} & \dfrac{\psi_3(P_1)}{\sqrt{W(P_1)}}\\[2ex]
\dfrac{e^{u/2}\,\psi_1(P_2)}{\sqrt{W(P_2)}} & \dfrac{e^{-u/2}\,\psi_2(P_2)}{\sqrt{W(P_2)}} & \dfrac{\psi_3(P_2)}{\sqrt{W(P_2)}}\\[2ex]
\dfrac{e^{u/2}\,\psi_1(P_3)}{\sqrt{W(P_3)}} & \dfrac{e^{-u/2}\,\psi_2(P_3)}{\sqrt{W(P_3)}} & \dfrac{\psi_3(P_3)}{\sqrt{W(P_1)}}
\end{matrix}\right\| \tag{4.13}$$

Resonant property (4.12), invariance of P_1, P_2, and P_3 under the action of τ, and the equalities (4.9) lead to the following relationship:

$$e^u\,\psi_1(P_i)\,\overline{\psi_1(P_j)}+e^{-u}\,\psi_2(P_i)\,\overline{\psi_2(P_j)}+\psi_3(P_i)\,\overline{\psi_3(P_j)}=W(P_i)\,\delta_{ij}.$$

This means that matrix U in (4.13) is unitary matrix. Moreover, this equality means that function (4.11) is real and non-negative on invariant cycles of τ. Therefore square roots in (4.13) are real numbers. Columns of unitary matrix U form an orthonormal frame in $\mathbb{C}^3$:

$$\mathbf{L}=U_1, \qquad\qquad \mathbf{M}=U_2, \qquad\qquad \mathbf{N}=U_3. \tag{4.14}$$

This frame consists of three unit vectors perpendicular to each other with respect to Hermitian metric (2.1).

5. Finite-gap embeddings of two-dimensional surfaces in $\mathbb{C}^3$.

Let's study the dynamics of orthonormal frame (4.14). In complex variable z and $\bar z$ it is determined by the equations (4.8):

$$\begin{aligned}
&\partial_z\mathbf{L}=-\frac{u_z}{2}\cdot\mathbf{L}+i\,\lambda\,e^{u/2}\cdot\mathbf{N}, &&\partial_{\bar z}\mathbf{L}=\frac{u_{\bar z}}{2}\cdot\mathbf{L}+i\,e^{-u}\,\cdot\mathbf{M},\\
&\partial_z\mathbf{M}=\frac{u_z}{2}\cdot\mathbf{M}+i\,e^{u}\cdot\mathbf{L}, &&\partial_{\bar z}\mathbf{M}=-\frac{u_{\bar z}}{2}\cdot\mathbf{M}+i\,e^{u/2}\,\cdot\mathbf{N},\\
&\partial_z\mathbf{N}=i\,e^{u/2}\cdot\mathbf{M}, &&\partial_{\bar z}\mathbf{N}=i\,\lambda^{-1}\,e^{u/2}\,\cdot\mathbf{L}.
\end{aligned}\tag{5.1}$$

Passing to real variables $x=x^1=\operatorname{Re}z$ and $y=x^2=\operatorname{Im}z$, from (5.1) we derive the following equalities for the dynamics of frame (4.14) with respect to x:

$$\begin{aligned}
&\partial_x\mathbf{L}=\frac{u_y}{2}\cdot\mathbf{L}+i\,\lambda\,e^{u/2}\cdot\mathbf{N}+i\,e^{-u}\cdot\mathbf{M},\\
&\partial_x\mathbf{M}=-i\,\frac{u_y}{2}\cdot\mathbf{M}+i\,e^{-u}\cdot\mathbf{L}+i\,e^{u/2}\cdot\mathbf{N},\\
&\partial_x\mathbf{N}=i\,e^{u/2}\cdot\mathbf{M}+i\,\lambda^{-1}\,e^{u/2}\cdot\mathbf{L},
\end{aligned}\tag{5.2}$$

Similar equations describe the dynamics of this frame with respect to y:

$$\begin{aligned}
&\partial_y\mathbf{L}=-\frac{u_x}{2}\cdot\mathbf{L}-\lambda\,e^{u/2}\cdot\mathbf{N}+e^{-u}\cdot\mathbf{M},\\
&\partial_y\mathbf{M}=i\,\frac{u_x}{2}\cdot\mathbf{M}-e^{u}\cdot\mathbf{L}+e^{u/2}\cdot\mathbf{N},\\
&\partial_y\mathbf{N}=-e^{u/2}\cdot\mathbf{M}+\lambda^{-1}\,e^{u/2}\cdot\mathbf{L}.
\end{aligned}\tag{5.3}$$

Let's define the embedding of the surface T into the sphere $S_R\subset\mathbb{C}^3$ parametrically by means of function

$$\mathbf{r}(x^1,x^2)=R\cdot\mathbf{N}(x^1,x^2)=R\cdot\mathbf{N}(x,y). \tag{5.4}$$

For tangent vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ in the case of such embedding we obtain

$$\begin{aligned}\mathbf{E}_1 &= i\,R\,e^{u/2}\cdot\mathbf{M} + i\,R\,\lambda^{-1}\,e^{u/2}\cdot\mathbf{L},\\ \mathbf{E}_2 &= -R\,e^{u/2}\cdot\mathbf{M} + R\,\lambda^{-1}\,e^{u/2}\cdot\mathbf{L}.\end{aligned}\tag{5.5}$$

Using Hermitian orthogonality of frame (4.14), one can easily find that vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ are both orthogonal to unit vector $\mathbf{N}$ with respect to Hermitian metric (2.1). Hence embedding (5.4) is complexly normal. Moreover, metric g_{ij} determined by (2.3) is diagonal and has conformally Euclidean form $\mathbf{g} = 2\,R^2\,e^u\,\big(dx^2 + dy^2\big)$.

Now let's apply the equations (5.2) and (5.3). Remember that λ is chosen to be complex number of unit modulus. Taking $\lambda = \cos\vartheta + i\,\sin\vartheta$, from (5.5) we derive the dynamics of vectors $\mathbf{E}_1$ and $\mathbf{E}_2$. Here it is:

$$\begin{aligned}\nabla_1\mathbf{E}_1 &= e^{-u}\,\cos\vartheta\cdot\mathbf{F}_1 - e^{-u}\,\sin\vartheta\cdot\mathbf{F}_2 - 2\,R\,e^u\cdot\mathbf{N},\\ \nabla_2\mathbf{E}_1 = \nabla_1\mathbf{E}_2 &= -e^{-u}\,\sin\vartheta\cdot\mathbf{F}_1 - e^{-u}\,\cos\vartheta\cdot\mathbf{F}_2,\\ \nabla_2\mathbf{E}_2 &= -e^{-u}\,\cos\vartheta\cdot\mathbf{F}_1 + e^{-u}\,\sin\vartheta\cdot\mathbf{F}_2 - 2\,R\,e^u\cdot\mathbf{N}.\end{aligned}\tag{5.6}$$

Components of metric connection for covariant derivatives in (5.6) are given by (3.2). Comparing (5.6) with (2.12) we obtain components of tensor T^k_{ij} for the embedding (5.4). They have exactly the same form as given by (3.4). Scalar invariants of this tensor are given by formulas (3.5). Compact finite-gap tori arise in the case of double-periodic function ψ_2. Problem of finding periodic solutions is standard in the theory of finite-gap integration. They are obtained by imposing some rather non-explicit restrictions to Riemann surface Γ, which are written as rationality condition for some quotients of Abelian integrals on it.

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