

V-REPRESENTATION FOR NORMALITY EQUATIONS
in geometry of generalized Legendre transformation.

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ABSTRACT. Normality equations describe Newtonian dynamical systems admitting normal shift of hypersurfaces. These equations were first derived in Euclidean geometry. Then very soon they were rederived in Riemannian and in Finslerian geometry. Recently I have found that normality equations can be derived in geometry given by classical and/or generalized Legendre transformation. However, in this case they appear to be written in \mathbf{p} -representation, i. e. in terms of momentum covector and its components. The goal of present paper is to transform normality equations back to \mathbf{v} -representation, which is more natural for Newtonian dynamical systems.

1. NEWTONIAN DYNAMICAL SYSTEMS
 IN VELOCITY AND MOMENTUM REPRESENTATIONS.

Traditionally first section of a paper is introduction. However for this paper it would be rather large. The matter is that theory of Newtonian dynamical systems admitting normal shift was discovered and was initially developed by me and my students (see papers [1–14] and theses [15] and [16]). Last few years it is developed by me alone (see papers [17–26]). In order to make easy reading my paper for people incognizant of this theory I have broken introductory information into several parts, each with its own title. In sections 1–5 below I give all necessary definitions along with historical overview, and I resume in brief results of previous paper [26].

Newtonian dynamical system describes the motion of mass point with unit mass according to Newton’s second law. It is given by ordinary differential equations

$$\dot{x}^i = v^i, \quad \dot{v}^i = \Phi^i, \quad (1.1)$$

where $\Phi^i = \Phi^i(x^1, \dots, x^n, v^1, \dots, v^n)$. In geometric interpretation of differential equations (1.1) variables x^1, \dots, x^n are interpreted as coordinates of moving point $p = p(t)$ of some n -dimensional manifold M in a local chart of this manifold. Variables v^1, \dots, v^n are components of velocity vector \mathbf{v} tangent to M at the point $p = p(t)$. Pair $q = (p, \mathbf{v})$ composed by a point $p \in M$ and by some tangent vector $\mathbf{v} \in T_p(M)$ is a point of tangent bundle TM . In such interpretation dynamical system (1.1) describes not only a moving mass point, but arbitrary mechanical system with holonomic constraints, i. e. any machine with numerous moving parts joined by cardan joints, thumbscrews, tooth gearings and so on (see details in

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For the sake of simplicity we shall assume generalized Legendre maps (1.4) given by functions (1.6) and (1.7) in local chart to be establishing diffeomorphism of TM and T^*M such that zero maps to zero in each fiber of these two bundles.

Now let's treat (1.7) as change of variables. Substituting (1.7) into differential equations (1.1), we can transform them to the following ones:

$$\dot{x}^i = V^i \qquad \dot{p}_i = \Theta_i. \qquad (1.8)$$

Here functions $V^i = V^i(x^1, \dots, x^n, p_1, \dots, p_n)$ are given by (1.7), while functions $\Theta^i = \Theta^i(x^1, \dots, x^n, p_1, \dots, p_n)$ form another set of n functions. They are determined implicitly through function Φ^1, \dots, Φ^n in (1.1) by the following equation:

$$\Phi^i \circ \lambda^{-1} = \sum_{s=1}^n \frac{\partial V^i}{\partial x^s} V^s + \sum_{s=1}^n \frac{\partial V^i}{\partial p_s} \Theta^s. \qquad (1.9)$$

If the equations (1.9) are fulfilled, then both (1.1) and (1.8) express the same dynamics, but in different representations: (1.8) is called **momentum representation** or **p-representation** for Newtonian dynamical system (1.1), while (1.1) is called **velocity representation** or **v-representation** for Newtonian dynamical system (1.8). Concluding this section, note that (1.9) can be written as

$$\Theta_i \circ \lambda = \sum_{s=1}^n \frac{\partial L_i}{\partial x^s} v^s + \sum_{s=1}^n \frac{\partial L_i}{\partial v^s} \Phi^s. \qquad (1.10)$$

Formula (1.10) is more explicit with respect to functions $\Theta_1, \dots, \Theta_n$ in (1.8).

2. EXTENDED TENSOR FIELDS.

Extended tensor fields are closely related to generalized Legendre maps (1.4). Indeed, functions (1.6) are components covector $\mathbf{L} \subset T_p^*(M)$. However, unlike to components of traditional covector field, they depend not only on coordinates of point $p \in M$, but also on components of velocity vector $\mathbf{v} \in T_p(M)$. Pair $q = (p, \mathbf{v})$ is a point of TM . Therefore \mathbf{L} is extended covector field in **v-representation** (see definition below). In a similar way, functions (1.7) are components of extended vector field \mathbf{V} in **p-representation**.

At each point p of base manifold M one can define tensor space $T_s^r(p, M)$. This is the following tensor product of several copies of $T_p(M)$ and $T_p^*(M)$:

$$T_s^r(p, M) = \overbrace{T_p(M) \otimes \dots \otimes T_p(M)}^{r \text{ times}} \otimes \underbrace{T_p^*(M) \otimes \dots \otimes T_p^*(M)}_{s \text{ times}}. \qquad (2.1)$$

Definition 2.1. Extended tensor field \mathbf{X} of type (r, s) in **v-representation** is a tensor-valued function $\mathbf{X} = \mathbf{X}(q)$ with argument $q = (p, \mathbf{v})$ in tangent bundle TM and with values in tensor space (2.1), where $p = \pi(q)$.

Definition 2.2. Extended tensor field \mathbf{X} of type (r, s) in **p-representation** is a tensor-valued function $\mathbf{X} = \mathbf{X}(q)$ with argument $q = (p, \mathbf{p})$ in cotangent bundle T^*M and with values in tensor space (2.1), where $p = \pi(q)$.

Each extended tensor field can be transformed from \mathbf{v} to \mathbf{p} -representation and back from \mathbf{p} to \mathbf{v} -representation by changing its argument. We use generalized Legendre maps (1.4) for this purpose. Extended covector field \mathbf{L} and extended vector field \mathbf{V} defining these two maps are also examples of extended tensor fields in \mathbf{v} and \mathbf{p} -representations respectively. Like traditional tensor fields, extended tensor field \mathbf{X} in local chart are represented by its components:

$$\mathbf{X} = \sum_{i_1=1}^n \dots \sum_{i_r=1}^n \sum_{j_1=1}^n \dots \sum_{j_s=1}^n X_{j_1 \dots j_s}^{i_1 \dots i_r} \frac{\partial}{\partial x^{i_1}} \otimes \dots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_s}.$$

Definition 2.3. Extended tensor field \mathbf{X} is called smooth if its components in any local chart of the manifold M are smooth functions of their arguments.

Let's denote by $T_s^r(M)$ the set of smooth extended tensor fields of type (r, s) either in \mathbf{v} or in \mathbf{p} -representation. They form a module over the ring of smooth scalar functions in TM or in T^*M , we denote these rings by $\mathfrak{F}(TM)$ and $\mathfrak{F}(T^*M)$ respectively. Then we can define the following direct sum:

$$\mathbf{T}(M) = \bigoplus_{r=0}^{\infty} \bigoplus_{s=0}^{\infty} T_s^r(M). \quad (2.2)$$

This direct sum (2.2) is a graded algebra over one of these two rings. It is called **extended algebra of tensor fields**. In $\mathbf{T}(M)$ we have all standard tensorial operations like summation, multiplication by scalars, tensor product, and contraction. Apart from these algebraic operations, in $\mathbf{T}(M)$ we have **canonical vertical differentiation** $\tilde{\nabla}$. In \mathbf{v} -representation it can be defined by the following formula:

$$\tilde{\nabla}_k X_{j_1 \dots j_s}^{i_1 \dots i_r} = \frac{\partial X_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial v^k}. \quad (2.3)$$

For extended tensor field \mathbf{X} in \mathbf{p} -representation we have formula similar to (2.3):

$$\tilde{\nabla}^k X_{j_1 \dots j_s}^{i_1 \dots i_r} = \frac{\partial X_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial p_k}. \quad (2.4)$$

Canonical vertical differentiation does not commute with generalized Legendre transformation λ . Here we have the following relationships:

$$\tilde{\nabla}_k X_{j_1 \dots j_s}^{i_1 \dots i_r} = \sum_{q=1}^n g_{qk} \cdot \tilde{\nabla}^q (X_{j_1 \dots j_s}^{i_1 \dots i_r} \circ \lambda^{-1}) \circ \lambda, \quad (2.5)$$

$$\tilde{\nabla}^k X_{j_1 \dots j_s}^{i_1 \dots i_r} = \sum_{q=1}^n (g^{qk} \circ \lambda^{-1}) \cdot \tilde{\nabla}_q (X_{j_1 \dots j_s}^{i_1 \dots i_r} \circ \lambda) \circ \lambda^{-1}. \quad (2.6)$$

By g_{qk} and g^{qk} in (2.5) and (2.6) we denote Jacobi matrices for maps (1.4):

$$g_{qk} = \tilde{\nabla}_k L_q, \quad g^{qk} = \tilde{\nabla}^k V^q \circ \lambda. \quad (2.7)$$

Matrices (2.7) are inverse to each other in the sense of the following equalities:

$$\sum_{r=1}^n g^{sr} g_{rk} = \delta_k^s, \quad \sum_{r=1}^n g_{kr} g^{rs} = \delta_k^s. \quad (2.8)$$

Extended tensor fields (2.7) are non-symmetric. But nevertheless, looking to (2.5), (2.6), and (2.8), we conclude that they are analogs of metric tensor and dual metric tensor in Riemannian geometry.

3. EXTENDED CONNECTIONS AND HORIZONTAL COVARIANT DERIVATIVES.

Normality equations we are going to study below are written in terms of covariant derivatives (see [26]). Apart from (2.4), there another covariant derivative is used. When applied to extended tensor field \mathbf{X} in \mathbf{p} -representation, it acts as follows:

$$\begin{aligned} \nabla_m X_{j_1 \dots j_s}^{i_1 \dots i_r} &= \frac{\partial X_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial x^m} + \sum_{a=1}^n \sum_{b=1}^n p_a \Gamma_{mb}^a \frac{\partial X_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial p_b} + \\ &+ \sum_{k=1}^r \sum_{a_k=1}^n \Gamma_{m a_k}^{i_k} X_{j_1 \dots a_k \dots j_s}^{i_1 \dots a_k \dots i_r} - \sum_{k=1}^s \sum_{b_k=1}^n \Gamma_{m j_k}^{b_k} X_{j_1 \dots b_k \dots j_s}^{i_1 \dots i_r}. \end{aligned} \quad (3.1)$$

Here Γ_{ij}^k are components of some extended affine connection¹ in \mathbf{p} -representation:

$$\Gamma_{ij}^k = \Gamma_{ij}^k(x^1, \dots, x^n, p_1, \dots, p_n). \quad (3.2)$$

Covariant differentiation ∇ in $\mathbf{T}(M)$ determined by extended affine connection (3.2) in formula (3.1) is called **horizontal covariant differentiation**. It is not canonical since it depends on the choice of Γ . Now we shall not discuss the concept of extended affine connection (see [26] and Chapter III of thesis [15]). Note only that by means of generalized Legendre map λ we can transform its components (3.2) to \mathbf{v} -representation: $\Gamma_{ij}^k \rightarrow \Gamma_{ij}^k \circ \lambda$. Then we have

$$\Gamma_{ij}^k = \Gamma_{ij}^k(x^1, \dots, x^n, v^1, \dots, v^n). \quad (3.3)$$

Using (3.3), one can define horizontal covariant differentiation in \mathbf{v} -representation:

$$\begin{aligned} \nabla_m X_{j_1 \dots j_s}^{i_1 \dots i_r} &= \frac{\partial X_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial x^m} - \sum_{a=1}^n \sum_{b=1}^n v^a \Gamma_{am}^b \frac{\partial X_{j_1 \dots j_s}^{i_1 \dots i_r}}{\partial v^b} + \\ &+ \sum_{k=1}^r \sum_{a_k=1}^n \Gamma_{m a_k}^{i_k} X_{j_1 \dots a_k \dots j_s}^{i_1 \dots a_k \dots i_r} - \sum_{k=1}^s \sum_{b_k=1}^n \Gamma_{m j_k}^{b_k} X_{j_1 \dots b_k \dots j_s}^{i_1 \dots i_r}. \end{aligned} \quad (3.4)$$

Horizontal covariant differentiation ∇ also does not commute with generalized Le-

¹Below we consider only symmetric connections $\Gamma_{ij}^k = \Gamma_{ji}^k$. This is sufficient for our purposes.

gendre map λ . Here for differentiation ∇ we have the following relationships:

$$\begin{aligned} \nabla_m X_{j_1 \dots j_s}^{i_1 \dots i_r} &= \nabla_m (X_{j_1 \dots j_s}^{i_1 \dots i_r} \circ \lambda) \circ \lambda^{-1} + \\ &+ \sum_{q=1}^n \nabla_m V^q \cdot \tilde{\nabla}_q (X_{j_1 \dots j_s}^{i_1 \dots i_r} \circ \lambda) \circ \lambda^{-1}, \end{aligned} \quad (3.5)$$

$$\begin{aligned} \nabla_m X_{j_1 \dots j_s}^{i_1 \dots i_r} &= \nabla_m (X_{j_1 \dots j_s}^{i_1 \dots i_r} \circ \lambda^{-1}) \circ \lambda + \\ &+ \sum_{q=1}^n \nabla_m L_q \cdot \tilde{\nabla}^q (X_{j_1 \dots j_s}^{i_1 \dots i_r} \circ \lambda^{-1}) \circ \lambda. \end{aligned} \quad (3.6)$$

These relationships (3.5) and (3.6) for ∇ are analogs of relationships (2.5) and (2.6).

4. FORCE VECTOR AND FORCE COVECTOR OF NEWTONIAN DYNAMICAL SYSTEM.

Note that functions $\Phi^i(x^1, \dots, x^n, v^1, \dots, v^n)$ in (1.1) are not components of vector field even if we understand it in the sense of definition 2.1. Similarly, functions $\Theta_i(x^1, \dots, x^n, p_1, \dots, p_n)$ in (1.8) are not components of covector field in the sense of definition 2.2. We can change this situation if we choose some symmetric extended affine connection Γ . Using components of Γ in \mathbf{v} -representation (3.3), we replace time derivative \dot{v}^i in (1.1) by covariant time derivative

$$\nabla_t v^i = \dot{v}^i + \sum_{j=1}^n \sum_{k=1}^n \Gamma_{jk}^i v^j v^k. \quad (4.1)$$

In a similar way we replace time derivative \dot{p}_i in (1.8) by covariant time derivative

$$\nabla_t p_i = \dot{p}_i - \sum_{j=1}^n \sum_{k=1}^n \Gamma_{ij}^k V^j p_k. \quad (4.2)$$

For this purpose we need components of Γ in \mathbf{p} -representation (3.2). Relying upon (4.1) and (4.2) we replace functions Φ^i and Θ_i by functions F^i and Q_i as follows:

$$F^i = \Phi^i + \sum_{j=1}^n \sum_{k=1}^n \Gamma_{jk}^i v^j v^k, \quad (4.3)$$

$$Q_i = \Theta_i - \sum_{j=1}^n \sum_{k=1}^n \Gamma_{ij}^k V^j p_k. \quad (4.4)$$

In terms of these newly introduced functions (4.3), (4.4) and covariant derivatives (4.1), (4.2) differential equations (1.1) and (1.8) are rewritten as

$$\dot{x}^i = v^i, \quad \nabla_t v^i = F^i, \quad (4.5)$$

$$\dot{x}^i = V^i, \quad \nabla_t p_i = Q_i. \quad (4.6)$$

We say that (4.5) and (4.6) are tensorial form of differential equations of Newtonian

dynamics in \mathbf{v} -representation and \mathbf{p} -representation respectively. Indeed, functions $F^i = F^i(x^1, \dots, x^n, v^1, \dots, v^n)$ in (4.5) are components of extended vector field \mathbf{F} . It is called **force vector** of Newtonian dynamical system (4.5). Functions $Q_i = Q_i(x^1, \dots, x^n, v^1, \dots, v^n)$ represent extended covector field \mathbf{Q} . It is called **force covector** of Newtonian dynamical system (4.6).

5. NORMAL SHIFT AND NORMALITY EQUATIONS.

In Euclidean and in Riemannian geometry normal shift is a continuous deformation of hypersurface when each its point moves along some trajectory and moving hypersurface in whole keeps orthogonality to trajectories of all its points. This means that normal vector of moving hypersurface is always collinear to tangent vector of shift trajectory. The same is true in Finslerian geometry (see Chapter VIII of thesis [15]). In non-metric geometries one cannot define normal vector of hypersurface. But here one can define **normal covector**. This is key idea of papers [24], [25], and [26]. Let σ be some hypersurface in M and let $p \in \sigma$ be a point of σ . Then $T_p(\sigma)$ is a hyperplane in tangent space $T_p(M)$. Denote by \mathbf{n} any nonzero covector in $T_p^*(M)$ whose null-space coincides with $T_p(\sigma)$:

$$T_p(\sigma) = \{\mathbf{X} \in T_p(P) : \langle \mathbf{n} | \mathbf{X} \rangle = 0\}. \tag{5.1}$$

By angular brackets in (5.1) we denote scalar product of vector \mathbf{X} and covector \mathbf{n} :

$$\langle \mathbf{n} | \mathbf{X} \rangle = \sum_{i=1}^n n_i X^i. \tag{5.2}$$

Covector $\mathbf{n} = \mathbf{n}(p)$ is determined by the condition (5.1) uniquely up to some nonzero scalar factor: $\mathbf{n} \rightarrow \nu \cdot \mathbf{n}$. It is called **normal covector** of hypersurface σ at the point p . One can choose $\mathbf{n} = \mathbf{n}(p)$ to be smooth covector function on σ (at least locally in some neighborhood of any point). Now we can associate with σ the following initial data for Newtonian dynamical system (1.8) in \mathbf{p} -representation:

$$x^i \Big|_{t=0} = x^i(p), \quad p_i \Big|_{t=0} = \nu(p) \cdot n_i(p). \tag{5.3}$$

Let's fix some point $p_0 \in \sigma$. Smooth normal covector $\mathbf{n} = \mathbf{n}(p)$ of hypersurface σ is defined in some neighborhood of the point p_0 . For scalar factor $\nu = \nu(p)$ in (5.3) we shall assume it to be smooth nonzero function in some possibly smaller neighborhood of the point p_0 normalized by the condition

$$\nu(p_0) = \nu_0, \tag{5.4}$$

where $\nu_0 \neq 0$ is some arbitrary predefined constant. We can apply initial data (5.3) to Newtonian dynamical system given either by differential equations (1.8) or (4.6). This initiates motion of hypersurface σ (or its smaller part including our fixed point p_0) when all points are moving along trajectories of this dynamical system.

Definition 5.1. Shift of hypersurface $\sigma \subset M$ along trajectories of a dynamical system in cotangent bundle T^*M determined by initial data (5.3) is called **normal shift** if normal covectors of all shifted hypersurfaces σ_t are collinear to momentum covector \mathbf{p} on shift trajectories.

Definition 5.2. Dynamical system in cotangent bundle T^*M satisfies **strong normality condition** if for any hypersurface $\sigma \subset M$, for any fixed point $p_0 \in \sigma$, and for any constant $\nu_0 \neq 0$ there is some open neighborhood of p_0 on σ and some smooth function $\nu(p)$ normalized by the condition (5.4) in this neighborhood such that it determines normal shift of σ in the sense of definition 5.1.

We say that dynamical system admits normal shift of hypersurfaces in M if it satisfies strong normality condition stated in definition 5.2. It is clear that when applied to Newtonian dynamical system (4.6), strong normality condition yields some restrictions for the functions V^1, \dots, V^n and Q_1, \dots, Q_n in (4.6). In [26] it was shown that these restrictions are written in form of so called **normality equations** for these functions. In order to write them we need to introduce several auxiliary extended tensor fields. First is a vector field \mathbf{W} with components

$$W^i = \sum_{s=1}^n p_s \tilde{\nabla}^i V^s. \quad (5.5)$$

This vector field is used to define projector-valued operator field \mathbf{P} with components

$$P_j^i = \delta_j^i - \frac{W^i p_j}{\Omega}, \quad (5.6)$$

where Ω is a scalar field defined as scalar product of covector \mathbf{p} and vector \mathbf{W} :

$$\Omega = \langle \mathbf{p} | \mathbf{W} \rangle = \sum_{s=1}^n p_s W^s \quad (5.7)$$

(compare with (5.2)). Next is covector field \mathbf{U} . Its components are given by formula

$$U_i = Q_i + \sum_{s=1}^n \nabla_i V^s p_s. \quad (5.8)$$

Formulas for vector field $\boldsymbol{\alpha}$ and covector field $\boldsymbol{\beta}$ are more complicated:

$$\begin{aligned} \alpha^k &= \sum_{r=1}^n \tilde{\nabla}^k V^r U_r + \sum_{r=1}^n \nabla_r W^k V^r + \sum_{r=1}^n \tilde{\nabla}^r W^k Q_r + \\ &+ \sum_{r=1}^n W^r \tilde{\nabla}^k Q_r - \sum_{r=1}^n \sum_{s=1}^n \sum_{q=1}^n p_s D_{rq}^{sk} W^r V^q. \\ \beta_k &= \sum_{r=1}^n \nabla_r U_k V^r + \sum_{r=1}^n \tilde{\nabla}^r U_k Q_r + \sum_{r=1}^n \nabla_k V^r U_r + \\ &+ \sum_{r=1}^n \nabla_k Q_r W^r - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n (R_{rsk}^s V^m - D_{rk}^{sm} Q_m) W^r p_s. \end{aligned} \quad (5.9)$$

$$(5.10)$$

Here in (5.9) and (5.10) we see components of two curvature tensors \mathbf{D} and \mathbf{R} :

$$D_{ij}^{kr} = -\frac{\partial \Gamma_{ij}^k}{\partial p_r}. \quad (5.11)$$

Tensor \mathbf{D} is dynamic curvature tensor. It is nonzero only for extended connections. As for tensor \mathbf{R} , it is analog of standard curvature tensor:

$$\begin{aligned} R_{rij}^k &= \frac{\partial \Gamma_{jr}^k}{\partial x^i} - \frac{\partial \Gamma_{ir}^k}{\partial x^j} + \sum_{m=1}^n \Gamma_{im}^k \Gamma_{jr}^m - \sum_{m=1}^n \Gamma_{jm}^k \Gamma_{ir}^m + \\ &+ \sum_{m=1}^n \sum_{s=1}^n p_s \Gamma_{mi}^s \frac{\partial \Gamma_{jr}^k}{\partial p^m} - \sum_{m=1}^n \sum_{s=1}^n p_s \Gamma_{mj}^s \frac{\partial \Gamma_{ir}^k}{\partial p^m}. \end{aligned} \quad (5.12)$$

Covector field $\boldsymbol{\eta}$ is defined by the following formula for its components:

$$\eta_k = \beta_k - \sum_{s=1}^n \frac{U_k \alpha^s p_s}{\Omega}. \quad (5.13)$$

As it was shown in [26] in multidimensional case $n = \dim M \geq 3$ complete system of normality equations naturally subdivides into two parts. First part is formed by so called **weak normality equations**. Relying upon all the above notations (5.5), (5.6), (5.7), (5.8), (5.9), (5.10), (5.11), (5.12), and (5.13), now we are able to write weak normality equations in a very concise form:

$$\sum_{r=1}^n \alpha^r P_r^k = 0, \quad \sum_{r=1}^n \eta_r P_k^r = 0. \quad (5.14)$$

Inspecting (5.14) with all the above notations in mind, one can find that weak normality equations form a system of $2n$ partial differential equations for components of two fields: extended vector field \mathbf{V} and extended covector field \mathbf{Q} . But rank of projection operator \mathbf{P} is $n - 1$. Therefore actual number of independent equations in (5.14) is equal to $2n - 2$.

Second part of normality equations in multidimensional case $n = \dim M \geq 3$ is formed by so called **additional normality equations**. In order to write them we need to use some more auxiliary notations:

$$A^{rs} = \tilde{\nabla}^r W^s, \quad (5.15)$$

$$\begin{aligned} B_s^r &= \tilde{\nabla}^r U_s + \sum_{m=1}^n \sum_{k=1}^n W^k p_m D_{ks}^{mr} - \\ &- \nabla_s W^r + \sum_{m=1}^n \frac{\tilde{\nabla}^m W^r - \tilde{\nabla}^r W^m}{\Omega} U_s p_m, \end{aligned} \quad (5.16)$$

$$\begin{aligned} C_{rs} &= \nabla_r U_s - \sum_{m=1}^n \frac{U_r \tilde{\nabla}^m U_s + U_s \nabla_r W^m}{\Omega} p_m - \\ &- \sum_{k=1}^n \sum_{q=1}^n \left(\sum_{m=1}^n \frac{D_{ks}^{mq} U_r}{\Omega} p_m + \frac{R_{krs}^q}{2} \right) W^k p_q. \end{aligned} \quad (5.17)$$

Here A^{rs} , B_s^r , and C_{rs} are components of three extended tensor fields \mathbf{A} , \mathbf{B} , and \mathbf{C} respectively. Now, using (5.15), (5.16), (5.17) and keeping in mind all previous

notations, we can write additional normality equations as well. They look like

$$\sum_{r=1}^n \sum_{s=1}^n (A^{rs} - A^{sr}) P_r^i P_s^j = 0, \quad (5.18)$$

$$\sum_{r=1}^n \sum_{s=1}^n P_r^i B_s^r P_j^s = \lambda P_j^i, \quad (5.19)$$

$$\sum_{r=1}^n \sum_{s=1}^n (C_{rs} - C_{sr}) P_i^r P_j^s = 0. \quad (5.20)$$

Here λ in (5.19) is a scalar factor. However, it is not a parameter with undetermined or deliberate value. Its value is determined by the equation (5.19) itself:

$$\lambda = \frac{\text{tr}(\lambda \mathbf{P})}{\text{tr} \mathbf{P}} = \sum_{r=1}^n \sum_{s=1}^n \frac{B_s^r P_r^s}{n-1}. \quad (5.21)$$

Due to formula (5.21) the equation (5.19) can be written as follows:

$$\sum_{r=1}^n \sum_{s=1}^n P_r^i B_s^r P_j^s = \sum_{r=1}^n \sum_{s=1}^n \frac{B_s^r P_r^s}{n-1} P_j^i. \quad (5.22)$$

Under some additional restriction for generalized Legendre map (see definition 6.1 in paper [26]) the role of weak and additional normality equations is described by the following result (see theorems 8.1, 11.1, and 12.1 in paper [26]).

Theorem 5.1. *Strong normality condition for Newtonian dynamical system (4.6) in multidimensional case $n \geq 3$ is equivalent to complete system of normality equations (5.14), (5.18), (5.19), (5.20) that should be fulfilled at all points $q = (p, \mathbf{p})$ of cotangent bundle T^*M , where $\mathbf{p} \neq 0$.*

Two-dimensional case is an exception. Here we have the following result.

Theorem 5.2. *Strong normality condition for Newtonian dynamical system (4.6) in the dimension $n = 2$ is equivalent to weak normality equations (5.14) that should be fulfilled at all points $q = (p, \mathbf{p})$ of cotangent bundle T^*M , where $\mathbf{p} \neq 0$.*

Weak normality equations (5.14) and additional normality equations (5.18), (5.19), (5.20), as well as the equations (5.22), are written in terms of extended tensor fields and covariant derivatives in \mathbf{p} -representation. Our goal below is to transform all of them to \mathbf{v} -representation.

6. TRANSFORMATION OF PROJECTION OPERATOR.

Projection operator \mathbf{P} with components (5.6) is present in each normality equation. Let's transform it to \mathbf{v} -representation. For W^i in (5.6) we have

$$W^i \circ \lambda = \sum_{s=1}^n (p_s \circ \lambda) (\tilde{\nabla}^i V^s \circ \lambda) = \sum_{s=1}^n L_s g^{si} = L^i. \quad (6.1)$$

Here we used formula (5.5) for W^i and notations (2.7). Formula $p_s \circ \lambda = L_s$ is quite obvious (see formulas (1.6), where generalized Legendre map λ is defined

by functions L_1, \dots, L_n in local chart). By L^i in (6.1) we denote components of extended vector field \mathbf{L} dual to extended covector field \mathbf{L} with respect to non-symmetric extended metric tensor (2.7).

Now let's transform extended scalar field Ω that forms denominator of fraction in formula (5.6). Applying λ to formula (5.7), we derive

$$\Omega \circ \lambda = \sum_{s=1}^n (p_s \circ \lambda) (W^s \circ \lambda) = \sum_{s=1}^n L_s L^s = |\mathbf{L}|^2. \quad (6.2)$$

For the last sum in (6.2) by similarity we used the same notation $|\mathbf{L}|^2$ as in Riemannian geometry, though the length of vector \mathbf{L} here is understood in the sense of extended metric (2.7). Combining (6.1) and (6.2), for components of \mathbf{P} we get

$$P_j^i \circ \lambda = \delta_j^i - \frac{L^i L_j}{|\mathbf{L}|^2}. \quad (6.3)$$

Note that formula (6.3) for P_j^i looks like formula for components of orthogonal projector in Riemannian geometry. Writing $\Omega \circ \lambda$ and $P_j^i \circ \lambda$ in (6.2) and (6.3) means that we take \mathbf{v} -representation of extended tensor fields Ω and \mathbf{P} , which are initially in \mathbf{p} -representation. But having transformed them to \mathbf{v} -representation, we can then omit λ symbol, i. e. we can write

$$\Omega = \sum_{s=1}^n L_s L^s = |\mathbf{L}|^2, \quad P_j^i = \delta_j^i - \frac{L^i L_j}{|\mathbf{L}|^2}, \quad (6.4)$$

assuming both sides of the equalities (6.4) to be in \mathbf{v} -representation. Below we shall use the same convention for other extended tensor fields \mathbf{U} , $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\eta}$, \mathbf{A} , \mathbf{B} , and \mathbf{C} .

7. WEAK NORMALITY EQUATIONS IN \mathbf{v} -REPRESENTATION.

Now let's continue transforming tensor fields used in writing weak normality equations (5.14). For $\nabla_i V^s$ in (5.8), applying formula (3.6), we derive

$$\nabla_i v^s = \nabla_i V^s \circ \lambda + \sum_{q=1}^n \nabla_i L_q (\tilde{\nabla}^q V^s \circ \lambda) = \nabla_i V^s \circ \lambda + \sum_{q=1}^n \nabla_i L_q g^{sq}. \quad (7.1)$$

Left hand side of (7.1) is equal to zero. Therefore we have the equality

$$\nabla_i V^s \circ \lambda = - \sum_{q=1}^n \nabla_i L_q g^{sq}. \quad (7.2)$$

Now we can substitute (7.2) into (5.8). Taking into account (6.1), then we get

$$U_i \circ \lambda = Q_i \circ \lambda - \sum_{q=1}^n L^q \nabla_i L_q. \quad (7.3)$$

Components of covector field \mathbf{Q} in \mathbf{p} -representation define Newtonian dynamical system (4.6). However, its components in \mathbf{v} -representation have no direct relation

to (4.5). This means that $Q_i \circ \lambda$ in (7.3) should be expressed through F^i . Combining (1.10) with formulas (4.3) and (4.4), we get the following expression for $Q_i \circ \lambda$:

$$Q_i \circ \lambda = \sum_{q=1}^n v^q \nabla_q L_i + \sum_{q=1}^n F^q \tilde{\nabla}_q L_i = \sum_{q=1}^n v^q \nabla_q L_i + \sum_{q=1}^n F^q g_{iq}. \quad (7.4)$$

By introducing extended covector field \mathbf{F} dual to extended vector field \mathbf{F} with respect to metric (2.7) we can simplify (7.4) a little bit more

$$Q_i \circ \lambda = \sum_{q=1}^n v^q \nabla_q L_i + F_i. \quad (7.5)$$

For L_i , L^q , F_i , and F^q we have the following relationships:

$$L^i = \sum_{q=1}^n L_q g^{qi}, \quad L_q = \sum_{i=1}^n L^i g_{iq}, \quad (7.6)$$

$$F^i = \sum_{q=1}^n g^{iq} F_q, \quad F_q = \sum_{i=1}^n g_{qi} F^i. \quad (7.7)$$

Comparing (7.6) and (7.7), we see that due to asymmetry of metric (2.7) they are slightly different. For the sake of certainty we say that L^i and L_q are right-dual to each other, while F^i and F_q are left-dual. Substituting (7.5) into (7.3), we get

$$U_i = \sum_{q=1}^n v^q \nabla_q L_i - \sum_{q=1}^n L^q \nabla_i L_q + F_i. \quad (7.8)$$

According to our convention, we omitted symbol λ in left hand side of (7.8) since this is ultimate expression for components of covector field \mathbf{U} in \mathbf{v} -representation.

Next step is to transform extended vector field $\boldsymbol{\alpha}$ with components (5.9). By means of formula (3.5) for covariant derivative $\nabla_r W^k$ we derive:

$$\nabla_r W^k \circ \lambda = \nabla_r L^k + \sum_{q=1}^n (\nabla_r V^q \circ \lambda) \tilde{\nabla}_q L^k. \quad (7.9)$$

Now, if we apply formula (7.2) to $\nabla_r V^q \circ \lambda$ in formula (7.9), we obtain

$$\nabla_r W^k \circ \lambda = \nabla_r L^k - \sum_{q=1}^n \sum_{s=1}^n \nabla_r L_s g^{qs} \tilde{\nabla}_q L^k. \quad (7.10)$$

Using formula (7.10), we can transform second term in right hand side of (5.9):

$$\sum_{r=1}^n (\nabla_r W^k V^r) \circ \lambda = \sum_{r=1}^n v^r \nabla_r L^k - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n v^r \nabla_r L_s g^{qs} \tilde{\nabla}_q L^k. \quad (7.11)$$

Using formulas (2.6) and (7.5), for third term in right hand side of (5.9) we derive

$$\begin{aligned} \sum_{r=1}^n (\tilde{\nabla}^r W^k Q_r) \circ \lambda &= \sum_{q=1}^n \sum_{r=1}^n g^{qr} \tilde{\nabla}_q L^k (Q_r \circ \lambda) = \\ &= \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n g^{qr} \tilde{\nabla}_q L^k v^s \nabla_s L_r + \sum_{q=1}^n F^q \tilde{\nabla}_q L^k. \end{aligned} \quad (7.12)$$

Transforming first term in right hand side of (5.9) needs no special efforts. Indeed, for $\tilde{\nabla}^k V^r$ we use formula (2.7). As a result we get

$$\sum_{r=1}^n (\tilde{\nabla}^k V^r U_r) \circ \lambda = \sum_{r=1}^n g^{rk} U_r. \quad (7.13)$$

Components of covector field \mathbf{U} in right hand side of (7.13) are already transformed to \mathbf{v} -representation, see formula (7.8).

In fourth term in right hand side of (5.9) we see $\tilde{\nabla}^k Q_r$. Applying formulas (2.6) and (7.5), for this derivative we obtain the following expression:

$$\begin{aligned} \tilde{\nabla}^k Q_r \circ \lambda &= \sum_{q=1}^n g^{qk} \tilde{\nabla}_q (Q_r \circ \lambda) = \sum_{q=1}^n g^{qk} \nabla_q L_r + \\ &+ \sum_{q=1}^n \sum_{s=1}^n g^{qk} v^s \tilde{\nabla}_q \nabla_s L_r + \sum_{q=1}^n \sum_{s=1}^n g^{qk} \tilde{\nabla}_q F_r. \end{aligned} \quad (7.14)$$

Using formula (7.14), for the fourth term in right hand side of (5.9) we derive

$$\begin{aligned} \sum_{r=1}^n (W^r \tilde{\nabla}^k Q_r) \circ \lambda &= \sum_{r=1}^n \sum_{q=1}^n L^r g^{qk} \nabla_q L_r + \\ &+ \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n L^r g^{qk} v^s \tilde{\nabla}_q \nabla_s L_r + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n L^r g^{qk} \tilde{\nabla}_q F_r. \end{aligned} \quad (7.15)$$

Last term in right hand side of (5.9) contains components of dynamic curvature tensor \mathbf{D} . In \mathbf{p} -representation they are given by formula (5.11). In \mathbf{v} -representation we have analogous formula for components of dynamic curvature tensor:

$$D_{rij}^k = -\frac{\partial \Gamma_{ir}^k}{\partial v^j}. \quad (7.16)$$

Note that (5.11) and (7.16) define two different extended tensor fields (not two representations of the same field). But there is very simple relationship binding components of these two fields in \mathbf{p} and \mathbf{v} -representations:

$$D_{ij}^{kr} \circ \lambda = \sum_{s=1}^n g^{sr} D_{ijs}^k. \quad (7.17)$$

On the base of (7.17) for last term in right hand side of (5.9) we derive:

$$\sum_{r=1}^n \sum_{s=1}^n \sum_{q=1}^n (p_s D_{rq}^{sk} W^r V^q) \circ \lambda = \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n g^{mk} L_s D_{rqm}^s L^r v^q. \quad (7.18)$$

Now we can summarize above calculations (7.11), (7.12), (7.13), (7.15), (7.18) and write formula for components of extended covector field α in \mathbf{v} -representation:

$$\begin{aligned} \alpha^k &= \sum_{r=1}^n g^{rk} U_r + \sum_{r=1}^n v^r \nabla_r L^k + \sum_{q=1}^n F^q \tilde{\nabla}_q L^k + \\ &+ \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n L^r g^{qk} v^s \tilde{\nabla}_q \nabla_s L_r + \sum_{q=1}^n \sum_{r=1}^n L^r g^{qk} \tilde{\nabla}_q F_r + \\ &+ \sum_{r=1}^n \sum_{q=1}^n L^r g^{qk} \nabla_q L_r - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n g^{mk} L_s D_{rqm}^s L^r v^q. \end{aligned} \quad (7.19)$$

Second normality equation in (5.14) is written in terms of components of covector field η . In order to use formula (5.13) for η_k we should first transform covector field β with components (5.10) to \mathbf{v} -representation. Using (3.5) and (7.2), we get

$$\nabla_r U_k \circ \lambda = \nabla_r U_k - \sum_{q=1}^n \sum_{s=1}^n \nabla_r L_q g^{sq} \tilde{\nabla}_s U_k. \quad (7.20)$$

For U_k in right hand side of (7.20) we should use (7.8) since covector field \mathbf{U} is already transformed to \mathbf{v} -representation. Substituting (7.20) into (5.10), for the first term in right hand side of (5.10) we derive

$$\sum_{r=1}^n (\nabla_r U_k V^r) \circ \lambda = \sum_{r=1}^n \nabla_r U_k v^r - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \nabla_r L_q g^{sq} \tilde{\nabla}_s U_k v^r. \quad (7.21)$$

In order to transform $\tilde{\nabla}^r U_k$ to \mathbf{v} -representation we apply formula (2.6). This yields

$$\tilde{\nabla}^r U_k \circ \lambda = \sum_{q=1}^n g^{qr} \tilde{\nabla}_q U_k. \quad (7.22)$$

Applying (7.22) and (7.5), for the second term in right hand side of (5.10) we get

$$\begin{aligned} \sum_{r=1}^n (\tilde{\nabla}^r U_k Q_r) \circ \lambda &= \sum_{q=1}^n \sum_{r=1}^n g^{qr} \tilde{\nabla}_q U_k F_r + \\ &+ \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n g^{qr} \tilde{\nabla}_q U_k v^s \nabla_s L_r. \end{aligned} \quad (7.23)$$

In order to transform third term in right hand side of (5.10) we use (7.2) and (7.8):

$$\begin{aligned} \sum_{r=1}^n \nabla_k V^r U_r &= - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \nabla_k L_q g^{rq} v^s \nabla_s L_r + \\ &+ \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \nabla_k L_q g^{rq} L^s \nabla_r L_s - \sum_{q=1}^n \sum_{r=1}^n \nabla_k L_q g^{rq} F_r. \end{aligned} \quad (7.24)$$

Further we need to transform $\nabla_k Q_r$. Using (3.5), (7.2), and (7.5), we get

$$\begin{aligned} \nabla_k Q_r \circ \lambda &= \nabla_k F_r + \sum_{m=1}^n v^m \nabla_k \nabla_m L_r - \\ &- \sum_{q=1}^n \sum_{s=1}^n \nabla_k L_q g^{sq} \tilde{\nabla}_s F_r - \sum_{q=1}^n \sum_{s=1}^n \nabla_k L_q g^{sq} \nabla_s L_r - \\ &- \sum_{q=1}^n \sum_{m=1}^n \sum_{s=1}^n \nabla_k L_q g^{sq} v^m \tilde{\nabla}_s \nabla_m L_r. \end{aligned} \quad (7.25)$$

Now let's substitute (7.25) into fourth term in right hand side of (5.10). This yields

$$\begin{aligned} \sum_{r=1}^n (\nabla_k Q_r W^r) \circ \lambda &= \sum_{r=1}^n L^r \nabla_k F_r + \sum_{m=1}^n \sum_{r=1}^n L^r v^m \nabla_k \nabla_m L_r - \\ &- \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n L^r \nabla_k L_q g^{sq} \tilde{\nabla}_s F_r - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n L^r \nabla_k L_q g^{sq} \nabla_s L_r - \\ &- \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n L^r \nabla_k L_q g^{sq} v^m \tilde{\nabla}_s \nabla_m L_r. \end{aligned} \quad (7.26)$$

Next steps are related to curvature tensors in formula (5.10). Curvature tensor \mathbf{R} in \mathbf{v} -representation is given by the following formula (see Chapter III of thesis [16]):

$$\begin{aligned} R_{rij}^k &= \frac{\partial \Gamma_{jr}^k}{\partial x^i} - \frac{\partial \Gamma_{ir}^k}{\partial x^j} + \sum_{m=1}^n \Gamma_{im}^k \Gamma_{jr}^m - \sum_{m=1}^n \Gamma_{jm}^k \Gamma_{ir}^m - \\ &- \sum_{m=1}^n \sum_{s=1}^n v^s \Gamma_{is}^m \frac{\partial \Gamma_{jr}^k}{\partial v^m} + \sum_{m=1}^n \sum_{s=1}^n v^s \Gamma_{js}^m \frac{\partial \Gamma_{ir}^k}{\partial v^m}. \end{aligned} \quad (7.27)$$

Like (5.11) and (7.16), formulas (5.12) and (7.26) express two different extended tensor field in different representations. However, the relation of these two tensor fields can be described by the following rather simple formula:

$$R_{rij}^k \circ \lambda = R_{rij}^k + \sum_{q=1}^n \sum_{s=1}^n \nabla_i L_q g^{sq} D_{jrs}^k - \sum_{q=1}^n \sum_{s=1}^n \nabla_j L_q g^{sq} D_{irs}^k. \quad (7.28)$$

Components of curvature tensor \mathbf{R} in right hand side of (7.28) are calculated according to the formula (7.27). Using (7.17) and (7.28), we can transform last two terms in right hand side of (5.10). As a result we get

$$\begin{aligned}
& - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n (R_{rmk}^s V^m W^r p_s) \circ \lambda = - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n R_{rmk}^s v^m L^r L_s - \\
& \quad - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n \nabla_m L_q g^{aq} D_{kra}^s v^m L^r L_s + \\
& \quad + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n \nabla_k L_q g^{aq} D_{mra}^s v^m L^r L_s,
\end{aligned} \tag{7.29}$$

$$\begin{aligned}
& \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n (D_{rk}^{sm} Q_m W^r p_s) \circ \lambda = \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n g^{am} D_{rka}^s \times \\
& \quad \times F_m L^r L_s + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n g^{am} D_{rka}^s v^q \nabla_q L_m L^r L_s.
\end{aligned} \tag{7.30}$$

Now we can summarize above calculations and write formula for components of covector field β . Combining (7.21), (7.23), (7.24), (7.26), (7.29), and (7.30), we get

$$\begin{aligned}
\beta_k &= \sum_{r=1}^n v^r \nabla_r U_k + \sum_{q=1}^n F^q \tilde{\nabla}_q U_k - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \nabla_k L_q g^{rq} v^s \nabla_s L_r - \\
& \quad - \sum_{q=1}^n \sum_{r=1}^n \nabla_k L_q g^{rq} F_r + \sum_{r=1}^n L^r \nabla_k F_r + \sum_{m=1}^n \sum_{r=1}^n L^r v^m \nabla_k \nabla_m L_r - \\
& \quad - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n L^r \nabla_k L_q g^{sq} \tilde{\nabla}_s F_r - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n L^r \nabla_k L_q g^{sq} v^m \times \\
& \quad \times \tilde{\nabla}_s \nabla_m L_r - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n R_{rmk}^s v^m L^r L_s + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n g^{aq} \times \\
& \quad \times \nabla_k L_q D_{mra}^s v^m L^r L_s + \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n g^{am} D_{rka}^s F_m L^r L_s.
\end{aligned} \tag{7.31}$$

Components of covector field \mathbf{L} in (7.31) are determined by generalized Legendre transformation (1.6), components of its dual vector field \mathbf{L} are given first formula (7.6), and components of covector field \mathbf{U} in (7.31) are determined by formula (7.8). Also we should keep in mind formulas (7.7) binding vectorial and covectorial form of force field \mathbf{F} and formulas (7.16) and (7.27) for curvature tensors.

Since α and β are already transformed to \mathbf{v} -representation, formula for components of covector field η is very simple. It is derived from (5.13):

$$\eta_k = \beta_k - \sum_{s=1}^n \frac{U_k \alpha^s L_s}{|\mathbf{L}|^2}. \tag{7.32}$$

Here $|\mathbf{L}|^2$ is \mathbf{v} -representation of scalar field Ω (see formula (6.2)). As for weak normality equations, in \mathbf{v} -representation we can use their initial form (5.14) since all extended tensor fields forming these equations are already transformed to \mathbf{v} -representation (see (6.4), (7.19), and (7.32)).

8. ADDITIONAL NORMALITY EQUATIONS IN \mathbf{v} -REPRESENTATION.

Transformation of additional normality equations to \mathbf{v} -representation consists in transforming extended tensor fields (5.15), (5.16), and (5.17). Components of projector field \mathbf{P} , which enter to (5.18), (5.19), (5.20), and (5.22), are already transformed to \mathbf{v} -representation (see formulas (6.3) or (6.4)). For A^{rs} we have

$$A^{rs} = \sum_{q=1}^n g^{qr} \tilde{\nabla}_q L^s. \quad (8.1)$$

In deriving (8.1) we used (2.6) and (6.1). Components of tensor field \mathbf{B} are given by a little bit more complicated formula. For them we have

$$\begin{aligned} B_s^r &= \sum_{q=1}^n g^{qr} \tilde{\nabla}_q U_s + \sum_{m=1}^n \sum_{k=1}^n \sum_{q=1}^n g^{qr} L^k L_m D_{ksq}^m - \nabla_s L^r + \\ &+ \sum_{q=1}^n \sum_{k=1}^n \nabla_s L_k g^{qk} \tilde{\nabla}_q L^r + \sum_{q=1}^n \sum_{m=1}^n \frac{g^{qm} \tilde{\nabla}_q L^r - g^{qr} \tilde{\nabla}_q L^m}{|\mathbf{L}|^2} U_s L_m. \end{aligned} \quad (8.2)$$

In deriving (8.2) we used formulas (7.22), (7.17), (7.10), and (2.6). In the last step we should transform formula (5.17) for components of extended tensor field \mathbf{C} to \mathbf{v} -representation. Using (7.20), (7.22), (7.10), (7.17), and (7.28), we get

$$\begin{aligned} C_{rs} &= \nabla_r U_s - \sum_{q=1}^n \sum_{k=1}^n \nabla_r L_q g^{kq} \tilde{\nabla}_k U_s - \sum_{m=1}^n \frac{U_s \nabla_r L^m}{|\mathbf{L}|^2} L_m - \\ &- \sum_{q=1}^n \sum_{m=1}^n \frac{U_r g^{qm} \tilde{\nabla}_q U_s}{|\mathbf{L}|^2} L_m + \sum_{m=1}^n \sum_{q=1}^n \sum_{k=1}^n \frac{U_s \nabla_r L_k g^{qk} \tilde{\nabla}_q L^m}{|\mathbf{L}|^2} L_m - \\ &- \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \frac{g^{aq} D_{ksa}^m U_r}{|\mathbf{L}|^2} L_m L^k L_q - \sum_{k=1}^n \sum_{q=1}^n \frac{R_{krs}^q}{2} L^k L_q - \\ &- \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \nabla_r L_m D_{ksa}^q g^{am} L^k L_q. \end{aligned} \quad (8.3)$$

Substituting (8.1), (8.2), (8.3) into (5.18), (5.19), (5.20), we get additional normality equations transformed to \mathbf{v} -representation. Formula (5.21) for scalar parameter λ remains unchanged in \mathbf{v} -representation as well.

9. CONNECTION INVARIANCE.

Let's recall that (4.5) is a tensorial form of differential equations (1.1). In order to write (4.5) we need to choose some extended connection Γ in M . However, strong normality condition in definition 5.2, from which weak and additional normality equation were derived in [26], is irrelative to the choice of Γ when applied

to Newtonian dynamical system (1.1). Therefore we should prove that complete system of normality equations is invariant under the change of connection components, provided functions Φ^i in differential equations (1.1) are unchanged. This means that we should consider gauge transformations

$$\begin{aligned} \Gamma_{ij}^k &\rightarrow \Gamma_{ij}^k + T_{ij}^k, \\ F^i &\rightarrow F^i + \sum_{k=1}^n \sum_{j=1}^n T_{kj}^i v^k v^j. \end{aligned} \quad (9.1)$$

Here T_{ij}^k are components of symmetric extended tensor field \mathbf{T} . Tensor fields \mathbf{L} , \mathbf{g} , $\Omega = |\mathbf{L}|^2$, and \mathbf{P} are obviously invariant under gauge transformations (9.1):

$$\begin{aligned} L_i &\rightarrow L_i, & g_{ij} &\rightarrow g_{ij}, \\ g^{ij} &\rightarrow g^{ij}, & L^i &\rightarrow L^i, \\ |\mathbf{L}|^2 &\rightarrow |\mathbf{L}|^2, & P_k^i &\rightarrow P_k^i. \end{aligned} \quad (9.2)$$

Applying (9.2) to (8.1), we find that extended tensor field \mathbf{A} with components (8.1) is invariant under gauge transformations (9.1):

$$A^{rs} \rightarrow A^{rs}. \quad (9.3)$$

Due to (9.3) and (9.2) additional normality equation (5.18) is invariant under gauge transformations (9.1).

Applying formulas (9.1) and (9.2) to (7.7), for covectorial form of force field of dynamical system (4.5) we derive the following transformation rule:

$$F_i \rightarrow F_i + \sum_{s=1}^n \sum_{k=1}^n \sum_{j=1}^n g_{is} T_{kj}^s v^k v^j. \quad (9.4)$$

Connection components are used in formula (3.4). Therefore we have

$$\begin{aligned} \nabla_s L_k &\rightarrow \nabla_s L_k - \sum_{a=1}^n T_{sk}^a L_a - \sum_{a=1}^n \sum_{b=1}^n g_{kb} T_{as}^b v^a, \\ \nabla_s L^k &\rightarrow \nabla_s L^k + \sum_{a=1}^n T_{sa}^k L^a - \sum_{a=1}^n \sum_{b=1}^n v^a T_{as}^b \tilde{\nabla}_b L^k. \end{aligned} \quad (9.5)$$

From (9.4) and (9.5) for covector field \mathbf{U} we derive transformation rule

$$U_i \rightarrow U_i + \sum_{q=1}^n \sum_{r=1}^n L^q T_{iq}^r L_r. \quad (9.6)$$

For dynamic curvature tensor \mathbf{D} by differentiating (9.1) we get

$$D_{rij}^k \rightarrow D_{rij}^k - \tilde{\nabla}_j T_{ir}^k. \quad (9.7)$$

Curvature tensor \mathbf{R} is determined by connection components according to formula (7.28). Here we have the following transformation rule:

$$\begin{aligned}
R_{rij}^k &\rightarrow R_{rij}^k + \nabla_i T_{jr}^k - \nabla_j T_{ir}^k - \sum_{s=1}^n \sum_{m=1}^n v^m T_{jm}^s D_{irs}^k + \\
&+ \sum_{s=1}^n \sum_{m=1}^n v^m T_{im}^s D_{jrs}^k + \sum_{m=1}^n (T_{im}^k T_{jr}^m - T_{jm}^k T_{ir}^m) + \\
&+ \sum_{s=1}^n \sum_{m=1}^n v^m T_{jm}^s \tilde{\nabla}_s T_{ir}^k - \sum_{s=1}^n \sum_{m=1}^n v^m T_{im}^s \tilde{\nabla}_s T_{jr}^k.
\end{aligned} \tag{9.8}$$

Further let's derive transformation rules for covariant derivatives of covector field \mathbf{U} . For covariant derivative $\tilde{\nabla}_q U_s$, applying formula (9.6), we obtain

$$\begin{aligned}
\tilde{\nabla}_q U_s &\rightarrow \tilde{\nabla}_q U_s + \sum_{k=1}^n \sum_{r=1}^n \tilde{\nabla}_q L^k T_{sk}^r L_r + \\
&+ \sum_{k=1}^n \sum_{r=1}^n L^k \tilde{\nabla}_q T_{sk}^r L_r + \sum_{k=1}^n \sum_{r=1}^n L^k T_{sk}^r \tilde{\nabla}_q L_r.
\end{aligned} \tag{9.9}$$

Transformation rule for covariant derivative $\nabla_q U_s$ is more complicated:

$$\begin{aligned}
\nabla_q U_s &\rightarrow \nabla_q U_s + \sum_{k=1}^n \sum_{r=1}^n \nabla_q L^k T_{sk}^r L_r + \sum_{k=1}^n \sum_{r=1}^n L^k \nabla_q T_{sk}^r L_r + \\
&+ \sum_{k=1}^n \sum_{r=1}^n L^k T_{sk}^r \nabla_q L_r - \sum_{m=1}^n T_{qs}^m U_m - \sum_{m=1}^n \sum_{k=1}^n \sum_{r=1}^n T_{qs}^m L^k T_{mk}^r L_r - \\
&- \sum_{m=1}^n \sum_{a=1}^n v^a T_{aq}^m \tilde{\nabla}_m U_s - \sum_{m=1}^n \sum_{a=1}^n \sum_{k=1}^n \sum_{r=1}^n v^a T_{aq}^m \tilde{\nabla}_m L^k T_{sk}^r L_r - \\
&- \sum_{m=1}^n \sum_{a=1}^n v^a T_{aq}^m \left(\sum_{k=1}^n \sum_{r=1}^n L^k \tilde{\nabla}_m T_{sk}^r L_r + \sum_{k=1}^n \sum_{r=1}^n L^k T_{sk}^r \tilde{\nabla}_m L_r \right).
\end{aligned} \tag{9.10}$$

Now we can combine (9.9), (9.7), (9.6), and (9.5) according to formula (8.2):

$$\begin{aligned}
B_s^r &\rightarrow B_s^r + \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n g^{qr} \tilde{\nabla}_q L^k T_{sk}^m L_m + \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n g^{qr} L^k \tilde{\nabla}_q T_{sk}^m L_m + \\
&+ \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n g^{qr} L^k T_{sk}^m \tilde{\nabla}_q L_m - \sum_{m=1}^n \sum_{k=1}^n \sum_{q=1}^n g^{qr} L^k L_m \tilde{\nabla}_q T_{sk}^m - \sum_{m=1}^n T_{sm}^r L^m + \\
&+ \sum_{a=1}^n \sum_{b=1}^n v^a T_{as}^b \tilde{\nabla}_b L^r - \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n T_{sk}^m L_m g^{qk} \tilde{\nabla}_q L^r - \sum_{q=1}^n \sum_{k=1}^n \sum_{a=1}^n \sum_{b=1}^n g_{kb} \times \\
&\times T_{as}^b v^a g^{qk} \tilde{\nabla}_q L^r + \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n \frac{A^{mr} - A^{rm}}{|\mathbf{L}|^2} L_m L^q T_{sq}^k L_k.
\end{aligned}$$

Using (2.7) for $\tilde{\nabla}_q L_m$ and canceling similar terms in the above formula, we get

$$B_s^r \rightarrow B_s^r + \sum_{m=1}^n \sum_{q=1}^n \sum_{k=1}^n L_m T_{sq}^m P_k^q (A^{rk} - A^{kr}). \quad (9.11)$$

Here we used formula (6.4) for P_k^q . Substituting (9.11) into normality equation (5.19) and taking into account another normality equation (5.18), we find that (5.19) is invariant under gauge transformations (9.1).

Now let's apply transformation rules (9.10), (9.9), (9.8), (9.7), (9.6), and (9.5) to formula (8.3) for components of covector field \mathbf{C} in \mathbf{v} -representation:

$$\begin{aligned} C_{rs} \rightarrow & C_{rs} + \dots + \sum_{k=1}^n \sum_{q=1}^n \nabla_r L^k T_{sk}^q L_q + \sum_{k=1}^n \sum_{q=1}^n L^k \nabla_r T_{sk}^q L_q + \sum_{k=1}^n \sum_{q=1}^n L^k \times \\ & \times T_{sk}^q \nabla_r L_q - \sum_{m=1}^n \sum_{a=1}^n v^a T_{ar}^m \tilde{\nabla}_m U_s - \sum_{m=1}^n \sum_{a=1}^n v^a T_{ar}^m \sum_{k=1}^n \sum_{q=1}^n \left(\tilde{\nabla}_m L^k T_{sk}^q L_q + \right. \\ & \left. + L^k \tilde{\nabla}_m T_{sk}^q L_q + L^k T_{sk}^q \tilde{\nabla}_m L_q \right) - \sum_{k=1}^n \sum_{q=1}^n \sum_{b=1}^n \sum_{a=1}^n \nabla_r L_q g^{kq} \tilde{\nabla}_k L^b T_{sb}^a L_a - \\ & - \sum_{k=1}^n \sum_{q=1}^n \sum_{b=1}^n \sum_{a=1}^n \nabla_r L_q g^{kq} L^b \tilde{\nabla}_k T_{sb}^a L_a - \sum_{k=1}^n \sum_{q=1}^n \sum_{b=1}^n \sum_{a=1}^n \nabla_r L_q g^{kq} L^b T_{sb}^a \times \\ & \times \tilde{\nabla}_k L_a + \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n T_{rq}^m L_m g^{kq} \tilde{\nabla}_k U_s + \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{b=1}^n \sum_{a=1}^n T_{rq}^m L_m g^{kq} \times \\ & \times \tilde{\nabla}_k L^b T_{sb}^a L_a + \sum_{b=1}^n \sum_{a=1}^n \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n T_{rq}^m L_m g^{kq} L^b \tilde{\nabla}_k T_{sb}^a L_a + \sum_{q=1}^n \sum_{m=1}^n T_{rq}^m \times \\ & \times L_m \sum_{b=1}^n \sum_{a=1}^n \sum_{k=1}^n g^{kq} L^b T_{sb}^a \tilde{\nabla}_k L_a + \sum_{k=1}^n \sum_{m=1}^n T_{mr}^k v^m \tilde{\nabla}_k U_s + \sum_{k=1}^n \sum_{m=1}^n T_{mr}^k \times \\ & \times \sum_{b=1}^n \sum_{a=1}^n v^m \tilde{\nabla}_k L^b T_{sb}^a L_a + \sum_{k=1}^n \sum_{b=1}^n \sum_{a=1}^n \sum_{m=1}^n T_{mr}^k v^m L^b \tilde{\nabla}_k T_{sb}^a L_a + \sum_{m=1}^n v^m \times \\ & \times \sum_{k=1}^n \sum_{b=1}^n \sum_{a=1}^n T_{mr}^k L^b T_{sb}^a \tilde{\nabla}_k L_a + \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n T_{rm}^b L_b g^{am} D_{ksa}^q L^k L_q + \\ & + \sum_{k=1}^n \sum_{q=1}^n \sum_{a=1}^n \sum_{b=1}^n T_{br}^a v^b D_{ksa}^q L^k L_q + \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \nabla_r L_m g^{am} \tilde{\nabla}_a T_{sk}^q L^k L_q - \\ & - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n T_{rm}^b L_b g^{am} \tilde{\nabla}_a T_{sk}^q L^k L_q - \sum_{k=1}^n \sum_{q=1}^n \sum_{a=1}^n \sum_{b=1}^n T_{br}^a v^b \tilde{\nabla}_a T_{sk}^q \times \\ & \times L^k L_q - \sum_{k=1}^n \sum_{q=1}^n \nabla_r T_{sk}^q L^k L_q - \sum_{k=1}^n \sum_{q=1}^n \sum_{a=1}^n \sum_{m=1}^n v^m T_{rm}^a D_{ska}^q L^k L_q - \sum_{k=1}^n L^k \times \\ & \times \sum_{q=1}^n \sum_{m=1}^n T_{rm}^q T_{sk}^m L_q + \sum_{k=1}^n \sum_{q=1}^n \sum_{a=1}^n \sum_{m=1}^n v^m T_{rm}^a \tilde{\nabla}_a T_{sk}^q L^k L_q + \frac{1}{|\mathbf{L}|^2} Z_{rs}. \end{aligned}$$

We used Z_{rs} in order to denote contribution of those terms in (8.3) which have denominator $|\mathbf{L}|^2$. By dots we denote terms symmetric in indices r and s . They do not affect the ultimate form of the equations (5.20). When collecting similar terms in the above formula some terms cancel each other. As a result we have

$$\begin{aligned}
C_{rs} &\rightarrow C_{rs} + \dots + \sum_{k=1}^n \sum_{q=1}^n \nabla_r L^k T_{sk}^q L_q - \sum_{k=1}^n \sum_{q=1}^n \sum_{b=1}^n \sum_{a=1}^n \nabla_r L_q \times \\
&\times g^{kq} \tilde{\nabla}_k L^b T_{sb}^a L_a + \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n T_{rq}^m L_m g^{kq} \tilde{\nabla}_k U_s + \sum_{k=1}^n \sum_{q=1}^n g^{kq} \times \\
&\times \sum_{m=1}^n \sum_{b=1}^n \sum_{a=1}^n T_{rq}^m L_m \tilde{\nabla}_k L^b T_{sb}^a L_a + \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n T_{rm}^b \times \\
&\times L_b g^{am} D_{ksa}^q L^k L_q + \frac{1}{|\mathbf{L}|^2} Z_{rs}.
\end{aligned} \tag{9.12}$$

In deriving (9.12) we used formula $g_{qk} = \tilde{\nabla}_k L_q$ (see (2.7) above). For Z_{rs} in (9.12) we have rather huge formula derived from formula (8.3) for C_{rs} :

$$\begin{aligned}
Z_{rs} &= - \sum_{m=1}^n \sum_{a=1}^n T_{ra}^m L^a L_m U_s + \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n v^a T_{ar}^b \tilde{\nabla}_b L^m L_m U_s - \\
&- \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n \nabla_r L^m L_m L^q T_{sq}^k L_k - \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n \sum_{a=1}^n T_{ra}^m L^a L_m L^q T_{sq}^k L_k + \\
&+ \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n v^a T_{ar}^b \tilde{\nabla}_b L^m L_m L^q T_{sq}^k L_k - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{c=1}^n U_r g^{qm} \times \\
&\times \tilde{\nabla}_q L^k T_{sk}^c L_c L_m - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{c=1}^n U_r g^{qm} L^k \tilde{\nabla}_q T_{sk}^c L_c L_m - \sum_{k=1}^n L^k \times \\
&\times \sum_{q=1}^n \sum_{m=1}^n \sum_{c=1}^n U_r g^{qm} T_{sk}^c \tilde{\nabla}_q L_c L_m - \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n L^a T_{ra}^b L_b g^{qm} \tilde{\nabla}_q U_s \times \\
&\times L_m - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n L^a T_{ra}^b L_b g^{qm} \tilde{\nabla}_q L^k T_{sk}^c L_c L_m - \sum_{k=1}^n L^k \times \\
&\times \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n L^a T_{ra}^b L_b g^{qm} \tilde{\nabla}_q T_{sk}^c L_c L_m - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n L^a \times \\
&\times \sum_{b=1}^n \sum_{c=1}^n T_{ra}^b L_b g^{qm} L^k T_{sk}^c \tilde{\nabla}_q L_c L_m - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{c=1}^n U_s T_{rk}^c L_c g^{qk} \times \\
&\times \tilde{\nabla}_q L^m L_m - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{c=1}^n \sum_{e=1}^n U_s g_{ke} T_{cr}^e v^c g^{qk} \tilde{\nabla}_q L^m L_m + \sum_{a=1}^n L^a \times \\
&\times \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{b=1}^n T_{sa}^b L_b \nabla_r L_k g^{qk} \tilde{\nabla}_q L^m L_m - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n L^a \times
\end{aligned}$$

$$\begin{aligned}
& \times T_{sa}^b L_b T_{rk}^c L_c g^{qk} \tilde{\nabla}_q L^m L_m - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{e=1}^n L^a T_{sa}^b L_b g_{ke} \times \\
& \times T_{cr}^e v^c g^{qk} \tilde{\nabla}_q L^m L_m + \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n g^{aq} \tilde{\nabla}_a T_{sk}^m U_r L_m L^k L_q - \sum_{k=1}^n L^k \times \\
& \times \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n g^{aq} D_{ksa}^m L^b T_{rb}^c L_c L_m L_q + \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n L^k \times \\
& \times \sum_{c=1}^n g^{aq} \tilde{\nabla}_a T_{sk}^m L^b T_{rb}^c L_c L_m L_q.
\end{aligned}$$

Upon collecting and canceling similar terms for Z_{rs} we get shorter expression

$$\begin{aligned}
Z_{rs} = & \dots - \sum_{q=1}^n \sum_{k=1}^n \sum_{m=1}^n \nabla_r L^m L_m L^q T_{sq}^k L_k - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n U_r \times \\
& \times \sum_{c=1}^n g^{qm} \tilde{\nabla}_q L^k T_{sk}^c L_c L_m - \sum_{q=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n L^a T_{ra}^b L_b g^{qm} \times \\
& \times \tilde{\nabla}_q U_s L_m - \sum_{a=1}^n \sum_{b=1}^n \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{c=1}^n T_{ra}^b L^a L_b g^{qm} \tilde{\nabla}_q L^k T_{sk}^c \times \\
& \times L_c L_m - \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{c=1}^n U_s T_{rk}^c L_c g^{qk} \tilde{\nabla}_q L^m L_m + \sum_{a=1}^n L^a \times \tag{9.13} \\
& \times \sum_{k=1}^n \sum_{q=1}^n \sum_{m=1}^n \sum_{b=1}^n T_{sa}^b L_b \nabla_r L_k g^{qk} \tilde{\nabla}_q L^m L_m - \sum_{a=1}^n \sum_{k=1}^n \sum_{q=1}^n L^a \times \\
& \times \sum_{m=1}^n \sum_{b=1}^n \sum_{c=1}^n T_{sa}^b L_b T_{rk}^c L_c g^{qk} \tilde{\nabla}_q L^m L_m - \sum_{a=1}^n \sum_{k=1}^n \sum_{q=1}^n g^{aq} \times \\
& \times \sum_{m=1}^n \sum_{b=1}^n \sum_{c=1}^n D_{ksa}^m L^b T_{rb}^c L_c L_m L^k L_q.
\end{aligned}$$

Now we can substitute (9.13) into (9.12) for further transformations. Keeping in mind formula (6.4) for components of projector field \mathbf{P} , we get

$$\begin{aligned}
C_{rs} \rightarrow & C_{rs} + \dots + \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n T_{rq}^b L_b P_m^q \left(\sum_{k=1}^n g^{km} \tilde{\nabla}_k U_s + \sum_{k=1}^n \sum_{a=1}^n \sum_{c=1}^n g^{am} \times \right. \\
& \times D_{ksa}^c L^k L_c - \nabla_s L^m + \sum_{k=1}^n \sum_{c=1}^n g^{kc} \tilde{\nabla}_k L^m \nabla_s L_c \left. \right) + \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n T_{rq}^b L_b L_m U_s \times \\
& \times \sum_{k=1}^n \frac{g^{km} \tilde{\nabla}_k L^q - g^{kq} \tilde{\nabla}_k L^m}{|\mathbf{L}|^2} + \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n \sum_{k=1}^n \sum_{a=1}^n \sum_{c=1}^n T_{rq}^b L_b P_m^q g^{km} \tilde{\nabla}_k L^a \times \\
& \times T_{sa}^c L_c - \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n \sum_{k=1}^n \sum_{a=1}^n \sum_{c=1}^n T_{rq}^b L_b \frac{g^{kq} \tilde{\nabla}_k L^m L_m T_{sc}^a L_a L_c}{|\mathbf{L}|^2}.
\end{aligned}$$

One can find four summands in right hand side of this relationship for C_{rs} (sums in round brackets are considered as a single term). Let's transform second summand containing fraction with numerator skew-symmetric in indices q and m :

$$\begin{aligned} \sum_{m=1}^n L_m \frac{g^{km} \tilde{\nabla}_k L^q - g^{kq} \tilde{\nabla}_k L^m}{|\mathbf{L}|^2} &= \sum_{m=1}^n \sum_{a=1}^n P_a^q L_m \frac{g^{km} \tilde{\nabla}_k L^a - g^{ka} \tilde{\nabla}_k L^m}{|\mathbf{L}|^2} + \\ &+ \sum_{m=1}^n \sum_{a=1}^n \frac{L^q L_a}{|\mathbf{L}|^2} L_m \frac{g^{km} \tilde{\nabla}_k L^a - g^{ka} \tilde{\nabla}_k L^m}{|\mathbf{L}|^2}. \end{aligned}$$

Here we used formula (6.4) for P_a^q again. Note that last term in right hand side of the above formula vanishes due to skew symmetry of the expression under summation with respect to indices m and a . Then for C_{rs} we obtain

$$\begin{aligned} C_{rs} \rightarrow C_{rs} + \dots + \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n T_{rq}^b L_b P_m^q B_s^m + \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n \sum_{k=1}^n \sum_{a=1}^n \sum_{c=1}^n T_{rq}^b L_b \times \\ \times P_m^q g^{km} \tilde{\nabla}_k L^a T_{sa}^c L_c - \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n \sum_{k=1}^n \sum_{a=1}^n \sum_{c=1}^n T_{rq}^b L_b \frac{g^{kq} \tilde{\nabla}_k L^m L_m T_{sc}^a L_a L^c}{|\mathbf{L}|^2}. \end{aligned}$$

In deriving this formula we used (8.2) for B_s^m . Now we shall apply similar trick with skew symmetry for transforming last term in the above formula:

$$\begin{aligned} \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n \sum_{k=1}^n \sum_{a=1}^n \sum_{c=1}^n T_{rq}^b L_b \frac{g^{kq} \tilde{\nabla}_k L^m L_m T_{sc}^a L_a L^c}{|\mathbf{L}|^2} = \\ \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n \sum_{k=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{e=1}^n T_{rq}^b L_b P_e^q \frac{g^{ke} \tilde{\nabla}_k L^m L_m T_{sc}^a L_a L^c}{|\mathbf{L}|^2} + \\ + \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n \sum_{k=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{e=1}^n T_{rq}^b L_b \frac{L^q L_e}{|\mathbf{L}|^2} \frac{g^{ke} \tilde{\nabla}_k L^m L_m T_{sc}^a L_a L^c}{|\mathbf{L}|^2}. \end{aligned} \quad (9.14)$$

Last term in (9.14) is symmetric with respect to indices r and s . Therefore we can denote it by dots when substituting (9.14) into formula for C_{rs} . As a result we get

$$\begin{aligned} C_{rs} \rightarrow C_{rs} + \dots + \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n T_{rq}^b L_b P_m^q B_s^m + \\ + \sum_{q=1}^n \sum_{b=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{e=1}^n T_{rq}^b L_b P_m^q A^{ma} P_a^c T_{sc}^e L_e. \end{aligned} \quad (9.15)$$

Here A^{ma} are given by formula (8.1). Substituting (9.15) into the equation (5.20) and taking into account equations (5.18) and (5.19), we get

$$\sum_{r=1}^n \sum_{s=1}^n (C_{rs} - C_{sr}) P_i^r P_j^s \rightarrow \sum_{r=1}^n \sum_{s=1}^n (C_{rs} - C_{sr}) P_i^r P_j^s.$$

This means that the equation (5.20) is invariant under gauge transformations (9.1). Summarizing this result and similar results for (5.18) and (5.19) obtained above, we can formulate them in the following theorem.

Theorem 9.1. *Additional normality equations (5.18), (5.19), and (5.20) transformed to \mathbf{v} -representation are invariant under gauge transformations (9.1).*

Now let's consider weak normality equations (5.14). In order to prove similar theorem for them we should derive transformation rules for vector field α and covector field η . For applying (9.1) to (7.19) we need to perform some preliminary calculations. From (9.5) for covariant derivative $\tilde{\nabla}_q \nabla_s L_r$ we derive

$$\begin{aligned} \tilde{\nabla}_q \nabla_s L_r \rightarrow \tilde{\nabla}_q \nabla_s L_r - \sum_{a=1}^n \tilde{\nabla}_q T_{sr}^a L_a - \sum_{a=1}^n T_{sr}^a g_{aq} - \\ - \sum_{a=1}^n \sum_{b=1}^n \tilde{\nabla}_q g_{rb} T_{as}^b v^a - \sum_{a=1}^n \sum_{b=1}^n g_{rb} \tilde{\nabla}_q T_{as}^b v^a - \sum_{b=1}^n g_{rb} T_{qs}^b. \end{aligned} \quad (9.16)$$

Now, using (7.19), (9.4), (9.5), (9.6), (9.7), and (9.16), for α^k we obtain

$$\begin{aligned} \alpha^k \rightarrow \alpha^k + \sum_{r=1}^n \sum_{q=1}^n \sum_{a=1}^n g^{rk} L^q T_{rq}^a L_a + \sum_{r=1}^n \sum_{a=1}^n v^r T_{ra}^k L^a - \sum_{r=1}^n \sum_{a=1}^n \sum_{b=1}^n v^r v^a \times \\ \times T_{ar}^b \tilde{\nabla}_b L^k + \sum_{q=1}^n \sum_{a=1}^n \sum_{j=1}^n T_{aj}^q v^a v^j \tilde{\nabla}_q L^k - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n L^r g^{qk} v^s \tilde{\nabla}_q T_{sr}^a L_a - \\ - \sum_{r=1}^n \sum_{s=1}^n L^r v^s T_{sr}^k - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n L^r g^{qk} v^s \tilde{\nabla}_q g_{rb} T_{as}^b v^a - \sum_{a=1}^n \sum_{b=1}^n L_b \times \\ \times \sum_{q=1}^n \sum_{s=1}^n g^{qk} v^s \tilde{\nabla}_q T_{as}^b v^a - \sum_{q=1}^n \sum_{s=1}^n \sum_{b=1}^n L_b g^{qk} v^s T_{qs}^b + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n L^r \times \\ \times g^{qk} \tilde{\nabla}_q g_{rs} T_{aj}^s v^a v^j + \sum_{q=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n L_s g^{qk} \tilde{\nabla}_q T_{aj}^s v^a v^j + \sum_{q=1}^n \sum_{j=1}^n \sum_{s=1}^n 2 L_s \times \\ \times g^{qk} T_{qj}^s v^j - \sum_{q=1}^n \sum_{r=1}^n \sum_{a=1}^n L^r g^{qk} T_{qr}^a L_a - \sum_{q=1}^n \sum_{r=1}^n \sum_{a=1}^n \sum_{b=1}^n L_b g^{qk} T_{aq}^b v^a - \\ - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n g^{mk} L_s \tilde{\nabla}_m T_{qr}^s L^r v^q. \end{aligned}$$

Looking attentively at the above formula, we see that almost all terms in right hand side do cancel each other. As a result we get the following transformation rule:

$$\alpha^k \rightarrow \alpha^k. \quad (9.17)$$

This means that first equation (5.14) is invariant under gauge transformations (9.1). In order to treat second equation (5.14) we need to derive transformation rule for covector fields β and η given by formulas (7.31) and (7.32). In (7.31) we have

the entry of second order covariant derivative $\nabla_k \nabla_m L_r$ and first order covariant derivative $\nabla_k F_r$. For $\nabla_k \nabla_m L_r$ we derive the following transformation rule:

$$\begin{aligned}
& \nabla_k \nabla_m L_r \rightarrow \nabla_k \nabla_m L_r - \sum_{a=1}^n \nabla_k T_{mr}^a L_a - \sum_{a=1}^n T_{mr}^a \nabla_k L_a - \sum_{a=1}^n v^a \times \\
& \times \sum_{b=1}^n \nabla_k g_{rb} T_{am}^b - \sum_{a=1}^n \sum_{b=1}^n g_{rb} \nabla_k T_{am}^b v^a - \sum_{c=1}^n T_{km}^c \nabla_c L_r + \sum_{c=1}^n T_{km}^c \times \\
& \times \sum_{a=1}^n T_{cr}^a L_a + \sum_{c=1}^n \sum_{a=1}^n \sum_{b=1}^n T_{km}^c g_{rb} T_{ac}^b v^a - \sum_{c=1}^n T_{kr}^c \nabla_m L_c + \sum_{c=1}^n T_{kr}^c \times \\
& \times \sum_{a=1}^n T_{mc}^a L_a + \sum_{c=1}^n \sum_{a=1}^n \sum_{b=1}^n T_{kr}^c g_{cb} T_{am}^b v^a - \sum_{c=1}^n \sum_{e=1}^n T_{ek}^c \tilde{\nabla}_c \nabla_m L_r \times \\
& \times v^e + \sum_{c=1}^n \sum_{e=1}^n \sum_{a=1}^n v^e T_{ek}^c \tilde{\nabla}_c T_{mr}^a L_a + \sum_{c=1}^n \sum_{e=1}^n \sum_{a=1}^n v^e T_{ek}^c T_{mr}^a \tilde{\nabla}_c L_a + \\
& + \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{e=1}^n v^e T_{ek}^c \tilde{\nabla}_c g_{rb} T_{am}^b v^a + \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{e=1}^n v^e T_{ek}^c g_{rb} \times \\
& \times \tilde{\nabla}_c T_{am}^b v^a + \sum_{a=1}^n \sum_{b=1}^n \sum_{e=1}^n v^e T_{ek}^a g_{rb} T_{am}^b.
\end{aligned} \tag{9.18}$$

In a similar way for $\nabla_k F_r$ we derive transformation rule which looks like:

$$\begin{aligned}
& \nabla_k F_r \rightarrow \nabla_k F_r + \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n \nabla_k g_{rs} T_{aj}^s v^a v^j + \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n g_{rs} \nabla_k T_{aj}^s \times \\
& \times v^a v^j - \sum_{c=1}^n T_{kr}^c F_c - \sum_{c=1}^n T_{kr}^c \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n g_{cs} T_{aj}^s v^a v^j - \sum_{c=1}^n \sum_{e=1}^n v^e \times \\
& \times T_{ke}^c \tilde{\nabla}_c F_r - \sum_{c=1}^n \sum_{e=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n v^e T_{ke}^c \tilde{\nabla}_c g_{rs} T_{aj}^s v^a v^j - \sum_{c=1}^n \sum_{e=1}^n v^e \times \\
& \times \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n T_{ke}^c g_{rs} \tilde{\nabla}_c T_{aj}^s v^a v^j - \sum_{e=1}^n \sum_{c=1}^n \sum_{s=1}^n \sum_{a=1}^n 2 v^e T_{ke}^c g_{rs} T_{ac}^s v^a.
\end{aligned} \tag{9.19}$$

Now let's combine (9.1), (9.2), (9.4), (9.5), (9.6), (9.7), (9.8), (9.9), (9.10), and two above formulas (9.18) and (9.19). Then for β_k given by formula (7.31) we obtain

$$\begin{aligned}
& \beta_k \rightarrow \beta_k + \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n v^r \nabla_r L^c T_{kc}^e L_e + \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n v^r L^c \nabla_r T_{kc}^e L_e + \sum_{r=1}^n v^r \times \\
& \times \sum_{c=1}^n \sum_{e=1}^n L^c T_{kc}^e \nabla_r L_e - \sum_{r=1}^n \sum_{m=1}^n v^r T_{rk}^m U_m - \sum_{r=1}^n \sum_{m=1}^n \sum_{c=1}^n \sum_{e=1}^n v^r T_{rk}^m L^c T_{mc}^e \times \\
& \times L_e - \sum_{r=1}^n \sum_{m=1}^n \sum_{a=1}^n v^r v^a T_{ar}^m \tilde{\nabla}_m U_k - \sum_{r=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{e=1}^n v^r v^a T_{ar}^m \tilde{\nabla}_m L^c L_e \times
\end{aligned}$$

$$\begin{aligned}
& \times T_{kc}^e - \sum_{r=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{e=1}^n v^r v^a T_{ar}^m L^c \tilde{\nabla}_m T_{kc}^e L_e - \sum_{m=1}^n \sum_{c=1}^n \sum_{e=1}^n T_{kc}^e \tilde{\nabla}_m L_e \times \\
& \times \sum_{r=1}^n \sum_{a=1}^n v^r v^a T_{ar}^m L^c + \sum_{q=1}^n \sum_{c=1}^n \sum_{r=1}^n F^q \tilde{\nabla}_q L^c T_{kc}^r L_r + \sum_{q=1}^n \sum_{c=1}^n \sum_{r=1}^n F^q \tilde{\nabla}_q T_{kc}^r \times \\
& \times L^c L_r + \sum_{q=1}^n \sum_{c=1}^n \sum_{r=1}^n F^q L^c T_{kc}^r \tilde{\nabla}_q L_r + \sum_{q=1}^n \sum_{i=1}^n \sum_{j=1}^n T_{ij}^q v^i v^j \tilde{\nabla}_q U_k + \sum_{i=1}^n v^i \times \\
& \times \sum_{c=1}^n \sum_{q=1}^n \sum_{r=1}^n \sum_{j=1}^n T_{ij}^q v^j \tilde{\nabla}_q L^c T_{kc}^r L_r + \sum_{q=1}^n \sum_{c=1}^n \sum_{r=1}^n \sum_{i=1}^n \sum_{j=1}^n T_{ij}^q v^i v^j L^c \tilde{\nabla}_q T_{kc}^r \times \\
& \times L_r + \sum_{q=1}^n \sum_{c=1}^n \sum_{r=1}^n \sum_{i=1}^n \sum_{j=1}^n T_{ij}^q v^i v^j L^c T_{kc}^r \tilde{\nabla}_q L_r + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n T_{kq}^c L_c g^{rq} \times \\
& \times v^s \nabla_s L_r + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n \sum_{e=1}^n g_{qe} T_{ck}^e v^c g^{rq} v^s \nabla_s L_r + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \nabla_k L_q v^s \times \\
& \times \sum_{a=1}^n g^{rq} T_{sr}^a L_a - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n \sum_{a=1}^n T_{kq}^c L_c g^{rq} v^s T_{sr}^a L_a - \sum_{q=1}^n \sum_{e=1}^n \sum_{r=1}^n g_{qe} \times \\
& \times \sum_{s=1}^n \sum_{c=1}^n \sum_{a=1}^n T_{ck}^e v^c g^{rq} v^s T_{sr}^a L_a + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n \nabla_k L_q g^{rq} v^s g_{rb} T_{as}^b \times \\
& \times v^a - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n T_{kq}^c L_c g^{rq} v^s g_{rb} T_{as}^b v^a - \sum_{q=1}^n \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n g_{qe} \times \\
& \times \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n T_{ck}^e v^c g^{rq} v^s g_{rb} T_{as}^b v^a + \sum_{q=1}^n \sum_{r=1}^n \sum_{a=1}^n T_{kq}^a L_a g^{rq} F_r + \sum_{q=1}^n \sum_{b=1}^n g_{qb} \times \\
& \times \sum_{r=1}^n \sum_{a=1}^n T_{ak}^b v^a g^{rq} F_r - \sum_{q=1}^n \sum_{r=1}^n \nabla_k L_q g^{rq} \sum_{s=1}^n \sum_{i=1}^n \sum_{j=1}^n g_{rs} T_{ij}^s v^i v^j + \sum_{a=1}^n L_a \times \\
& \times \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{i=1}^n \sum_{j=1}^n T_{kq}^a g^{rq} g_{rs} T_{ij}^s v^i v^j + \sum_{q=1}^n \sum_{r=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{s=1}^n \sum_{i=1}^n \sum_{j=1}^n g_{qb} v^a \times \\
& \times T_{ak}^b g^{rq} g_{rs} T_{ij}^s v^i v^j + \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n L^r \nabla_k g_{rs} T_{aj}^s v^a v^j + \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n g_{rs} \times \\
& \times \sum_{j=1}^n L^r \nabla_k T_{aj}^s v^a v^j - \sum_{r=1}^n \sum_{c=1}^n L^r T_{kr}^c F_c - \sum_{r=1}^n \sum_{c=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n L^r T_{kr}^c g_{cs} v^a \times \\
& \times T_{aj}^s v^j - \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n L^r v^e T_{ke}^c \tilde{\nabla}_c F_r - \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n L^r v^e T_{ke}^c T_{aj}^s \times \\
& \times \tilde{\nabla}_c g_{rs} v^a v^j - \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{j=1}^n L^r v^e T_{ke}^c g_{rs} \tilde{\nabla}_c T_{aj}^s v^a v^j - \sum_{r=1}^n \sum_{e=1}^n L^r \times \\
& \times \sum_{c=1}^n \sum_{s=1}^n \sum_{a=1}^n 2 v^e T_{ke}^c g_{rs} T_{ac}^s v^a - \sum_{m=1}^n \sum_{r=1}^n \sum_{a=1}^n L^r v^m \nabla_k T_{mr}^a L_a - \sum_{m=1}^n \sum_{r=1}^n L^r \times
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{a=1}^n v^m T_{mr}^a \nabla_k L_a - \sum_{m=1}^n \sum_{r=1}^n \sum_{a=1}^n \sum_{b=1}^n L^r v^m v^a \nabla_k g_{rb} T_{am}^b - \sum_{m=1}^n \sum_{r=1}^n L^r v^m \times \\
& \times \sum_{a=1}^n \sum_{b=1}^n g_{rb} \nabla_k T_{am}^b v^a - \sum_{m=1}^n \sum_{r=1}^n \sum_{c=1}^n L^r v^m T_{km}^c \nabla_c L_r + \sum_{m=1}^n \sum_{r=1}^n \sum_{c=1}^n \sum_{a=1}^n L^r \times \\
& \times v^m T_{km}^c T_{cr}^a L_a + \sum_{m=1}^n \sum_{r=1}^n \sum_{c=1}^n \sum_{a=1}^n \sum_{b=1}^n L^r v^m T_{km}^c g_{rb} T_{ac}^b v^a - \sum_{m=1}^n \sum_{r=1}^n L^r v^m \times \\
& \times \sum_{c=1}^n T_{kr}^c \nabla_m L_c + \sum_{m=1}^n \sum_{r=1}^n \sum_{c=1}^n \sum_{a=1}^n L^r v^m T_{kr}^c T_{mc}^a L_a + \sum_{m=1}^n \sum_{r=1}^n \sum_{c=1}^n L^r v^m T_{kr}^c \times \\
& \times \sum_{a=1}^n \sum_{b=1}^n g_{cb} T_{am}^b v^a - \sum_{m=1}^n \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n L^r v^m T_{ek}^c \tilde{\nabla}_c \nabla_m L_r v^e + \sum_{m=1}^n \sum_{r=1}^n L^r v^m \times \\
& \times \sum_{c=1}^n \sum_{e=1}^n \sum_{a=1}^n v^e T_{ek}^c \tilde{\nabla}_c T_{mr}^a L_a + \sum_{m=1}^n \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n \sum_{a=1}^n L^r v^m v^e T_{ek}^c T_{mr}^a \tilde{\nabla}_c L_a + \\
& + \sum_{m=1}^n \sum_{r=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{e=1}^n L^r v^m v^e T_{ek}^c \tilde{\nabla}_c g_{rb} T_{am}^b v^a + \sum_{m=1}^n \sum_{r=1}^n \sum_{a=1}^n \sum_{b=1}^n L^r v^m \times \\
& \times \sum_{c=1}^n \sum_{e=1}^n v^e T_{ek}^c g_{rb} \tilde{\nabla}_c T_{am}^b v^a + \sum_{m=1}^n \sum_{r=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{e=1}^n L^r v^m v^e T_{ek}^c g_{rb} T_{am}^b - \left| \right. \\
& - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n \sum_{i=1}^n \sum_{j=1}^n L^r \nabla_k L_q g^{sq} \tilde{\nabla}_s g_{rc} T_{ij}^c v^i v^j - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n \sum_{i=1}^n L^r \times \\
& \times \sum_{j=1}^n \nabla_k L_q g^{sq} g_{rc} \tilde{\nabla}_s T_{ij}^c v^i v^j - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n \sum_{j=1}^n 2 L^r \nabla_k L_q g^{sq} g_{rc} T_{sj}^c v^j + \\
& + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n L^r T_{kq}^a L_a g^{sq} \tilde{\nabla}_s F_r + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{i=1}^n \sum_{j=1}^n L^r T_{kq}^a L_a \times \\
& \times g^{sq} \tilde{\nabla}_s g_{rc} T_{ij}^c v^i v^j + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{i=1}^n \sum_{j=1}^n L^r T_{kq}^a L_a g^{sq} g_{rc} \tilde{\nabla}_s T_{ij}^c v^i v^j + \\
& + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{j=1}^n 2 L^r T_{kq}^a L_a g^{sq} g_{rc} T_{sj}^c v^j + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n L^r g_{qb} \times \\
& \times T_{ak}^b v^a g^{sq} \tilde{\nabla}_s F_r + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{i=1}^n \sum_{j=1}^n L^r g_{qb} T_{ak}^b v^a g^{sq} \tilde{\nabla}_s g_{rc} T_{ij}^c \times \\
& \times v^i v^j + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{i=1}^n \sum_{j=1}^n L^r g_{qb} T_{ak}^b v^a g^{sq} g_{rc} \tilde{\nabla}_s T_{ij}^c v^i v^j + \sum_{r=1}^n 2 \times \\
& \times \sum_{q=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n \sum_{j=1}^n L^r g_{qb} T_{ak}^b v^a g^{sq} g_{rc} T_{sj}^c v^j \left| + \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n L^r \times \right. \\
& \times \nabla_k L_q g^{sq} v^m \tilde{\nabla}_s T_{mr}^c L_c + \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n L^r \nabla_k L_q g^{sq} v^m T_{mr}^c g_{cs} + \sum_{r=1}^n L^r \times
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{q=1}^n \sum_{m=1}^n \sum_{s=1}^n \sum_{c=1}^n \sum_{e=1}^n \nabla_k L_q g^{sq} v^m \tilde{\nabla}_s g_{re} T_{cm}^e v^c + \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n \sum_{e=1}^n L^r \times \\
& \times \nabla_k L_q g^{sq} v^m g_{re} \tilde{\nabla}_s T_{cm}^e v^c + \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{e=1}^n L^r \nabla_k L_q g^{sq} v^m g_{re} T_{sm}^e + \\
& + \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n L^r T_{kq}^a L_a g^{sq} v^m \tilde{\nabla}_s \nabla_m L_r - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{c=1}^n L^r \times \\
& \times T_{kq}^a L_a g^{sq} v^m \tilde{\nabla}_s T_{mr}^c L_c - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{c=1}^n L^r T_{kq}^a L_a g^{sq} v^m T_{mr}^c g_{cs} - \\
& - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{e=1}^n L^r T_{kq}^a L_a g^{sq} v^m \tilde{\nabla}_s g_{re} T_{cm}^e v^c - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n L^r \times \\
& \times \sum_{s=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{e=1}^n T_{kq}^a L_a g^{sq} v^m g_{re} \tilde{\nabla}_s T_{cm}^e v^c - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{e=1}^n L^r T_{kq}^a \times \\
& \times L_a g^{sq} v^m g_{re} T_{sm}^e + \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n L^r g_{qb} T_{ak}^b v^a g^{sq} v^m \tilde{\nabla}_s \nabla_m L_r - \\
& - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n L^r g_{qb} T_{ak}^b v^a g^{sq} v^m \tilde{\nabla}_s T_{mr}^c L_c - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n L^r \times \\
& \times \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{c=1}^n g_{qb} T_{ak}^b v^a g^{sq} v^m T_{mr}^c g_{cs} - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n L^r g_{qb} T_{ak}^b \times \\
& \times \sum_{c=1}^n \sum_{e=1}^n v^a g^{sq} v^m \tilde{\nabla}_s g_{re} T_{cm}^e v^c - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n L^r g_{qb} T_{ak}^b v^a g^{sq} \times \\
& \times v^m \sum_{c=1}^n \sum_{e=1}^n g_{re} \tilde{\nabla}_s T_{cm}^e v^c - \sum_{q=1}^n \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{e=1}^n L^r g_{qb} T_{ak}^b v^a g^{sq} v^m \times \\
& \times g_{re} T_{sm}^e \Big| - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \nabla_m T_{kr}^s v^m L^r L_s + \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \nabla_k T_{mr}^s v^m L^r L_s + \\
& + \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{b=1}^n \sum_{a=1}^n v^a T_{ka}^b D_{mrb}^s v^m L^r L_s - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{b=1}^n \sum_{a=1}^n v^a T_{ma}^b v^m \times \\
& \times D_{krb}^s L^r L_s - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n T_{ma}^s T_{kr}^a v^m L^r L_s + \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n T_{ka}^s T_{mr}^a \times \\
& \times v^m L^r L_s - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{b=1}^n \sum_{a=1}^n v^a T_{ka}^b \tilde{\nabla}_b T_{mr}^s v^m L^r L_s + \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n v^a \times \\
& \times \sum_{b=1}^n T_{ma}^b \tilde{\nabla}_b T_{kr}^s v^m L^r L_s \Big| - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{c=1}^n \sum_{a=1}^n g^{cq} T_{kq}^a L_a D_{mrc}^s v^m L^r L_s - \\
& - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{c=1}^n \sum_{b=1}^n g^{cq} g_{qb} T_{ak}^b v^a D_{mrc}^s v^m L^r L_s - \sum_{q=1}^n \sum_{r=1}^n \sum_{c=1}^n g^{cq} \times
\end{aligned}$$

$$\begin{aligned}
& \times \sum_{m=1}^n \sum_{s=1}^n \nabla_k L_q \tilde{\nabla}_c T_{rm}^s v^m L^r L_s + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{c=1}^n \sum_{a=1}^n g^{cq} T_{kq}^a L_a \tilde{\nabla}_c T_{rm}^s \times \\
& \times v^m L^r L_s + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{c=1}^n \sum_{a=1}^n \sum_{b=1}^n g^{cq} g_{qb} T_{ak}^b v^a \tilde{\nabla}_c T_{rm}^s v^m L^r L_s + \sum_{i=1}^n v^i \times \\
& \times \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{j=1}^n g^{am} D_{rka}^s g_{mb} T_{ij}^b v^j L^r L_s - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n g^{am} \times \\
& \times \tilde{\nabla}_a T_{kr}^s F_m L^r L_s - \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{a=1}^n \sum_{b=1}^n \sum_{i=1}^n \sum_{j=1}^n g^{am} \tilde{\nabla}_a T_{kr}^s g_{mb} T_{ij}^b v^i v^j L^r L_s.
\end{aligned}$$

Totally in right hand side of the above formula we have 97 terms apart from β_k . Lets denote them by $T[i]$. Then formula can be written as

$$\beta_k \rightarrow \beta_k + \sum_{i=1}^{97} T[i]. \quad (9.20)$$

Terms $T[1], \dots, T[97]$ are divided into 11 groups separated from each other by vertical bars. These groups represent contributions from 11 terms in formula (7.31):

$$97 = 9 + 7 + 8 + 5 + 8 + 16 + 11 + 17 + 8 + 5 + 3.$$

One can find that most of terms $T[1], \dots, T[97]$ in formula (9.20) do cancel each other. In the following table we collect mutually canceling terms:

$$\begin{array}{lll}
T[2] + T[82] = 0, & T[3] + T[45] = 0, & T[5] + T[43] = 0, \\
T[6] + T[13] = 0, & T[7] + T[14] = 0, & T[8] + T[15] = 0, \\
T[9] + T[16] = 0, & T[11] + T[96] = 0, & T[12] + T[32] = 0, \\
T[21] + T[44] = 0, & T[22] + T[27] = 0, & T[23] + T[28] = 0, \\
T[24] + T[29] = 0, & T[30] + T[40] = 0, & T[31] + T[41] = 0, \\
T[33] + T[47] = 0, & T[34] + T[61] = 0, & T[35] + T[51] = 0, \\
T[36] + T[52] = 0, & T[37] + T[64] = 0, & T[38] + T[83] = 0, \\
T[39] + T[66] = 0, & T[46] + T[86] = 0, & T[48] + T[76] = 0, \\
T[49] + T[77] = 0, & T[50] + T[78] = 0, & T[53] + T[81] = 0, \\
T[54] + T[67] = 0, & T[55] + T[68] = 0, & T[58] + T[73] = 0, \\
T[59] + T[74] = 0, & T[62] + T[79] = 0, & T[63] + T[80] = 0, \\
T[65] + T[92] = 0, & T[71] + T[93] = 0, & T[72] + T[87] = 0, \\
T[84] + T[91] = 0, & T[85] + T[95] = 0, & T[88] + T[94] = 0,
\end{array}$$

$$\begin{array}{ll}
T[19] + T[56] + T[69] = 0, & T[20] + T[60] + T[75] = 0, \\
T[89] + T[97] = 0, & T[4] + T[18] + T[26] + T[42] = 0.
\end{array}$$

After all above cancellations only 7 terms $T[i]$ survive in formula (9.20) for β_k :

$$\beta_k \rightarrow \beta_k + T[1] + T[10] + T[17] + T[25] + T[57] + T[70] + T[90]. \quad (9.21)$$

Now we can write (9.21) in explicit form. This form is rather observable one:

$$\begin{aligned}
\beta_k \rightarrow & \beta_k + \sum_{r=1}^n \sum_{c=1}^n \sum_{e=1}^n v^r \nabla_r L^c T_{kc}^e L_e + \sum_{q=1}^n \sum_{c=1}^n \sum_{r=1}^n F^q \tilde{\nabla}_q L^c T_{kc}^r L_r + \\
& + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n T_{kq}^c L_c g^{rq} v^s \nabla_s L_r + \sum_{q=1}^n \sum_{r=1}^n \sum_{a=1}^n T_{kq}^a L_a g^{rq} F_r + \\
& + \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{a=1}^n L^r T_{kq}^a L_a g^{sq} \tilde{\nabla}_s F_r + \sum_{q=1}^n \sum_{r=1}^n \sum_{m=1}^n \sum_{s=1}^n \sum_{a=1}^n L^r T_{kq}^a L_a \times \\
& \times g^{sq} v^m \tilde{\nabla}_s \nabla_m L_r - \sum_{q=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{m=1}^n \sum_{c=1}^n \sum_{a=1}^n g^{cq} T_{kq}^a L_a D_{mrc}^s v^m L^r L_s.
\end{aligned} \tag{9.22}$$

Looking attentively at the above formula (9.22), one can find that it can be further transformed. Namely, one can extract common factors in all terms:

$$\begin{aligned}
\beta_k \rightarrow & \beta_k + \sum_{e=1}^n \sum_{q=1}^n T_{kq}^e L_e \left(\sum_{r=1}^n v^r \nabla_r L^q + \sum_{r=1}^n F^r \tilde{\nabla}_r L^q + \right. \\
& + \sum_{s=1}^n \sum_{r=1}^n \sum_{c=1}^n L^r g^{cq} v^s \tilde{\nabla}_c \nabla_s L_r + \sum_{r=1}^n g^{rq} F_r + \sum_{r=1}^n \sum_{s=1}^n L^r g^{sq} \tilde{\nabla}_s F_r + \\
& \left. + \sum_{r=1}^n \sum_{s=1}^n g^{rq} v^s \nabla_s L_r - \sum_{m=1}^n \sum_{r=1}^n \sum_{s=1}^n \sum_{c=1}^n g^{mq} L_s D_{crm}^s L^r v^c \right).
\end{aligned} \tag{9.23}$$

Due to the equality (7.8) we can derive the following relationship for U_r and F_r :

$$\sum_{r=1}^n g^{rq} \left(F_r + \sum_{s=1}^n v^s \nabla_s L_r \right) = \sum_{r=1}^n g^{rq} \left(U_r + \sum_{s=1}^n L^s \nabla_r L_s \right). \tag{9.24}$$

Combining (9.24) with (9.23) and applying formula (7.19), we can provide substantial simplification of formula (7.23). Now it looks like

$$\beta_k \rightarrow \beta_k + \sum_{e=1}^n \sum_{q=1}^n T_{kq}^e L_e \alpha^q. \tag{9.25}$$

Let's remember formula (7.32) for η_k . Applying (9.6), (9.17), and (9.25) to (7.32) and using formula (6.4) for projector components, we derive

$$\eta_k \rightarrow \eta_k + \sum_{e=1}^n \sum_{q=1}^n \sum_{s=1}^n T_{kq}^e L_e P_s^q \alpha^s. \tag{9.26}$$

If we recall first equation (5.14), we see that (9.26) is equivalent to

$$\eta_k \rightarrow \eta_k \tag{9.27}$$

This means that covector field $\boldsymbol{\eta}$ is invariant under gauge transformations (9.1). Due to (9.17) and (9.27) both weak normality equations (5.14) are invariant under gauge transformations (9.1). This result can be stated as a theorem.

Theorem 9.2. *Weak normality equations (5.14) transformed to \mathbf{v} -representation are invariant under gauge transformations (9.1).*

Theorems 9.1 and 9.2 mean that normality equations (5.14), (5.18), (5.19), and (5.20), when applied to Newtonian dynamical system (1.1), do not actually depend on connection components Γ_{ij}^k . Therefore we can substitute

$$\Gamma_{ij}^k = 0, \quad F^i = \Phi^i, \quad \nabla_i = \frac{\partial}{\partial x^i}, \quad \tilde{\nabla}_i = \frac{\partial}{\partial v^i}$$

and write normality equations in connection free form. In such form each term of these equations has no separate tensorial interpretation and one should find invariant (coordinate-free) interpretation for the equations in whole. However, this is separate problem which will be studied later.

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¹Electronic Archive at Los Alamos National Laboratory of USA (LANL). Archive is accessible through Internet <http://arXiv.org>, it has mirror site <http://ru.arXiv.org> at the Institute for Theoretical and Experimental Physics (ITEP, Moscow).

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