SPINOR FUNCTIONS OF SPINORS AND THE CONCEPT OF EXTENDED SPINOR FIELDS.

Однажды старик-египтянин призвал к себе своего старшего сына и говорит ему:
— Сынок, я открою тебе страшную тайну. Жрецы из храма солнца Амон-Ра — не боги!
— Отец, как можно! Они же говорят со звездами и предсказывают затмения! — изумился сын. Но старик продолжил невозмутимо:
— Ещё в детстве я тайком проник в их храм и видел, что они едят и справляют нужду совсем так же, как мы с тобой.

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Abstract. Spinor fields depending on tensor fields and other spinor fields are considered. The concept of extended spinor fields is introduced and the theory of differentiation for such fields is developed.

1. Introduction.

The *space-time* is a stage for all events in general relativity. Saying the space-time, we understand a smooth 4-dimensional manifold M equipped with a pseudo-Euclidean metric $\mathbf{g}$ of the Minkowski-type signature $(+,-,-,-)$. Word lines of particles, tangent spaces of M and their light cones, local charts and local coordinates, tensors and tensor fields in M, the metric connection Γ with the components

$$\Gamma^k_{ij}=\sum_{s=0}^{3}\frac{g^{ks}}{2}\left(\frac{\partial g_{sj}}{\partial x^i}+\frac{\partial g_{is}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^s}\right),$$

the covariant differentiation ∇ determined by Γ, and some other basic concepts of general relativity are assumed to be known to the reader. The free textbooks [1–4] are recommended as an introductory material for getting acquainted with that basic concepts. The concepts of spinors and spinor fields are less commonly known. Therefore, we consider them in full details in the next eight sections 2 through 8. These sections form the preliminary part of the present paper.

The concept of an extended tensor field in its full generality was introduced in paper [5]. It arises if one consider a tensor-valued function

$$\mathbf{X}=\mathbf{X}(p,\mathbf{T}[1],\ldots,\mathbf{T}[Q]) \tag{1.1}$$

with one point argument $p\in M$ and several tensorial arguments $\mathbf{T}[1], \ldots, \mathbf{T}[Q]$. The paper [6] illustrates how extended tensor fields are applied in the theory of continuous media. When passing to the quantum area, one should also take into account the specifically quantum phenomenon of *spin* which does not reduce to the pure rotation. Therefore, we need to add some spin-tensors $\mathbf{S}[1], \ldots, \mathbf{S}[J]$ to the arguments of the tensor-valued function $\mathbf{X}$ in (1.1)

$$\mathbf{X}=\mathbf{X}(p,\mathbf{S}[1],\ldots,\mathbf{S}[J],\mathbf{T}[1],\ldots,\mathbf{T}[Q]) \tag{1.2}$$

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and extend the theory to the spin-tensorial functions with the same kind arguments:

$$\mathbf{U} = \mathbf{U}(p, \mathbf{S}[1], \ldots, \mathbf{S}[J], \mathbf{T}[1], \ldots, \mathbf{T}[Q]). \tag{1.3}$$

This is the main purpose of the present paper. We reproduce the results of [5] as applied to the extended tensor fields (1.2) and extended spin-tensorial fields (1.3).

2. Spinors and spin-tensors.

Let M be the space-time. Denote by SM some smooth 2-dimensional complex vector bundle over M. We shall call it the *spinor* bundle. Its smooth global and local sections are called *spinor fields*. Apart from SM, we consider the tangent bundle TM. For the sake of uniformity we denote the canonical projections for both bundles TM and SM with the same symbol π:

$$\pi\colon\, TM \to M, \qquad\qquad \pi\colon\, SM \to M. \tag{2.1}$$

Let $p \in M$ be a point of the space-time M. We denote by $T_p(M)$ and $S_p(M)$ the *fibers* of the bundles TM and SM over the point p. In other words, $T_p(M)$ and $S_p(M)$ are the total preimages of the point p under the mappings (2.1). Note that $T_p(M)$ is a 4-dimensional vector space over the real numbers $\mathbb{R}$, while $S_p(M)$ is a 2-dimensional vector space over the complex numbers $\mathbb{C}$.

Let's denote by $T^*_p(M)$ and $S^*_p(M)$ the conjugate spaces for $T^*_p(M)$ and $S^*_p(M)$ respectively. The elements of $T_p(M)$ are called *vectors*, the elements of $T^*_p(M)$ are called *covectors*. Similarly, the elements of $S_p(M)$ are called *spinors*, the elements of $S^*_p(M)$ are called *cospinors*. Let's introduce the following spaces:

$$T^r_s(p, M) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r\text{ times}} \otimes \underbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}_{s\text{ times}}, \tag{2.2}$$

$$S^\alpha_\beta(p, M) = \overbrace{S_p(M) \otimes \ldots \otimes S_p(M)}^{\alpha\text{ times}} \otimes \underbrace{S^*_p(M) \otimes \ldots \otimes S^*_p(M)}_{\beta\text{ times}}. \tag{2.3}$$

The elements of $T^r_s(p, M)$ are called *tensors of the type* (r, s). Similarly, the elements of $S^\alpha_\beta(p, M)$ are called *spin-tensors of the type* (r, s). The space (2.2) is a linear space over the real numbers, while (2.3) is a complex linear space. Apart from (2.2) and (2.3), one can consider their tensor product

$$S^\alpha_\beta T^r_s(p, M) = S^\alpha_\beta(p, M) \otimes T^r_s(p, M). \tag{2.4}$$

The elements of the tensor product (2.4) are also called *spin-tensors*. Note that $S^\alpha_\beta T^r_s(p, M)$ is a complex linear space. Remember also that for any real linear space V the complex linear space $\mathbb{C} \otimes V$ is defined, it is called the *complexification* of V. Let's denote by $\mathbb{C}T_p(M)$ the complexification of the tangent space:

$$\mathbb{C}T_p(M) = \mathbb{C} \otimes T_p(M). \tag{2.5}$$

Then let's use (2.5) in the following tensor product:

$$\mathbb{C}T^r_s(p,M)=\overbrace{\mathbb{C}T_p(M)\otimes\ldots\otimes\mathbb{C}T_p(M)}^{r\text{times}}\otimes\underbrace{\mathbb{C}T^*_p(M)\otimes\ldots\otimes\mathbb{C}T^*_p(M)}_{s\text{ times}}. \tag{2.6}$$

The tensor product (2.4) now can be written as

$$S^\alpha_\beta T^r_s(p,M)=S^\alpha_\beta(p,M)\otimes\mathbb{C}T^r_s(p,M). \tag{2.7}$$

This representation (2.7) of $S^\alpha_\beta T^r_s(p,M)$ is absolutely equivalent to (2.4).

For the conjugate spaces $\mathbb{C}T^*_p(M)$ and $T^*_p(M)$ in (2.6) and (2.2) and for the tensor products (2.6) and (2.2) themselves we have the following obvious equalities:

$$\mathbb{C}T^*_p(M)=\mathbb{C}\otimes T^*_p(M),\qquad\qquad \mathbb{C}T^r_s(p,M)=\mathbb{C}\otimes T^r_s(p,M). \tag{2.8}$$

Due to (2.8) we have the semilinear isomorphism of complex conjugation:

$$\tau\colon\ \mathbb{C}T^r_s(p,M)\to\mathbb{C}T^r_s(p,M). \tag{2.9}$$

It is introduced by the formula $\tau(\alpha\otimes\mathbf{X})=\overline{\alpha}\otimes\mathbf{X}$, where $\mathbf{X}\in T^r_s(p,M)$, $\alpha\in\mathbb{C}$, and $\overline{\alpha}$ is the conjugate number for α.

The complexified tangent spaces (2.5) are glued into a complex vector bundle $\mathbb{C}TM$. It is called the *complexified tangent bundle*. Its local and global sections are called *complexified vector fields*. However, below we shall often call them vector fields for the sake of brevity.

3. Semilinear functions and Hermitian conjugate spaces.

Let V be some finite-dimensional linear space over the field of complex numbers. Its dual space V^* by definition is the set of linear functions $f=f(\mathbf{v})$, where $\mathbf{v}\in V$. This means that each $f\in V^*$ satisfies the following two linearity conditions:

(1) $f(\mathbf{v}_1+\mathbf{v}_2)=f(\mathbf{v}_1)+f(\mathbf{v}_2)$ for any two vectors $\mathbf{v}_1,\mathbf{v}_2\in V$;
(2) $f(\alpha\,\mathbf{v})=\alpha\,f(\mathbf{v})$ for any $\mathbf{v}\in V$ and for any $\alpha\in\mathbb{C}$.

In geometric terminology linear functions $f\in V^*$ are called *covectors*, their values, when applied to a vector $\mathbf{v}\in V$, are written in the form of a scalar product:

$$f(\mathbf{v})=(f,\,\mathbf{v}). \tag{3.1}$$

Semilinear functions in V are defined by the following two conditions:

(1) $f(\mathbf{v}_1+\mathbf{v}_2)=f(\mathbf{v}_1)+f(\mathbf{v}_2)$ for any two vectors $\mathbf{v}_1,\mathbf{v}_2\in V$;
(2) $f(\alpha\,\mathbf{v})=\bar\alpha\,f(\mathbf{v})$ for any $\mathbf{v}\in V$ and for any $\alpha\in\mathbb{C}$.

By means of $\bar\alpha$ in (2) above we denote the conjugate complex number $\bar\alpha=x-iy$ for a complex number $\alpha=x+iy$. Semilinear functions form another linear vector space $V^\dagger$, it is called the *Hermitian conjugate space* or *Hermitian dual space* for V. Like in (3.1), the value of a function $f\in V^\dagger$ is denoted as a scalar product:

$$f(\mathbf{v})=\langle\mathbf{v},\,f\rangle. \tag{3.2}$$

However, (3.2) is a Hermitian scalar product. For this reason we used the angular brackets for it. The elements of $V^\dagger$ are called *conjugate covectors*.

Each linear function $f\in V^*$ can be converted to a semilinear function $\bar f\in V^\dagger$ by passing to conjugate numbers in its values:

$$\bar f(\mathbf{v})=\overline{f(\mathbf{v})} \tag{3.3}$$

Conversely, if $f\in V^\dagger$, then $\bar f\in V^*$. Thus, the formula (3.3) defines the canonical semilinear isomorphism: $V^*\rightleftarrows V^\dagger$. We shall use the same symbol τ for both semilinear mappings implementing this isomorphism:

$$\tau\colon\ V^*\to V^\dagger, \qquad\qquad \tau\colon\ V^\dagger\to V^*. \tag{3.4}$$

Let's denote by $V^{*\dagger}$ the conjugate dual for the dual space V^*. It is clear that $V^{\dagger *}$ is canonically isomorphic to $V^{*\dagger}$, so that one can treat them as the same space. Indeed, for any vector $\mathbf{v}\in V$ the following function is defined:

$$w_{\mathbf{v}}(f)=\langle\mathbf{v},\,f\rangle,\ \text{ where }\ f\in V^\dagger. \tag{3.5}$$

The formula (3.5) defines the canonical semilinear isomorphism $V^{\dagger *}\rightleftarrows V$:

$$\tau\colon\ V\to V^{\dagger *}, \qquad\qquad \tau\colon\ V^{\dagger *}\to V. \tag{3.6}$$

Similarly, for any vector $\mathbf{v}\in V$ the following function is defined:

$$w_{\mathbf{v}}(f)=\overline{(f,\,\mathbf{v})},\ \text{ where }\ f\in V^*. \tag{3.7}$$

The formula (3.7) defines the canonical semilinear isomorphism $V^{*\dagger}\rightleftarrows V$:

$$\tau\colon\ V\to V^{*\dagger}, \qquad\qquad \tau\colon\ V^{*\dagger}\to V. \tag{3.8}$$

All canonical isomorphisms (3.4), (3.6), and (3.8) can be extended to various tensor products of V, V^*, $V^\dagger$, and $V^{*\dagger}$. In coordinate representation they are expressed as passing to the conjugate numbers in the coordinates of tensors.

4. Conjugate spinors and spin-tensors.

The construction of the Hermitian conjugate space described in the previous section 3 can be applied to the spinor spaces $S_p(M)$, where $p\in M$. As a result we get the spaces $S^\dagger_p(M)$ and $S^{\dagger *}_p(M)=S^{*\dagger}_p(M)$ in addition to $S_p(M)$ and $S^*_p(M)$. These new spaces are glued into smooth complex bundles $S^\dagger M$ and $S^{\dagger *}M$ over the base manifold M. They are called the *Hermitian conjugate spinor bundles*.

Like in (2.3), one can construct a tensor product of multiple copies of Hermitian conjugate spinor spaces $S^\dagger_p(M)$ and $S^{\dagger *}_p(M)$:

$$\bar S^\nu_\gamma(p,M)=\overbrace{S^{\dagger *}_p(M)\otimes\ldots\otimes S^{\dagger *}_p(M)}^{\nu\text{ times}}\otimes\underbrace{S^\dagger_p(M)\otimes\ldots\otimes S^\dagger_p(M)}_{\gamma\text{ times}}. \tag{4.1}$$

Then, using (4.1), one can extend the tensor product (2.7) as follows:

$$S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{r}_{s}(p,M)=S^{\alpha}_{\beta}(p,M)\otimes\bar{S}^{\nu}_{\gamma}(p,M)\otimes\mathbb{C}T^{r}_{s}(p,M).\tag{4.2}$$

Elements of the spaces (4.2) are also called *spin-tensors*. The spaces (4.2) with p running over M constitute a smooth complex bundle over the base manifold M. Local and global smooth sections of this bundle are called *spin-tensorial fields*. Upon choosing some local chart U of M over which the spinor bundle SM trivializes one can express any spin-tensorial field $\mathbf{Y}$ in the coordinate form

$$Y^{i_1\ldots\, i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\, h_r}_{j_1\ldots\, j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\, k_s}=Y^{i_1\ldots\, i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\, h_r}_{j_1\ldots\, j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\, k_s}(x^0,x^1,x^2,x^3),\tag{4.3}$$

where x^0, x^1, x^2, x^3 are local coordinates of a point $p\in M$. The barred indices $\bar{i}_1,\ldots\ \bar{i}_\nu$ and $\bar{j}_1,\ldots,\bar{j}_\gamma$ in (4.3) are treated as separate variables independent of $i_1,\ldots,i_\alpha$ and $j_1,\ldots,j_\beta$. In some books (see [7]) dotted indices are used instead of barred indices.

The semilinear isomorphism (2.9) and the semilinear isomorphisms (3.4), (3.6), and (3.8) with $V=S_p(M)$ induce the following semilinear isomorphisms of the spin-tensorial spaces (4.2) and corresponding bundles:

$$\tau\colon\ S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{r}_{s}(p,M)\to S^{\nu}_{\gamma}\bar{S}^{\alpha}_{\beta}T^{r}_{s}(p,M),\tag{4.4}$$

$$\tau\colon\ S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{r}_{s}M\to S^{\nu}_{\gamma}\bar{S}^{\alpha}_{\beta}T^{r}_{s}M.\tag{4.5}$$

In local coordinates the isomorphisms (4.4) and (4.5) are expressed by the formula

$$Z^{i_1\ldots\, i_\nu\bar{i}_1\ldots\,\bar{i}_\alpha h_1\ldots\, h_r}_{j_1\ldots\, j_\gamma\bar{j}_1\ldots\,\bar{j}_\beta k_1\ldots\, k_s}=\overline{Y^{\bar{i}_1\ldots\,\bar{i}_\alpha i_1\ldots\, i_\nu h_1\ldots\, h_r}_{\bar{j}_1\ldots\,\bar{j}_\beta j_1\ldots\, j_\gamma k_1\ldots\, k_s}},\tag{4.6}$$

where $\mathbf{Z}=\tau(\mathbf{Y})$. The formula (4.6) means that τ acts as the complex conjugation upon the components of spin-tensors simultaneously exchanging barred and non-barred spinor indices in upper and lower positions.

5. Local trivializations of the spinor bundle.

The spinor bundle SM, as it was introduced above in section 2, is not the actual spinor bundle. It is yet an arbitrary 2-dimensional complex bundle over the space-time M. In order to get the actual spinor bundle we should relate SM to TM through the metric tensor $\mathbf{g}$, which is the basic structure of the space-time manifold M.

Let p_0 be some point of M and let U be a local chart covering that point p_0. Then any point $p\in U$ is represented by its local coordinates

$$x^0=x^0(p),\quad x^1=x^1(p),\quad x^2=x^2(p),\quad x^3=x^3(p)\tag{5.1}$$

and any tangent vector $\mathbf{v}\in T_pM$ is represented by its components v^0, v^1, v^2, v^3:

$$\mathbf{v}=v^0\,\frac{\partial}{\partial x^0}+v^1\,\frac{\partial}{\partial x^1}+v^2\,\frac{\partial}{\partial x^2}+v^3\,\frac{\partial}{\partial x^3}.\tag{5.2}$$

The coordinate vectors

$$\mathbf{E}_0 = \frac{\partial}{\partial x^0}, \quad \mathbf{E}_1 = \frac{\partial}{\partial x^1}, \quad \mathbf{E}_2 = \frac{\partial}{\partial x^2}, \quad \mathbf{E}_3 = \frac{\partial}{\partial x^3}. \tag{5.3}$$

in the expansion (5.2) form the *holonomic moving frame* associated with the local coordinates (5.1) in U. The pair $q = (p, \mathbf{v})$ is a point of the tangent bundle TM and $p = \pi(q)$, see the formula (2.1) above. Therefore, the quantities

$$\begin{aligned} &x^0 = x^0(q), \quad x^1 = x^1(q), \quad x^2 = x^2(q), \quad x^3 = x^3(q),\\ &v^0 = v^0(q), \quad v^1 = v^1(q), \quad v^2 = v^2(q), \quad v^3 = v^3(q) \end{aligned} \tag{5.4}$$

determine the trivialization of TM over U in the holonomic frame (5.3). Suppose that $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_4$ are some smooth vector fields in U being linearly independent at each point $p \in U$. In this case we can write the expansion similar to (5.2):

$$\mathbf{v} = v^0\,\boldsymbol{\Upsilon}_0 + v^1\,\boldsymbol{\Upsilon}_1 + v^2\,\boldsymbol{\Upsilon}_2 + v^3\,\boldsymbol{\Upsilon}_3. \tag{5.5}$$

Replacing (5.2) by the expansion (5.5), we say that the quantities (5.4) determine the trivialization of TM over U in the *non-holonomic frame* $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$.

In the holonomic frame (5.3) the metric tensor $\mathbf{g}$ is represented by a square 4×4 matrix. In general case of a non-flat space-time M this matrix is non-diagonal:

$$g_{ij} = g_{j\,i} = \begin{Vmatrix} g_{00} & g_{01} & g_{02} & g_{03}\\ g_{01} & g_{11} & g_{12} & g_{13}\\ g_{02} & g_{12} & g_{22} & g_{23}\\ g_{03} & g_{13} & g_{23} & g_{33} \end{Vmatrix} \tag{5.6}$$

This matrix (5.6) can be diagonalized at any fixed point $p_0 \in M$, but it cannot be diagonalized in a whole neighborhood of the point p_0 if we use only holonomic frames, i. e. frames (5.3) associated with some local coordinates x^0, x^1, x^2, x^3. In the case of non-holonomic frames one can easily prove the following theorem.

Theorem 5.1. *For any local chart $U \subset M$ and for any point $p_0 \in U$ there is some smaller neighborhood $\tilde{U} \subset U$ of the point p_0 and there is some smooth non-holonomic frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ within $\tilde{U}$ such that the metric tensor $\mathbf{g}$ is represented by the standard diagonal matrix of the Minkowski-type metric:*

$$g_{ij} = g_{j\,i} = \begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{5.7}$$

Definition 5.1. A frame of four smooth vector fields $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ in which the metric tensor $\mathbf{g}$ is given by the matrix (5.7) is called an *orthonormal frame*.

Theorem 5.2. *Any two orthonormal frames of the space-time* $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ *and* $\tilde{\boldsymbol{\Upsilon}}_0$, $\tilde{\boldsymbol{\Upsilon}}_1$, $\tilde{\boldsymbol{\Upsilon}}_2$, $\tilde{\boldsymbol{\Upsilon}}_3$ *at the same point* $p \in M$ *are related to each other by some Lorentzian transition matrices* $S \in \mathrm{O}(1,3)$ *and* $T \in \mathrm{O}(1,3)$*:*

$$\tilde{\boldsymbol{\Upsilon}}_i = \sum_{j=0}^{3} S_i^j \, \boldsymbol{\Upsilon}_j, \qquad\qquad \boldsymbol{\Upsilon}_i = \sum_{j=0}^{3} T_i^j \, \tilde{\boldsymbol{\Upsilon}}_j. \tag{5.8}$$

The matrices S *and* T *in the formulas* (5.8) *are inverse to each other:* $T = S^{-1}$.

Apart from the metric tensor $\mathbf{g}$, the space-time manifold M carries other two structures: the *orientation* and the *polarization*. The orientation in M means that we can distinguish *right oriented frames* and *left oriented frames*. If two frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\tilde{\boldsymbol{\Upsilon}}_0$, $\tilde{\boldsymbol{\Upsilon}}_1$, $\tilde{\boldsymbol{\Upsilon}}_2$, $\tilde{\boldsymbol{\Upsilon}}_3$ are of the same orientation, both right or both left, then the determinant of the transition matrix S in (5.8) is positive: $\det S > 0$. Otherwise, for two frames of different orientation $\det S < 0$. Remember that for any Lorentzian matrix $S \in \mathrm{O}(1,3)$ we have $\det S = \pm 1$. Lorentzian matrices with $\det S = 1$ form a subgroup in the Lorentz group $\mathrm{O}(1,3)$. This is the *special Lorentz group* $\mathrm{SO}(1,3)$.

The polarization is a geometric structure of the space-time M that distinguishes «the Future» and «the Past» (see [4]). Remember that a time-like vector is a tangent vector $\mathbf{v} \in T_p(M)$ at a point $p \in M$ such that $g(\mathbf{v}, \mathbf{v}) > 0$ with respect to the Minkowski metric $\mathbf{g}$. At each point $p \in M$ the time-like vectors form the interior of two cones (they are called the *light cones*). The polarization marks one of these two cones as «the Future cone» and the other as «the Past cone». The polarization of the space-time M is a smooth structure. This means that for any smooth parametric curve $p = p(\tau)$ in M there is a smooth vector-valued function $\mathbf{v} = \mathbf{v}(\tau)$ on this curve such that all its values $\mathbf{v}(\tau)$ are time-like vectors from the Future cone. A Lorentzian matrix S is called an *orthochronous* matrix if $S_0^0 > 1$. Orthochronous Lorentzian matrices form a subgroup in $\mathrm{O}(1,3)$. This subgroup is denoted as $\mathrm{O}^+(1,3)$. The *special orthochronous Lorentz group* is defined as

$$\mathrm{SO}^+(1,3) = \mathrm{SO}(1,3) \cap \mathrm{O}^+(1,3). \tag{5.9}$$

Suppose that $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\tilde{\boldsymbol{\Upsilon}}_0$, $\tilde{\boldsymbol{\Upsilon}}_1$, $\tilde{\boldsymbol{\Upsilon}}_2$, $\tilde{\boldsymbol{\Upsilon}}_3$ are two orthonormal frames. Then $\boldsymbol{\Upsilon}_0$ and $\tilde{\boldsymbol{\Upsilon}}_0$ both are time-like unit vectors. If they belong to the same light cone (both are in the the Future cone or both are in the Past cone), then the Lorentz matrices S and T relating the frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\tilde{\boldsymbol{\Upsilon}}_0$, $\tilde{\boldsymbol{\Upsilon}}_1$, $\tilde{\boldsymbol{\Upsilon}}_2$, $\tilde{\boldsymbol{\Upsilon}}_3$ in (5.8) are orthochronous Lorentzian matrices. Let's consider the following Pauli matrices:

$$\boldsymbol{\sigma}_1 = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \qquad \boldsymbol{\sigma}_2 = \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix} \qquad \boldsymbol{\sigma}_3 = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} \tag{5.10}$$

We complement the Pauli matrices (5.10) with the unit matrix $\boldsymbol{\sigma}_0$:

$$\boldsymbol{\sigma}_0 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}. \tag{5.11}$$

Using the matrices (5.10) and (5.11), one can map each 4-vector $\mathbf{w} \in \mathbb{R}^4$ to a

Hermitian 2×2 matrix by means of the following formula:

$$\begin{Vmatrix} w^0\\ w^1\\ w^2\\ w^3 \end{Vmatrix} \mapsto h(\mathbf{w})=\sum_{m=1}^{3} w^m\,\boldsymbol{\sigma}_m=\begin{Vmatrix} w^0+w^3 & w^1-i\,w^2\\ w^1+i\,w^2 & w^0-w^3 \end{Vmatrix}. \tag{5.12}$$

Conversely, each hermitian matrix $\mathbf{h}$ can be mapped back to a 4-vector $\mathbf{w}\in\mathbb{R}^4$:

$$\begin{Vmatrix} h^{11} & h^{12}\\ \overline{h^{12}} & h^{22} \end{Vmatrix} \mapsto w(\mathbf{h})=\frac{1}{2}\begin{Vmatrix} h^{11}+h^{22}\\ h^{12}+\overline{h^{12}}\\ i\,h^{12}-i\,\overline{h^{12}}\\ h^{11}-h^{22} \end{Vmatrix}. \tag{5.13}$$

It is easy to see that the above mappings (5.12) and (5.13) are inverse to each other. They define an isomorphism of two linear spaces

$$\mathbb{R}^4 \ \underset{w}{\overset{h}{\rightleftarrows}}\ \mathbb{HC}^{2\times 2}. \tag{5.14}$$

Note that the determinant of the matrix $\mathbf{h}=h(\mathbf{w})$ coincides with the square of the norm of $\mathbf{w}$ measured in the diagonal Minkowski metric (5.7):

$$\det h(\mathbf{w})=(w^0)^2-(w^1)^2-(w^2)^2-(w^3)^2=g(\mathbf{w},\mathbf{w})=\sum_{i=0}^{3}\sum_{j=0}^{3} g_{ij}\,w^i\,w^j. \tag{5.15}$$

Let $\mathbf{U}$ be a complex 2×2 matrix and let $\det\mathbf{U}=1$. Then due to the isomorphism (5.14) we can write the following equality:

$$\mathbf{U}\,h(\mathbf{w})\,\mathbf{U}^{\dagger}=h(\mathbf{v}). \tag{5.16}$$

Here $\mathbf{U}^{\dagger}$ is the Hermitian transposition of $\mathbf{U}$. Due to the equality $\det\mathbf{U}=1$ and due to (5.15) for the components of the vector $\mathbf{v}$ in (5.16) we derive

$$v^i=\sum_{j=0}^{3} S^i_j\,w^j. \tag{5.17}$$

The components of the Lorentzian matrix S in (5.17) can be explicitly expressed through the components of the matrix $\mathbf{U}\in\mathrm{SL}(2,\mathbb{C})$ in (5.16) (see [8]). Here is the formula for S^0_0. It is derived by direct calculations:

$$S^0_0=\frac{|U^1_1|^2+|U^1_2|^2+|U^2_1|^2+|U^2_2|^2}{2}>0. \tag{5.18}$$

We shall not write the explicit expressions for other components of the matrix S, however, using them, we calculate the determinant of S:

$$\det S=(\det\mathbf{U})^2\,(\det\mathbf{U}^{\dagger})^2=1. \tag{5.19}$$

From (5.18) and (5.19) we get $S \in \mathrm{SO}^+(1,3)$ (see (5.9)). Thus, we have a mapping:

$$\varphi\colon\ \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3,\mathbb{R}). \tag{5.20}$$

Theorem 5.3. *The mapping φ in (5.20) is a group homomorphism. Its kernel consists of the following two matrices*

$$\boldsymbol{\sigma}_0 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \qquad\qquad -\boldsymbol{\sigma}_0 = \begin{Vmatrix} -1 & 0 \\ 0 & -1 \end{Vmatrix}. \tag{5.21}$$

Topologically, the mapping φ is a two-sheeted non-ramified covering of the 6-dimensional real manifold $\mathrm{SO}^+(1,3,\mathbb{R})$ by the other 6-dimensional real manifold $\mathrm{SL}(2,\mathbb{C})$.

More details concerning the theorem 5.3 and its proof can be found in [9]. This theorem is a base for defining the spinor bundle over the space-time M. Remember that any bundle is determined by its local trivializations and by transition functions relating any two trivializations with intersecting domains (see [10]). In the case of the tangent bundle TM of the space-time manifold M any local chart $U \subset M$ equipped with an orthonormal frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ determines a local trivialization of the bundle TM over U. Indeed, a point $q = (p, \mathbf{v})$ of the open set $\pi^{-1}(U)$ in this case is associated with 8 numbers

$$x^0,\ x^1,\ x^2,\ x^3,\ v^0,\ v^1,\ v^2,\ v^2,$$

where x^0, x^1, x^2, x^3 are the local coordinates of the point p in the local chart U and v^0, v^1, v^2, v^3 are the coefficients in the expansion (5.5) for the vector $\mathbf{v}$. Such a trivialization of TM is called an *orthonormal trivialization*. The theorem 5.1 says that orthonormal trivializations of TM do exist and their domains $\pi^{-1}(U)$ cover TM. Therefore, the orthonormal trivializations are sufficient for defining the bundle TM in whole. Any two orthonormal trivializations of the tangent bundle are related to each other by transition functions:

$$v^i = \sum_{j=0}^{3} S^i_j\,\tilde{v}^j, \qquad\qquad \tilde{v}^i = \sum_{j=0}^{3} T^i_j\,v^j. \tag{5.22}$$

Due to theorem 5.2 the transition functions (5.22) are given by the components of two mutually inverse Lorentzian matrices $S = S(p)$ and $T = T(p)$.

An orthonormal frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ is called a positively polarized right orthonormal frame if it is right frame in the sense of the orientation in the space-time M and if $\boldsymbol{\Upsilon}_0$ is a time-like unit vector in the Future cone in the sense of the polarization in the space-time M. By means of the proper choice of sign $\boldsymbol{\Upsilon}_i \to \pm\boldsymbol{\Upsilon}_i$ one can transform any orthonormal frame into a positively polarized right orthonormal frame. Therefore, the positively polarized right orthonormal trivializations of TM are sufficient to describe completely the tangent bundle TM. The transition functions (5.22) in this case are given by mutually inverse Lorentzian matrices

$$S(p) \in \mathrm{SO}^+(1,3,\mathbb{R}), \qquad\qquad T(p) \in \mathrm{SO}^+(1,3,\mathbb{R}). \tag{5.23}$$

Definition 5.2. A two-dimensional complex vector bundle SM over the four-dimensional real space-time manifold M is called the *spinor bundle* if for each positively polarized right orthonormal trivialization $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ of the tangent bundle TM there is a trivialization $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ of SM such that for any two positively polarized right orthonormal trivializations of the tangent bundle TM related to each other by means of the formulas

$$\tilde{\boldsymbol{\Upsilon}}_i=\sum_{j=0}^{3}S^j_i\,\boldsymbol{\Upsilon}_j,\qquad\qquad \boldsymbol{\Upsilon}_i=\sum_{j=0}^{3}T^j_i\,\tilde{\boldsymbol{\Upsilon}}_j \tag{5.24}$$

the associated trivializations $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2)$ of SM are related to each other by means of the formulas

$$\tilde{\boldsymbol{\Psi}}_i=\pm\sum_{j=1}^{2}\mathfrak{S}^j_i\,\boldsymbol{\Psi}_j,\qquad\qquad \boldsymbol{\Psi}_i=\pm\sum_{j=1}^{2}\mathfrak{T}^j_i\,\tilde{\boldsymbol{\Psi}}_j, \tag{5.25}$$

where $S=\varphi(\mathfrak{S})$, $T=\varphi(\mathfrak{T})$, and φ is the group homomorphism (5.20).

The definition 5.2 is self-consistent due to the theorem 5.3. However, there is the uncertainty in sign in the formulas (5.25) because of the nontrivial kernel (5.21) of the group homomorphism (5.20). Therefore, there could be some global topological obstructions for the existence of the spinor bundle SM over a particular manifold M. They are discussed in [8]. Below we shall assume that the topology of the actual physical space-time M is such that the spinor bundle SM over M introduced by the definition 5.2 does exist and is unique up to an isomorphism.

A remark. The trivialization $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ of the spinor bundle SM which is canonically associated with a positively polarized right orthonormal trivialization $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ of the tangent bundle TM in the definition 5.2 is not unique. There are exactly two trivializations $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ and $(U,\,-\boldsymbol{\Psi}_1,\,-\boldsymbol{\Psi}_2)$ associated with each such trivialization $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ of TM.

Let's return to the coordinate representations of spinors and spin-tensors that we discussed in section 4. Below we shall assume that all coordinate representations of spin-tensorial fields (4.3) are relative to some non-holonomic positively polarized right orthonormal frame $\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3$ in TM and its associated frame $\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2$ in SM. Let's denote by $\boldsymbol{\eta}^0,\,\boldsymbol{\eta}^1,\,\boldsymbol{\eta}^2,\,\boldsymbol{\eta}^3$ the dual frame for $\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3$. This means that $\boldsymbol{\eta}^0,\,\boldsymbol{\eta}^1,\,\boldsymbol{\eta}^2,\,\boldsymbol{\eta}^3$ are four covectorial fields such that

$$\bigl(\boldsymbol{\eta}^i,\,\boldsymbol{\Upsilon}_j\bigr)=\eta^i(\boldsymbol{\Upsilon}_j)=\delta^i_j. \tag{5.26}$$

Similarly, let's denote by $\boldsymbol{\vartheta}^{\,1},\,\boldsymbol{\vartheta}^{\,2}$ the dual frame for $\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2$. Here we have

$$\bigl(\boldsymbol{\vartheta}^{\,i},\,\boldsymbol{\Psi}_j\bigr)=\vartheta^{\,i}(\boldsymbol{\Psi}_j)=\delta^i_j \tag{5.27}$$

(see (3.1) for comparison). And finally, let's denote

$$\overline{\boldsymbol{\Psi}}_i=\tau(\boldsymbol{\Psi}_i),\qquad\qquad \overline{\boldsymbol{\vartheta}}^{\,i}=\tau(\boldsymbol{\vartheta}^{\,i}), \tag{5.28}$$

where by τ we designate the semilinear mappings considered above (see (2.9), (3.4), (3.6), (3.8), (4.4), (4.5)). The frames $\overline{\mathbf{\Psi}}_1$, $\overline{\mathbf{\Psi}}_2$ and $\overline{\boldsymbol{\vartheta}}^{\,1}$, $\overline{\boldsymbol{\vartheta}}^{\,2}$ are dual to each other:

$$\bigl(\overline{\boldsymbol{\vartheta}}^{\,i},\,\overline{\mathbf{\Psi}}_j\bigr)=\overline{\vartheta}^{\,i}(\mathbf{\Psi}_j)=\delta^i_j \tag{5.29}$$

Now let's consider the following tensor products:

$$\mathbf{\Upsilon}^{k_1\ldots\,k_s}_{h_1\ldots\,h_r}=\mathbf{\Upsilon}_{h_1}\otimes\ldots\otimes\mathbf{\Upsilon}_{h_r}\otimes\boldsymbol{\eta}^{k_1}\otimes\ldots\otimes\boldsymbol{\eta}^{k_s}, \tag{5.30}$$

$$\mathbf{\Psi}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}=\mathbf{\Psi}_{i_1}\otimes\ldots\otimes\mathbf{\Psi}_{i_\alpha}\otimes\boldsymbol{\vartheta}^{j_1}\otimes\ldots\otimes\boldsymbol{\vartheta}^{j_\beta}, \tag{5.31}$$

$$\overline{\mathbf{\Psi}}^{\bar{j}_1\ldots\,\bar{j}_\gamma}_{\bar{i}_1\ldots\,\bar{i}_\nu}=\overline{\mathbf{\Psi}}_{\bar{i}_1}\otimes\ldots\otimes\overline{\mathbf{\Psi}}_{\bar{i}_\nu}\otimes\overline{\boldsymbol{\vartheta}}^{\bar{j}_1}\otimes\ldots\otimes\overline{\boldsymbol{\vartheta}}^{\bar{j}_\gamma}. \tag{5.32}$$

Using (5.30), (5.31), and (5.32), we introduce the following tensor product:

$$\mathbf{\Psi}^{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_s}_{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_r}=\mathbf{\Psi}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}\otimes\overline{\mathbf{\Psi}}^{\bar{j}_1\ldots\,\bar{j}_\gamma}_{\bar{i}_1\ldots\,\bar{i}_\nu}\otimes\mathbf{\Upsilon}^{k_1\ldots\,k_s}_{h_1\ldots\,h_r}. \tag{5.33}$$

Then the coordinate representation (4.3) of a spin-tensorial field $\mathbf{Y}$ means that it is represented as an expansion in the basis of spin-tensorial fields (5.33):

$$\mathbf{Y}=\sum^{2}_{\substack{i_1,\,\ldots,\,i_\alpha\\ j_1,\,\ldots,\,j_\beta}}\!\!\ldots\sum^{2}\sum^{2}_{\substack{\bar{i}_1,\,\ldots,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots,\,\bar{j}_\gamma}}\!\!\ldots\sum^{2}\sum^{3}_{\substack{h_1,\,\ldots,\,h_r\\ k_1,\,\ldots,\,k_s}}\!\!\ldots\sum^{3} Y^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_r}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_s}\;\mathbf{\Psi}^{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_s}_{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_r}. \tag{5.34}$$

The coordinate representation (4.6) of τ is based on the expansion (5.34) and on the formulas (5.28) defining $\overline{\mathbf{\Psi}}_i$ and $\overline{\boldsymbol{\vartheta}}^{\,i}$. Due to the duality (biorthogonality) relations (5.26), (5.27), and (5.29) we can perform the contraction of (4.3) with respect to any pair within three groups of indices in (4.3), i. e. we can contract h_m with k_n, i_m with j_n, and $\bar{i}_m$ with $\bar{j}_n$.

6. The spin-metric.

Let's look at the definition 5.2 again. The transition matrices S and T in (5.24) belong to the special orthochronous Lorentz group (see (5.23)). This fact reflects three basic structures available in the space-time manifold M: the metric, the orientation, and the polarization. The matrices $\mathfrak{S}$ and $\mathfrak{T}$ in (5.25) are not complex 2×2 matrices of general form. They belong to the special linear group:

$$\mathfrak{S}(p)\in\mathrm{SL}(2,\mathbb{C}),\qquad\qquad \mathfrak{T}(p)\in\mathrm{SL}(2,\mathbb{C}). \tag{6.1}$$

There is a special structure in SM responsible for (6.1). Let's denote by $\mathbf{d}$ the skew-symmetric spin-tensor with the components

$$d_{ij}=\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix} \tag{6.2}$$

in each trivialization $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2)$ of SM associated with some positively polarized right orthonormal trivialization of TM. In other words, let's define $\mathbf{d}$ as

$$\mathbf{d}=\boldsymbol{\vartheta}^{1}\otimes\boldsymbol{\vartheta}^{2}-\boldsymbol{\vartheta}^{2}\otimes\boldsymbol{\vartheta}^{1}. \tag{6.3}$$

This definition (6.3) of $\mathbf{d}$ is self-consistent because of the well-known identities valid for the matrix (6.2) and arbitrary 2×2 matrices $\mathfrak{S}$ and $\mathfrak{T}$:

$$\begin{aligned}&\sum_{i=1}^{2}\sum_{j=1}^{2} d_{ij}\,\mathfrak{S}^i_k\,\mathfrak{S}^j_q=\det\mathfrak{S}\cdot d_{kq},\\ &\sum_{i=1}^{2}\sum_{j=1}^{2} d_{ij}\,\mathfrak{T}^i_k\,\mathfrak{T}^j_q=\det\mathfrak{T}\cdot d_{kq}.\end{aligned}\tag{6.4}$$

Due to (6.1) and (6.4) the formula (6.3) for $\mathbf{d}$ is compatible with the transition formulas (5.25). The skew-symmetric spin-tensor $\mathbf{d}$ plays the same role for the spinor bundle SM as the metric tensor $\mathbf{g}$ for TM. In particular, the matrix (6.2) is used for lowering spinor indices of spin-tensors:

$$Y^{\dots\,\dots\,\dots}_{\dots\,j\,\dots}=\sum_{i=1}^{2}Y^{\dots\,i\,\dots}_{\dots\,\dots\,\dots}\;d_{ij}.\tag{6.5}$$

For this reason $\mathbf{d}$ is called the *spin-metric tensor*. The *dual spin-metric tensor* is defined by the matrix inverse to (6.2):

$$d^{ij}=\begin{Vmatrix}0 & -1\\ 1 & 0\end{Vmatrix}.\tag{6.6}$$

It is denoted by the same symbol $\mathbf{d}$. Therefore, we have the formula

$$\mathbf{d}=-\boldsymbol{\Psi}_1\otimes\boldsymbol{\Psi}_2+\boldsymbol{\Psi}_2\otimes\boldsymbol{\Psi}_1.\tag{6.7}$$

The dual spin-metric tensor given by the formula (6.6) or by the equivalent formula (6.7) is used for raising spinor indices of spin-tensors:

$$Y^{\dots\,j\,\dots}_{\dots\,\dots\,\dots}=\sum_{i=1}^{2}Y^{\dots\,\dots\,\dots}_{\dots\,i\,\dots}\;d^{ij}.\tag{6.8}$$

Applying τ to (6.3) and (6.7) we get the *conjugate spin-metric tensors*:

$$\bar{\mathbf{d}}=\overline{\boldsymbol{\vartheta}}^1\otimes\overline{\boldsymbol{\vartheta}}^2-\overline{\boldsymbol{\vartheta}}^2\otimes\overline{\boldsymbol{\vartheta}}^1,\tag{6.9}$$

$$\bar{\mathbf{d}}=-\overline{\boldsymbol{\Psi}}_1\otimes\overline{\boldsymbol{\Psi}}_2+\overline{\boldsymbol{\Psi}}_2\otimes\overline{\boldsymbol{\Psi}}_1.\tag{6.10}$$

The conjugate spin-metric tensors (6.9) and (6.10) are represented by the same matrices (6.2) and (6.6), but we use the barred d for denoting their components:

$$\bar{d}_{\bar{i}\bar{j}}=\begin{Vmatrix}0 & 1\\ -1 & 0\end{Vmatrix},\qquad\qquad \bar{d}^{\bar{i}\bar{j}}=\begin{Vmatrix}0 & -1\\ 1 & 0\end{Vmatrix}.\tag{6.11}$$

The matrices (6.11) are used for raising and lowering the barred spinor indices:

$$Y^{\dots\,\dots\,\dots}_{\dots\,\bar{j}\,\dots}=\sum_{\bar{i}=1}^{2}Y^{\dots\,\bar{i}\,\dots}_{\dots\,\dots\,\dots}\;\bar{d}_{\bar{i}\bar{j}}\,,\tag{6.12}$$

$$Y^{\dots\,\bar{j}\,\dots}_{\dots\,\dots\,\dots}=\sum_{i=1}^{2}Y^{\dots\,\dots\,\dots}_{\dots\,\bar{i}\,\dots}\;\bar{d}^{\bar{i}\bar{j}}.\tag{6.13}$$

These formulas (6.12) and (6.13) are analogs of the formulas (6.5) and (6.8).

7. Tensors to spin-tensors conversion.

As we can see in (4.3), a general spin-tensorial field $\mathbf{Y}$ has three groups of indices: regular spinor indices, barred spinor indices, and tensorial indices. In this section we shall show that tensorial indices can be converted into spinor and barred spinor indices. For this purpose let's return to the formula (5.16). Let's denote $h = h(\mathbf{v})$, $\tilde{h} = h(\mathbf{w})$, and let's substitute $\mathbf{U} = \mathfrak{S}$ into (5.16). Then we find that (5.16) is equivalent to the following equality:

$$h^{i\bar{i}} = \sum_{\bar{q}=1}^{2}\sum_{q=1}^{2} \mathfrak{S}^i_q\;\tilde{h}^{q\bar{q}}\;\overline{\mathfrak{S}^{\bar{i}}_{\bar{q}}}. \tag{7.1}$$

This formula (7.1) coincides with the transformation rule for the components of a spin-tensor with one regular upper index and one barred upper index under the change of frame given by the formulas (5.25). The sign uncertainty in (5.25) does not affect the formula (7.1).

Similarly, if we denote $\mathbf{w} = \tilde{\mathbf{v}}$, we can write the formula (5.17) in the form coinciding with the first formula in (5.22):

$$v^i = \sum_{j=0}^{3} S^i_j\,\tilde{v}^j. \tag{7.2}$$

The formula (7.2) expresses the transformation rule for the components of a tensor with one upper index under the change of a frame given by the formulas (5.24). Due to (7.1) and (7.2) the upper mapping h in (5.14) is interpreted as the mapping

$$h\colon\, T_p(M) \to S_p(M) \otimes S^{\dagger *}_p(M). \tag{7.3}$$

Note that $h(\mathbf{w})$ in (5.12) is a Hermitian matrix. For the spin-tensor $\mathbf{h}$ represented by this matrix in (7.1) this yields $\tau(\mathbf{h}) = \mathbf{h}$, where τ is the mapping defined by the formula (4.6). Indeed, we can easily verify that

$$h^{i\bar{i}} = \overline{h^{\bar{i}i}} \tag{7.4}$$

Due to (7.4) we can take the Hermitian symmetric part of the tensor product in the right hand side of (7.3) and thus write (5.14) as follows:

$$T_p(M) \underset{w}{\overset{h}{\rightleftarrows}} \operatorname{HSym}(S_p(M) \otimes S^{\dagger *}_p(M)) \tag{7.5}$$

The upper mapping h in (7.5) is given by a special spin-tensorial field $\mathbf{G}$ with two upper spinor indices and one lower tensorial index:

$$h^{i\bar{i}} = \sum_{q=0}^{3} G^{i\bar{i}}_q\,w^q. \tag{7.6}$$

Its components $G^{i\bar{i}}_j$ in (7.6) are called *Infeld-van der Waerden symbols*. The numeric values of these symbols are calculated through the components of Pauli matrices (5.10) and (5.11) because of the formula (5.12):

$$\begin{aligned}
&G^{11}_0 = 1, && G^{11}_1 = 0, && G^{11}_2 = 0, && G^{11}_3 = 1,\\
&G^{12}_0 = 0, && G^{12}_1 = 1, && G^{12}_2 = -i, && G^{12}_3 = 0,\\
&G^{21}_0 = 0, && G^{21}_1 = 1, && G^{21}_2 = i, && G^{21}_3 = 0,\\
&G^{22}_0 = 1, && G^{22}_1 = 0, && G^{22}_2 = 0, && G^{22}_3 = -1.
\end{aligned}\tag{7.7}$$

The inverse mapping w in (7.5) is also given by some special spin-tensorial field. We use the same symbol $\mathbf{G}$ for this field:

$$w^q = \sum_{i=1}^{2}\sum_{\bar{i}=1}^{2} G^{\,q}_{i\bar{i}}\ h^{i\bar{i}}.\tag{7.8}$$

Its components $G^{\,q}_{i\bar{i}}$ in (7.8) are called the *inverse Infeld-van der Waerden symbols*. By means of (5.13) one can calculate them in explicit form:

$$\begin{aligned}
&G^0_{11} = \frac{1}{2}, && G^0_{12} = 0, && G^0_{21} = 0, && G^0_{22} = \frac{1}{2},\\
&G^1_{11} = 0, && G^1_{12} = \frac{1}{2}, && G^1_{21} = \frac{1}{2}, && G^1_{22} = 0,\\
&G^2_{11} = 0, && G^2_{12} = \frac{i}{2}, && G^2_{21} = -\frac{i}{2}, && G^2_{22} = 0,\\
&G^3_{11} = \frac{1}{2}, && G^3_{12} = 0, && G^3_{21} = 0, && G^3_{22} = -\frac{1}{2}.
\end{aligned}\tag{7.9}$$

The following symmetry properties of the Infeld-van der Waerden symbols are obvious. They are derived from (7.7) and (7.9):

$$G^{i\bar{i}}_q = \overline{G^{\bar{i}i}_q}, \qquad\qquad G^{\,q}_{i\bar{i}} = \overline{G^{\,q}_{\bar{i}i}}.\tag{7.10}$$

The transformation (7.6) can be applied not only to a tensorial field with one upper index. We can apply this transformation to any spin-tensorial with at least one non-spinor upper index. By analogy to (7.6) we can do it in the following way

$$\hat{Y}^{\ldots\, i \ldots\, \bar{i} \ldots}_{\ldots\ldots\ldots} = \sum_{q=0}^{3} Y^{\ldots\, q\, \ldots}_{\ldots\ldots\ldots}\ G^{i\bar{i}}_q.\tag{7.11}$$

In (7.11) with the use of the Infeld-van der Waerden symbols (7.7) the tensorial index q is converted into the pair of spinor indices i and $\bar{i}$. The spin-tensorial field $\mathbf{Y}$ in (7.11) can be recovered from $\hat{\mathbf{Y}}$ with the use of the inverse Infeld-van der Waerden symbols (7.9) by means of the formula

$$Y^{\ldots\, q\, \ldots}_{\ldots\ldots\ldots} = \sum_{i=1}^{2}\sum_{\bar{i}=1}^{2} \hat{Y}^{\ldots\, i \ldots\, \bar{i} \ldots}_{\ldots\ldots\ldots}\ G^{\,q}_{i\bar{i}}\,.\tag{7.12}$$

Similarly, if we apply the transformation (7.12) to an arbitrary spin-tensorial field $\hat{\mathbf{Y}}$ with two upper spinor indices i and $\bar{i}$, then we can recover it from $\mathbf{Y}$ by means of the formula (7.11). The Infeld-van der Waerden symbols satisfy the identities

$$\sum_{i=1}^{2}\sum_{\bar{i}=1}^{2} G^{i\bar{i}}_{p}\, G^{q}_{i\bar{i}} = \delta^{q}_{p}, \qquad\qquad \sum_{q=0}^{3} G^{i\bar{i}}_{q}\, G^{q}_{j\bar{j}} = \delta^{i}_{j}\,\delta^{\bar{i}}_{\bar{j}}. \tag{7.13}$$

The identities (7.13) are compatible with (7.10). They are derived by means of direct calculations from (7.7) and (7.9) and used in order to prove that the transformations (7.11) and (7.12) are inverse to each other.

As an example of their usage, let's apply the conversion formulas (7.11) and (7.12) to the metric tensor $\mathbf{g}$. The result is given by the following identities:

$$\sum_{p=0}^{3}\sum_{q=0}^{3} g_{p\,q}\, G^{p}_{i\bar{i}}\, G^{q}_{j\bar{j}} = \frac{d_{ij}\,\bar{d}_{\bar{i}\bar{j}}}{2}, \tag{7.14}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} \frac{d_{ij}\,\bar{d}_{\bar{i}\bar{j}}}{2}\, G^{i\bar{i}}_{p}\, G^{j\bar{j}}_{q} = g_{p\,q}. \tag{7.15}$$

Applying the formulas (7.11) and (7.12) to the dual metric tensor, we get

$$\sum_{p=0}^{3}\sum_{q=0}^{3} g^{p\,q}\, G^{i\bar{i}}_{p}\, G^{j\bar{j}}_{q} = 2\, d^{\,ij}\,\bar{d}^{\,\bar{i}\bar{j}}, \tag{7.16}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} 2\, d^{\,ij}\,\bar{d}^{\,\bar{i}\bar{j}}\, G^{p}_{i\bar{i}}\, G^{q}_{j\bar{j}} = g^{p\,q}. \tag{7.17}$$

The identities (7.14), (7.15), (7.16), and (7.17) are derived from (7.7) and (7.9) by direct calculations.

A remark. The mapping (7.3) can be extended to the complexified tangent space $\mathbb{C}T_p(M)$. In this case the symmetry condition (7.4) is not valid and we have

$$\mathbb{C}T_p(M) \underset{w}{\overset{h}{\rightleftarrows}} S_p(M)\otimes S^{\dagger *}_p(M). \tag{7.18}$$

This formula (7.18) is the replacement of the formula (7.5) for the case of the complexified tangent space $\mathbb{C}T_p(M)$.

8. Dirac spinors.

The definition 5.2 introducing the concept of the spinor bundle over the space-time manifold M is a self-consistent and self-sufficient definition. However, the transition matrices S and T in this definition are restricted to the special orthochronous Lorentz group $\mathrm{SO}^{+}(1,3,\mathbb{R})$ (see (5.23)). In order to extend $\mathrm{SO}^{+}(1,3,\mathbb{R})$ up to the complete Lorentz group $\mathrm{O}(1,3,\mathbb{R})$ one should add the following two matrices to it:

$$\theta = \begin{Vmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{Vmatrix}, \qquad\qquad P = \begin{Vmatrix} 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

For this purpose the other spinor bundle is introduced. It is constructed as the direct sum of SM and its Hermitian conjugate bundle $S^{\dagger}M$:

$$DM = SM \oplus S^{\dagger}M. \tag{8.1}$$

The elements of this 4-dimensional complex bundle (8.1) are called *Dirac spinors*. Though the Dirac spinors are very popular in physics, we shall not consider them in this paper since due to (8.1) they are not primary objects — they are reduced to the standard 2-components spinors from SM.

9. Composite spin-tensorial bundles.

Let's remember the section 4. There we have introduced the spin-tensorial bundle $S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{m}_{n}M$. It is composed by spin tensors of the type $(\alpha, \beta|\nu, \gamma|m, n)$. Suppose that we have several bundles of that sort over the space-time M. Let's denote them

$$S^{\alpha_1}_{\beta_1}\bar{S}^{\nu_1}_{\gamma_1}T^{m_1}_{n_1}M, \ \ldots\ , S^{\alpha_J}_{\beta_J}\bar{S}^{\nu_J}_{\gamma_J}T^{m_J}_{n_J}M. \tag{9.1}$$

In addition to (9.1) we consider several tensor bundles

$$T^{r_1}_{s_1}M, \ \ldots\ , T^{r_Q}_{s_Q}M. \tag{9.2}$$

Then we construct the direct sum of the bundles (9.1) and (9.2):

$$N = S^{\alpha_1}_{\beta_1}\bar{S}^{\nu_1}_{\gamma_1}T^{m_1}_{n_1}M \oplus \ldots \oplus S^{\alpha_J}_{\beta_J}\bar{S}^{\nu_J}_{\gamma_J}T^{m_J}_{n_J}M \oplus T^{r_1}_{s_1}M \oplus \ldots \oplus T^{r_Q}_{s_Q}M \tag{9.3}$$

We shall call N in (9.3) the *composite spin-tensorial bundle*. By definition a point q of the composite spin-tensorial bundle (9.3) is a list

$$q = (p,\, \mathbf{S}[1],\ \ldots\ , \mathbf{S}[J],\, \mathbf{T}[1],\ \ldots\ , \mathbf{T}[Q]), \tag{9.4}$$

where p is a point of the space-time M, $\mathbf{S}[1], \ldots, \mathbf{S}[J]$ are spin-tensors of the types $(\alpha_1, \beta_1|\nu_1, \gamma_1|m_1, n_1), \ldots, (\alpha_J, \beta_J|\nu_J, \gamma_J|m_J, n_J)$, and $\mathbf{T}[1], \ldots, \mathbf{T}[Q]$ are tensors of the types $(r_1, s_1), \ldots, (r_Q, s_Q)$ at the point p. Comparing (9.4) with (1.2) and (1.3), we find that the composite tensor bundle is a suitable geometric object for choosing it as the domain of the functions (1.2) and (1.3). However, it is more convenient to extend this domain. Indeed, note that the real tensor bundles (9.2) can be extended up to the complexified tensor bundles

$$\mathbb{C}T^{r_1}_{s_1}M, \ \ldots\ , \mathbb{C}T^{r_Q}_{s_Q}M. \tag{9.5}$$

Then remember that the complexified tensor bundles (9.5) coincide with the following spin-tensorial bundles (see formula (4.2)):

$$S^0_0\bar{S}^0_0T^{r_1}_{s_1}M, \ \ldots\ , S^0_0\bar{S}^0_0T^{r_Q}_{s_Q}M \tag{9.6}$$

Due to (9.6) we can replace the direct sum (9.3) by the other direct sum

$$N = S^{\alpha_1}_{\beta_1}\bar{S}^{\nu_1}_{\gamma_1}T^{m_1}_{n_1}M \oplus \ldots \oplus S^{\alpha_{J+Q}}_{\beta_{J+Q}}\bar{S}^{\nu_{J+Q}}_{\gamma_{J+Q}}T^{m_{J+Q}}_{n_{J+Q}}M, \tag{9.7}$$

where $\alpha_P = \beta_P = \nu_P = \gamma_P = 0$, $m_P = r_{P-J}$, $n_P = s_{P-J}$ for $J < P \leqslant J + Q$. Passing from (9.3) to the composite tensor bundle (9.7), we can treat in a more uniform way the components of the list (9.4). Indeed, now we can write

$$q = (p,\, \mathbf{S}[1],\, \dots\,,\, \mathbf{S}[J+Q]), \tag{9.8}$$

where $\mathbf{T}[1] = \mathbf{S}[J{+}1],\, \dots\,,\, \mathbf{T}[Q] = \mathbf{S}[J{+}Q]$. In other words, upon complexification, we treat the tensors $\mathbf{T}[1],\, \dots\,,\, \mathbf{T}[Q]$ as the spin-tensors $\mathbf{S}[J{+}1],\, \dots\,,\, \mathbf{S}[J{+}Q]$. The *canonical projection* of the composite bundle (9.7) is defined as the map

$$\pi\colon\, N \to M \tag{9.9}$$

that takes a point q of the form (9.8) to the point $p \in M$. Despite the complexification, the complex bundle N is a real smooth manifold. Its dimension

$$\dim_{\mathbb{R}} N = 2 \sum_{i=1}^{J+Q} 2^{\,\alpha_i+\nu_i+\beta_i+\gamma_i}\ 4^{m_i+n_i} + 4. \tag{9.10}$$

Below we shall use local charts of some special sort in N. In order to construct a local chart of that sort one should choose some local chart U of M equipped with a positively polarized right oriented orthonormal frame $\boldsymbol{\Upsilon}_0,\, \boldsymbol{\Upsilon}_1,\, \boldsymbol{\Upsilon}_2,\, \boldsymbol{\Upsilon}_3$ and with its associated spinor frame $\boldsymbol{\Psi}_1,\, \boldsymbol{\Psi}_2$. The first part of the local coordinates of a point q in (9.8) is the local coordinates of its image $p = \pi(q)$ under the projection (9.9):

$$x^0,\, x^1,\, x^2,\, x^3. \tag{9.11}$$

The second part of the local coordinates of q is composed by the components of the spin-tensors $\mathbf{S}[1],\, \dots\,,\, \mathbf{S}[J+Q]$ in (9.8):

$$\begin{gathered} S^{1\,\dots\,1\,1\,\dots\,1\,0\,\dots\,0}_{1\,\dots\,1\,1\,\dots\,1\,0\,\dots\,0}[1],\, \dots\,,\, S^{2\,\dots\,2\,2\,\dots\,2\,3\,\dots\,3}_{2\,\dots\,2\,2\,\dots\,2\,3\,\dots\,3}[1],\, \dots\\ \dots\,,\, S^{1\,\dots\,1\,1\,\dots\,1\,0\,\dots\,0}_{1\,\dots\,1\,1\,\dots\,1\,0\,\dots\,0}[J+Q],\, \dots\,,\, S^{2\,\dots\,2\,2\,\dots\,2\,3\,\dots\,3}_{2\,\dots\,2\,2\,\dots\,2\,3\,\dots\,3}[J+Q]. \end{gathered} \tag{9.12}$$

Each part of the variables yields its own contribution to the total dimension of N in (9.10). Note that (9.12) are complex variables. For this reason the contribution of (9.12) in (9.10) is taken with the factor 2.

Under a change of local charts the coordinates (9.11) are transformed traditionally by means of the transition functions

$$\begin{cases} \tilde{x}^0 = \tilde{x}^1(x^0,\, \dots\,, x^3),\\ \dots\dots\dots\dots\dots\\ \tilde{x}^3 = \tilde{x}^n(x^0,\, \dots\,, x^3). \end{cases} \qquad \begin{cases} x^0 = \tilde{x}^1(\tilde{x}^0,\, \dots\,, \tilde{x}^3),\\ \dots\dots\dots\dots\dots\\ x^3 = \tilde{x}^n(\tilde{x}^0,\, \dots\,, \tilde{x}^3). \end{cases} \tag{9.13}$$

The coordinates (9.12) are transformed as the components of spin-tensors

$$\begin{cases} \tilde{S}^{i_1\dots\, i_\alpha \bar{i}_1\dots\, \bar{i}_\nu h_1\dots\, h_m}_{j_1\dots\, j_\beta \bar{j}_1\dots\, \bar{j}_\gamma k_1\dots\, k_n}[P] = \sum^{2}_{\substack{a_1,\,\dots\,,\,a_\alpha\\ b_1,\,\dots\,,\,b_\beta}} \!\!\dots\! \sum^{2} \sum^{2}_{\substack{\bar{a}_1,\,\dots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\dots\,,\,\bar{b}_\gamma}} \!\!\dots\! \sum^{2} \sum^{3}_{\substack{c_1,\,\dots\,,\,c_m\\ d_1,\,\dots\,,\,d_n}} \!\!\dots\! \sum^{3} \mathfrak{T}^{i_1}_{a_1} \dots\, \mathfrak{T}^{i_\alpha}_{a_\alpha} \times\\ \qquad \times\ \mathfrak{S}^{b_1}_{j_1} \dots\, \mathfrak{S}^{b_\beta}_{j_\beta}\ \overline{\mathfrak{T}^{\bar{i}_1}_{\bar{a}_1}} \dots\, \overline{\mathfrak{T}^{\bar{i}_\nu}_{\bar{a}_\nu}}\ \ \overline{\mathfrak{S}^{\bar{b}_1}_{\bar{j}_1}} \dots\, \overline{\mathfrak{S}^{\bar{b}_\gamma}_{\bar{j}_\gamma}}\ T^{h_1}_{c_1} \dots\, T^{h_m}_{c_m} \times\\ \qquad \times\ S^{d_1}_{k_1} \dots\, S^{d_n}_{k_n}\ S^{\,a_1\dots\, a_\alpha \bar{a}_1\dots\, \bar{a}_\nu\, c_1\dots\, c_m}_{\,b_1\dots\, b_\beta \bar{b}_1\dots\, \bar{b}_\gamma\, d_1\dots\, d_n}[P], \end{cases} \tag{9.14}$$

$$
\begin{cases}
S^{i_1\dots\,i_\alpha\bar i_1\dots\,\bar i_\nu h_1\dots\,h_m}_{j_1\dots\,j_\beta\bar j_1\dots\,\bar j_\gamma k_1\dots\,k_n}[P]=\sum^{2}_{\substack{a_1,\,\dots\,,\,a_\alpha\\ b_1,\,\dots\,,\,b_\beta}}\!\!\dots\!\sum\;\sum^{2}_{\substack{\bar a_1,\,\dots\,,\,\bar a_\nu\\ \bar b_1,\,\dots\,,\,\bar b_\gamma}}\!\!\dots\!\sum\;\sum^{3}_{\substack{c_1,\,\dots\,,\,c_m\\ d_1,\,\dots\,,\,d_n}}\!\!\dots\!\sum\mathfrak{S}^{i_1}_{a_1}\dots\,\mathfrak{S}^{i_\alpha}_{a_\alpha}\,\times\\
\qquad\times\;\mathfrak{T}^{b_1}_{j_1}\dots\,\mathfrak{T}^{b_\beta}_{j_\beta}\;\overline{\mathfrak{S}^{\bar i_1}_{\bar a_1}}\dots\,\overline{\mathfrak{S}^{\bar i_\nu}_{\bar a_\nu}}\;\;\overline{\mathfrak{T}^{\bar b_1}_{\bar j_1}}\dots\,\overline{\mathfrak{T}^{\bar b_\gamma}_{\bar j_\gamma}}\;S^{h_1}_{c_1}\dots\,S^{h_m}_{c_m}\,\times\\
\qquad\quad\times\;T^{d_1}_{k_1}\dots\,T^{d_n}_{k_n}\;\tilde S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P],
\end{cases}
\tag{9.15}
$$

where $\alpha=\alpha_P$, $\beta=\beta_P$, $\nu=\nu_P$, $\gamma=\gamma_P$, $m=m_P$, $n=n_P$, and the integer number P runs from 1 to $J+Q$. The coordinates (9.12) are complex numbers. Actually, their real and imaginary parts form local coordinates of a point q of the real manifold N. However, using the complex numbers (9.12) is preferable at least because the transition functions (9.14) and (9.15) look more simple in terms of these complex numbers.

The components of the transition matrices $\mathfrak{S}$, $\mathfrak{T}$, S, and T in (9.14) and (9.15) are frame relative, not coordinate relative. For this reason they do not depend on the transition functions in (9.13). The components of the matrices $\mathfrak{S}\in\mathrm{SL}(2,\mathbb{C})$ and $\mathfrak{T}\in\mathrm{SL}(2,\mathbb{C})$ are taken from the frame relationships

$$
\tilde{\boldsymbol\Psi}_i=\sum^{2}_{j=1}\mathfrak{S}^j_i\,\boldsymbol\Psi_j,\qquad\qquad \boldsymbol\Psi_i=\sum^{2}_{j=1}\mathfrak{T}^j_i\,\tilde{\boldsymbol\Psi}_j.
\tag{9.16}
$$

The Lorentzian matrices $S=\varphi(\mathfrak{S})\in\mathrm{SO}^+(1,3,\mathbb{R})$ and $T=\varphi(\mathfrak{T})\in\mathrm{SO}^+(1,3,\mathbb{R})$ are obtained by applying the homomorphism (5.20). Their components could be taken from the corresponding frame relationships for the associated frames:

$$
\tilde{\boldsymbol\Upsilon}_i=\sum^{3}_{j=0}S^j_i\,\boldsymbol\Upsilon_j,\qquad\qquad \boldsymbol\Upsilon_i=\sum^{3}_{j=0}T^j_i\,\tilde{\boldsymbol\Upsilon}_j.
\tag{9.17}
$$

Compare (9.16) and (9.17) with (5.24) and (5.25). Note that the sign uncertainty in (9.16) now is absent.

Let's consider the tangent space $T_q(N)$ of the manifold (9.7) at some of its points $q\in N$. Its complexification $\mathbb{C}\otimes T_q(N)$ is the span of the vectors

$$
\mathbf{X}_i=\frac{\partial}{\partial x^i},
\tag{9.18}
$$

$$
\mathbf{W}^{j_1\dots\,j_\beta\,\bar j_1\dots\,\bar j_\gamma\,k_1\dots\,k_n}_{i_1\dots\,i_\alpha\,\bar i_1\dots\,\bar i_\nu\,h_1\dots\,h_m}[P]=\frac{\partial}{\partial S^{\,i_1\dots\,i_\alpha\,\bar i_1\dots\,\bar i_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar j_1\dots\,\bar j_\gamma\,k_1\dots\,k_n}[P]},
\tag{9.19}
$$

$$
\bar{\mathbf{W}}^{\bar j_1\dots\,\bar j_\gamma\,j_1\dots\,j_\beta\,k_1\dots\,k_n}_{\bar i_1\dots\,\bar i_\nu\,i_1\dots\,i_\alpha\,h_1\dots\,h_m}[P]=\frac{\partial}{\partial\overline{S^{\,i_1\dots\,i_\alpha\,\bar i_1\dots\,\bar i_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar j_1\dots\,\bar j_\gamma\,k_1\dots\,k_n}[P]}}.
\tag{9.20}
$$

Note that $\alpha=\alpha_P$, $\beta=\beta_P$, $\nu=\nu_P$, $\gamma=\gamma_P$, $m=m_P$, $n=n_P$, and P runs from 1 to $J+Q$ in (9.19) and in (9.20). Under the change of local coordinates given by the formulas (9.13), (9.14), and (9.15) the tangent vectors (9.18), (9.19), and (9.20) are transformed according to the next five formulas:

$$
\begin{aligned}
\tilde{\mathbf{X}}_j = {} & \sum_{i=0}^{3} \frac{\partial x^i}{\partial \tilde{x}^j} \mathbf{X}_i + \sum_{P=1}^{J+Q} \sum_{\substack{i_1,\ldots,i_\alpha\\ j_1,\ldots,j_\beta\\ a_1,\ldots,a_\alpha\\ b_1,\ldots,b_\beta}}^{2} \!\!\ldots\! \sum^{2} \sum_{\substack{\bar{i}_1,\ldots,\bar{i}_\nu\\ \bar{j}_1,\ldots,\bar{j}_\gamma\\ \bar{a}_1,\ldots,\bar{a}_\nu\\ \bar{b}_1,\ldots,\bar{b}_\gamma}}^{2} \!\!\ldots\! \sum^{2} \sum_{\substack{h_1,\ldots,h_m\\ k_1,\ldots,k_n\\ c_1,\ldots,c_m\\ d_1,\ldots,d_n}}^{3} \!\!\ldots\! \sum^{3} \mathfrak{S}^{i_1}_{a_1} \ldots \mathfrak{S}^{i_\alpha}_{a_\alpha} \times \\
& \times \mathfrak{T}^{b_1}_{j_1} \ldots \mathfrak{T}^{b_\beta}_{j_\beta} \, \overline{\mathfrak{S}^{\bar{i}_1}_{\bar{a}_1}} \ldots \overline{\mathfrak{S}^{\bar{i}_\nu}_{\bar{a}_\nu}} \;\; \overline{\mathfrak{T}^{\bar{b}_1}_{\bar{j}_1}} \ldots \overline{\mathfrak{T}^{\bar{b}_\gamma}_{\bar{j}_\gamma}} \, S^{h_1}_{c_1} \ldots S^{h_m}_{c_m} \, T^{d_1}_{k_1} \ldots T^{d_n}_{k_n} \times \\
& \times \Biggl(\sum_{\mu=1}^{\alpha} \sum_{v_\mu=1}^{2} \Biggl(\sum_{a=1}^{2} \mathfrak{T}^{a_\mu}_{a} \frac{\partial \mathfrak{S}^{a}_{v_\mu}}{\partial \tilde{x}^j} \Biggr) \tilde{S}^{\,a_1 \ldots\, v_\mu \ldots\, a_\alpha \, \bar{a}_1 \ldots\, \bar{a}_\nu \, c_1 \ldots\, c_m}_{\,b_1 \ldots \ldots \ldots\, b_\beta \, \bar{b}_1 \ldots\, \bar{b}_\gamma \, d_1 \ldots\, d_n} [P] + \\
& + \sum_{\mu=1}^{\beta} \sum_{w_\mu=1}^{2} \Biggl(\sum_{a=1}^{2} \frac{\partial \mathfrak{T}^{w_\mu}_{a}}{\partial \tilde{x}^j} \mathfrak{S}^{a}_{b_\mu} \Biggr) \tilde{S}^{\,a_1 \ldots \ldots \ldots\, a_\alpha \, \bar{a}_1 \ldots\, \bar{a}_\nu \, c_1 \ldots\, c_m}_{\,b_1 \ldots\, w_\mu \ldots\, b_\beta \, \bar{b}_1 \ldots\, \bar{b}_\gamma \, d_1 \ldots\, d_n} [P] + \\
& + \sum_{\mu=1}^{\nu} \sum_{v_\mu=1}^{2} \Biggl(\sum_{\bar{a}=1}^{2} \overline{\mathfrak{T}^{\bar{a}_\mu}_{\bar{a}}} \frac{\partial \overline{\mathfrak{S}^{\bar{a}}_{v_\mu}}}{\partial \tilde{x}^j} \Biggr) \tilde{S}^{\,a_1 \ldots\, a_\alpha \, \bar{a}_1 \ldots\, v_\mu \ldots\, \bar{a}_\nu \, c_1 \ldots\, c_m}_{\,b_1 \ldots\, b_\beta \, \bar{b}_1 \ldots \ldots \ldots\, \bar{b}_\gamma \, d_1 \ldots\, d_n} [P] + \\
& + \sum_{\mu=1}^{\gamma} \sum_{w_\mu=1}^{2} \Biggl(\sum_{\bar{a}=1}^{2} \frac{\partial \overline{\mathfrak{T}^{w_\mu}_{\bar{a}}}}{\partial \tilde{x}^j} \overline{\mathfrak{S}^{\bar{a}}_{\bar{b}_\mu}} \Biggr) \tilde{S}^{\,a_1 \ldots\, a_\alpha \, \bar{a}_1 \ldots \ldots \ldots\, \bar{a}_\nu \, c_1 \ldots\, c_m}_{\,b_1 \ldots\, b_\beta \, \bar{b}_1 \ldots\, w_\mu \ldots\, \bar{b}_\gamma \, d_1 \ldots\, d_n} [P] + \\
& + \sum_{\mu=1}^{m} \sum_{v_\mu=0}^{3} \Biggl(\sum_{a=0}^{3} T^{c_\mu}_{a} \frac{\partial S^{a}_{v_m}}{\partial \tilde{x}^j} \Biggr) \tilde{S}^{\,a_1 \ldots\, a_\alpha \, \bar{a}_1 \ldots\, \bar{a}_\nu \, c_1 \ldots\, v_\mu \ldots\, c_m}_{\,b_1 \ldots\, b_\beta \, \bar{b}_1 \ldots\, \bar{b}_\gamma \, d_1 \ldots \ldots \ldots\, d_n} [P] + \\
& + \sum_{\mu=1}^{n} \sum_{w_\mu=0}^{3} \Biggl(\sum_{a=0}^{3} \frac{\partial T^{w_\mu}_{a}}{\partial \tilde{x}^j} S^{a}_{d_\mu} \Biggr) \tilde{S}^{\,a_1 \ldots\, a_\alpha \, \bar{a}_1 \ldots\, \bar{a}_\nu \, c_1 \ldots \ldots \ldots\, c_m}_{\,b_1 \ldots\, b_\beta \, \bar{b}_1 \ldots\, \bar{b}_\gamma \, d_1 \ldots\, w_\mu \ldots\, d_n} [P] \Biggr) \times \\
& \times \mathbf{W}^{j_1 \ldots\, j_\beta \, \bar{j}_1 \ldots\, \bar{j}_\gamma \, k_1 \ldots\, k_n}_{i_1 \ldots\, i_\alpha \, \bar{i}_1 \ldots\, \bar{i}_\nu \, h_1 \ldots\, h_m} [P] + \sum_{P=1}^{J+Q} \sum_{\substack{i_1,\ldots,i_\alpha\\ j_1,\ldots,j_\beta\\ a_1,\ldots,a_\alpha\\ b_1,\ldots,b_\beta}}^{2} \!\!\ldots\! \sum^{2} \sum_{\substack{\bar{i}_1,\ldots,\bar{i}_\nu\\ \bar{j}_1,\ldots,\bar{j}_\gamma\\ \bar{a}_1,\ldots,\bar{a}_\nu\\ \bar{b}_1,\ldots,\bar{b}_\gamma}}^{2} \!\!\ldots\! \sum^{2} \sum_{\substack{h_1,\ldots,h_m\\ k_1,\ldots,k_n\\ c_1,\ldots,c_m\\ d_1,\ldots,d_n}}^{3} \!\!\ldots\! \sum^{3} \overline{\mathfrak{S}^{i_1}_{a_1}} \ldots \overline{\mathfrak{S}^{i_\alpha}_{a_\alpha}} \times \\
& \times \overline{\mathfrak{T}^{b_1}_{j_1}} \ldots \overline{\mathfrak{T}^{b_\beta}_{j_\beta}} \, \mathfrak{S}^{\bar{i}_1}_{\bar{a}_1} \ldots \mathfrak{S}^{\bar{i}_\nu}_{\bar{a}_\nu} \;\; \mathfrak{T}^{\bar{b}_1}_{\bar{j}_1} \ldots \mathfrak{T}^{\bar{b}_\gamma}_{\bar{j}_\gamma} \, S^{h_1}_{c_1} \ldots S^{h_m}_{c_m} \, T^{d_1}_{k_1} \ldots T^{d_n}_{k_n} \times \\
& \times \Biggl(\sum_{\mu=1}^{\alpha} \sum_{v_\mu=1}^{2} \Biggl(\sum_{a=1}^{2} \overline{\mathfrak{T}^{a_\mu}_{a}} \frac{\partial \overline{\mathfrak{S}^{a}_{v_\mu}}}{\partial \tilde{x}^j} \Biggr) \overline{\tilde{S}^{\,a_1 \ldots\, v_\mu \ldots\, a_\alpha \, \bar{a}_1 \ldots\, \bar{a}_\nu \, c_1 \ldots\, c_m}_{\,b_1 \ldots \ldots \ldots\, b_\beta \, \bar{b}_1 \ldots\, \bar{b}_\gamma \, d_1 \ldots\, d_n} [P]} + \\
& + \sum_{\mu=1}^{\beta} \sum_{w_\mu=1}^{2} \Biggl(\sum_{a=1}^{2} \frac{\partial \overline{\mathfrak{T}^{w_\mu}_{a}}}{\partial \tilde{x}^j} \overline{\mathfrak{S}^{a}_{b_\mu}} \Biggr) \overline{\tilde{S}^{\,a_1 \ldots \ldots \ldots\, a_\alpha \, \bar{a}_1 \ldots\, \bar{a}_\nu \, c_1 \ldots\, c_m}_{\,b_1 \ldots\, w_\mu \ldots\, b_\beta \, \bar{b}_1 \ldots\, \bar{b}_\gamma \, d_1 \ldots\, d_n} [P]} + \\
& + \sum_{\mu=1}^{\nu} \sum_{v_\mu=1}^{2} \Biggl(\sum_{\bar{a}=1}^{2} \mathfrak{T}^{\bar{a}_\mu}_{\bar{a}} \frac{\partial \mathfrak{S}^{\bar{a}}_{v_\mu}}{\partial \tilde{x}^j} \Biggr) \overline{\tilde{S}^{\,a_1 \ldots\, a_\alpha \, \bar{a}_1 \ldots\, v_\mu \ldots\, \bar{a}_\nu \, c_1 \ldots\, c_m}_{\,b_1 \ldots\, b_\beta \, \bar{b}_1 \ldots \ldots \ldots\, \bar{b}_\gamma \, d_1 \ldots\, d_n} [P]} + \\
& + \sum_{\mu=1}^{\gamma} \sum_{w_\mu=1}^{2} \Biggl(\sum_{\bar{a}=1}^{2} \frac{\partial \mathfrak{T}^{w_\mu}_{\bar{a}}}{\partial \tilde{x}^j} \mathfrak{S}^{\bar{a}}_{\bar{b}_\mu} \Biggr) \overline{\tilde{S}^{\,a_1 \ldots\, a_\alpha \, \bar{a}_1 \ldots \ldots \ldots\, \bar{a}_\nu \, c_1 \ldots\, c_m}_{\,b_1 \ldots\, b_\beta \, \bar{b}_1 \ldots\, w_\mu \ldots\, \bar{b}_\gamma \, d_1 \ldots\, d_n} [P]} + \\
& + \sum_{\mu=1}^{m} \sum_{v_\mu=0}^{3} \Biggl(\sum_{a=0}^{3} T^{c_\mu}_{a} \frac{\partial S^{a}_{v_m}}{\partial \tilde{x}^j} \Biggr) \overline{\tilde{S}^{\,a_1 \ldots\, a_\alpha \, \bar{a}_1 \ldots\, \bar{a}_\nu \, c_1 \ldots\, v_\mu \ldots\, c_m}_{\,b_1 \ldots\, b_\beta \, \bar{b}_1 \ldots\, \bar{b}_\gamma \, d_1 \ldots \ldots \ldots\, d_n} [P]} +
\end{aligned}
$$

$$+\sum_{\mu=1}^{n}\sum_{w_\mu=0}^{3}\left(\sum_{a=0}^{3}\frac{\partial T^{w_\mu}_{a}}{\partial\tilde{x}^j}\,S^{a}_{d_\mu}\right)\overline{\tilde{S}^{\,a_1\dots\,a_\alpha\,\bar{a}_1\dots\,\bar{a}_\nu\,c_1\dots\dots\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar{b}_1\dots\,\bar{b}_\gamma\,d_1\dots\,w_\mu\dots\,d_n}[P]}\Bigg)\times$$
$$\times\,\bar{\mathbf{W}}^{\bar{j}_1\dots\,\bar{j}_\gamma\,j_1\dots\,j_\beta\,k_1\dots\,k_n}_{\bar{i}_1\dots\,\bar{i}_\nu\,i_1\dots\,i_\alpha\,h_1\dots\,h_m}[P].$$

As we see, the transformation rule for the vector (9.18) is expressed by the formula bigger than the page size. For this reason we give it without its inverse counterpart.

$$\left\{\begin{aligned}&\tilde{\mathbf{W}}^{j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}_{i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}[P]=\sum^{2}_{\substack{a_1,\,\dots\,,\,a_\alpha\\ b_1,\,\dots\,,\,b_\beta}}\!\!\dots\sum^{2}\;\sum^{2}_{\substack{\bar{a}_1,\,\dots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\dots\,,\,\bar{b}_\gamma}}\!\!\dots\sum^{2}\;\sum^{3}_{\substack{c_1,\,\dots\,,\,c_m\\ d_1,\,\dots\,,\,d_n}}\!\!\dots\sum^{3}\mathfrak{S}^{a_1}_{i_1}\dots\,\mathfrak{S}^{a_\alpha}_{i_\alpha}\times\\ &\qquad\times\,\mathfrak{T}^{j_1}_{b_1}\dots\,\mathfrak{T}^{j_\beta}_{b_\beta}\,\overline{\mathfrak{S}^{\bar{a}_1}_{\bar{i}_1}}\dots\,\overline{\mathfrak{S}^{\bar{a}_\nu}_{\bar{i}_\nu}}\;\;\overline{\mathfrak{T}^{\bar{j}_1}_{\bar{b}_1}}\dots\,\overline{\mathfrak{T}^{\bar{j}_\gamma}_{\bar{b}_\gamma}}\,S^{c_1}_{h_1}\dots\,S^{c_m}_{h_m}\times\\ &\qquad\quad\times\,T^{k_1}_{d_1}\dots\,T^{k_n}_{d_n}\,\mathbf{W}^{\,b_1\dots\,b_\beta\,\bar{b}_1\dots\,\bar{b}_\gamma\,d_1\dots\,d_n}_{\,a_1\dots\,a_\alpha\,\bar{a}_1\dots\,\bar{a}_\nu\,c_1\dots\,c_m}[P],\end{aligned}\right.\tag{9.21}$$

$$\left\{\begin{aligned}&\mathbf{W}^{j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}_{i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}[P]=\sum^{2}_{\substack{a_1,\,\dots\,,\,a_\alpha\\ b_1,\,\dots\,,\,b_\beta}}\!\!\dots\sum^{2}\;\sum^{2}_{\substack{\bar{a}_1,\,\dots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\dots\,,\,\bar{b}_\gamma}}\!\!\dots\sum^{2}\;\sum^{3}_{\substack{c_1,\,\dots\,,\,c_m\\ d_1,\,\dots\,,\,d_n}}\!\!\dots\sum^{3}\mathfrak{T}^{a_1}_{i_1}\dots\,\mathfrak{T}^{a_\alpha}_{i_\alpha}\times\\ &\qquad\times\,\mathfrak{S}^{j_1}_{b_1}\dots\,\mathfrak{S}^{j_\beta}_{b_\beta}\,\overline{\mathfrak{T}^{\bar{a}_1}_{\bar{i}_1}}\dots\,\overline{\mathfrak{T}^{\bar{a}_\nu}_{\bar{i}_\nu}}\;\;\overline{\mathfrak{S}^{\bar{j}_1}_{\bar{b}_1}}\dots\,\overline{\mathfrak{S}^{\bar{j}_\gamma}_{\bar{b}_\gamma}}\,T^{c_1}_{h_1}\dots\,T^{c_m}_{h_m}\times\\ &\qquad\quad\times\,S^{k_1}_{d_1}\dots\,S^{k_n}_{d_n}\,\tilde{\mathbf{W}}^{\,b_1\dots\,b_\beta\,\bar{b}_1\dots\,\bar{b}_\gamma\,d_1\dots\,d_n}_{\,a_1\dots\,a_\alpha\,\bar{a}_1\dots\,\bar{a}_\nu\,c_1\dots\,c_m}[P],\end{aligned}\right.\tag{9.22}$$

$$\left\{\begin{aligned}&\tilde{\bar{\mathbf{W}}}^{\bar{j}_1\dots\,\bar{j}_\gamma\,j_1\dots\,j_\beta\,k_1\dots\,k_n}_{\bar{i}_1\dots\,\bar{i}_\nu\,i_1\dots\,i_\alpha\,h_1\dots\,h_m}[P]=\sum^{2}_{\substack{a_1,\,\dots\,,\,a_\alpha\\ b_1,\,\dots\,,\,b_\beta}}\!\!\dots\sum^{2}\;\sum^{2}_{\substack{\bar{a}_1,\,\dots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\dots\,,\,\bar{b}_\gamma}}\!\!\dots\sum^{2}\;\sum^{3}_{\substack{c_1,\,\dots\,,\,c_m\\ d_1,\,\dots\,,\,d_n}}\!\!\dots\sum^{3}\overline{\mathfrak{S}^{a_1}_{i_1}}\dots\,\overline{\mathfrak{S}^{a_\alpha}_{i_\alpha}}\times\\ &\qquad\times\,\overline{\mathfrak{T}^{j_1}_{b_1}}\dots\,\overline{\mathfrak{T}^{j_\beta}_{b_\beta}}\,\mathfrak{S}^{\bar{a}_1}_{\bar{i}_1}\dots\,\mathfrak{S}^{\bar{a}_\nu}_{\bar{i}_\nu}\;\;\mathfrak{T}^{\bar{j}_1}_{\bar{b}_1}\dots\,\mathfrak{T}^{\bar{j}_\gamma}_{\bar{b}_\gamma}\,S^{c_1}_{h_1}\dots\,S^{c_m}_{h_m}\times\\ &\qquad\quad\times\,T^{k_1}_{d_1}\dots\,T^{k_n}_{d_n}\,\bar{\mathbf{W}}^{\,\bar{b}_1\dots\,\bar{b}_\gamma\,b_1\dots\,b_\beta\,d_1\dots\,d_n}_{\,\bar{a}_1\dots\,\bar{a}_\nu\,a_1\dots\,a_\alpha\,c_1\dots\,c_m}[P],\end{aligned}\right.\tag{9.23}$$

$$\left\{\begin{aligned}&\bar{\mathbf{W}}^{\bar{j}_1\dots\,\bar{j}_\gamma\,j_1\dots\,j_\beta\,k_1\dots\,k_n}_{\bar{i}_1\dots\,\bar{i}_\nu\,i_1\dots\,i_\alpha\,h_1\dots\,h_m}[P]=\sum^{2}_{\substack{a_1,\,\dots\,,\,a_\alpha\\ b_1,\,\dots\,,\,b_\beta}}\!\!\dots\sum^{2}\;\sum^{2}_{\substack{\bar{a}_1,\,\dots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\dots\,,\,\bar{b}_\gamma}}\!\!\dots\sum^{2}\;\sum^{3}_{\substack{c_1,\,\dots\,,\,c_m\\ d_1,\,\dots\,,\,d_n}}\!\!\dots\sum^{3}\overline{\mathfrak{T}^{a_1}_{i_1}}\dots\,\overline{\mathfrak{T}^{a_\alpha}_{i_\alpha}}\times\\ &\qquad\times\,\overline{\mathfrak{S}^{j_1}_{b_1}}\dots\,\overline{\mathfrak{S}^{j_\beta}_{b_\beta}}\,\mathfrak{T}^{\bar{a}_1}_{\bar{i}_1}\dots\,\mathfrak{T}^{\bar{a}_\nu}_{\bar{i}_\nu}\;\;\mathfrak{S}^{\bar{j}_1}_{\bar{b}_1}\dots\,\mathfrak{S}^{\bar{j}_\gamma}_{\bar{b}_\gamma}\,T^{c_1}_{h_1}\dots\,T^{c_m}_{h_m}\times\\ &\qquad\quad\times\,S^{k_1}_{d_1}\dots\,S^{k_n}_{d_n}\,\tilde{\bar{\mathbf{W}}}^{\,\bar{b}_1\dots\,\bar{b}_\gamma\,b_1\dots\,b_\beta\,d_1\dots\,d_n}_{\,\bar{a}_1\dots\,\bar{a}_\nu\,a_1\dots\,a_\alpha\,c_1\dots\,c_m}[P].\end{aligned}\right.\tag{9.24}$$

The transformation rules (9.21), (9.22), (9.23), and (9.24) are subdivided into pairs: each direct transformation rule is paired with an inverse one. Note also that the coefficients in (9.21) and (9.22) differ from those in (9.23) and (9.24) only by complex conjugation. Unlike the corresponding formula in [5], the transition matrices S and T here are not Jacobi matrices. In particular, we have

$$\frac{\partial x^i}{\partial\tilde{x}^j}\neq S^i_j.\tag{9.25}$$

In order to replace the partial derivatives (9.25) by the components of the transition matrix S, we should pass from the holonomic frame given by the vectors (9.18), (9.19), and (9.20) to some specially designed non-holonomic frame in N. For this purpose let's keep the vectors (9.19) and (9.20) unchanged, but replace the tangent vectors (9.18) by the following ones:

$$\mathbf{U}_i = \sum_{j=0}^{3} \Upsilon_i^j \, \mathbf{X}_j. \tag{9.26}$$

The coefficients Υ_i^j in the formula (9.26) are the coordinates of the frame vectors $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$, i. e. they are taken from the expansion

$$\boldsymbol{\Upsilon}_i = \sum_{j=0}^{3} \Upsilon_i^j \, \mathbf{E}_j, \tag{9.27}$$

where $\mathbf{E}_0$, $\mathbf{E}_1$, $\mathbf{E}_2$, $\mathbf{E}_3$ are the coordinate vectors (5.3) forming a holonomic frame in M. Despite to the similarity of the formulas (9.26) and (9.27) they are different. The formula (9.27) is the expansion in the tangent space to the space-time manifold M, while the formula (9.26) is the expansion in the tangent space to the composite spin-tensorial bundle (9.7).

Upon passing from (9.18) to the new vectors $\mathbf{U}_0$, $\mathbf{U}_1$, $\mathbf{U}_2$, $\mathbf{U}_3$ in (9.26) we introduce the following θ-parameters defined through the transition matrices:

$$\tilde{\theta}_{ij}^k = \sum_{a=0}^{3} T_a^k \, L_{\tilde{\boldsymbol{\Upsilon}}_i}(S_j^a) = \sum_{a=0}^{3}\sum_{v=0}^{3} T_a^k \, \tilde{\Upsilon}_i^v \, \frac{\partial S_j^a}{\partial \tilde{x}^v} = -\sum_{a=0}^{3} L_{\tilde{\boldsymbol{\Upsilon}}_i}(T_a^k) \, S_j^a, \tag{9.28}$$

$$\tilde{\vartheta}_{ij}^k = \sum_{a=1}^{2} \mathfrak{T}_a^k \, L_{\tilde{\boldsymbol{\Upsilon}}_i}(\mathfrak{S}_j^a) = \sum_{a=1}^{2}\sum_{v=0}^{3} \mathfrak{T}_a^k \, \tilde{\Upsilon}_i^v \, \frac{\partial \mathfrak{S}_j^a}{\partial \tilde{x}^v} = -\sum_{a=1}^{2} L_{\tilde{\boldsymbol{\Upsilon}}_i}(\mathfrak{T}_a^k) \, \mathfrak{S}_j^a. \tag{9.29}$$

The θ-parameters without tilde are introduced in a similar way:

$$\theta_{ij}^k = \sum_{a=0}^{3} S_a^k \, L_{\boldsymbol{\Upsilon}_i}(T_j^a) = \sum_{a=0}^{3}\sum_{v=0}^{3} S_a^k \, \Upsilon_i^v \, \frac{\partial T_j^a}{\partial x^v} = -\sum_{a=0}^{3} L_{\boldsymbol{\Upsilon}_i}(S_a^k) \, T_j^a, \tag{9.30}$$

$$\vartheta_{ij}^k = \sum_{a=1}^{2} \mathfrak{S}_a^k \, L_{\boldsymbol{\Upsilon}_i}(\mathfrak{T}_j^a) = \sum_{a=1}^{2}\sum_{v=0}^{3} \mathfrak{S}_a^k \, \Upsilon_i^v \, \frac{\partial \mathfrak{T}_j^a}{\partial \tilde{x}^v} = -\sum_{a=1}^{2} L_{\boldsymbol{\Upsilon}_i}(\mathfrak{S}_a^k) \, \mathfrak{T}_j^a. \tag{9.31}$$

Here $L_{\boldsymbol{\Upsilon}_i}$ and $L_{\tilde{\boldsymbol{\Upsilon}}_i}$ are Lie derivatives (see [11]). Using (9.28), (9.29), (9.30), (9.31) one easily derives the following identities similar to those in [5]:

$$\theta_{ij}^k = -\sum_{a=0}^{3}\sum_{b=0}^{3}\sum_{c=0}^{3} T_i^a \, \tilde{\theta}_{ab}^c \, S_c^k \, T_j^b, \tag{9.32}$$

$$\tilde{\theta}_{ij}^k = -\sum_{a=0}^{3}\sum_{b=0}^{3}\sum_{c=0}^{3} S_i^a \, \theta_{ab}^c \, T_c^k \, S_j^b, \tag{9.33}$$

$$\vartheta_{ij}^k = -\sum_{a=0}^{3}\sum_{b=1}^{2}\sum_{c=1}^{2} T_i^a \, \tilde{\vartheta}_{ab}^c \, \mathfrak{S}_c^k \, \mathfrak{T}_j^b, \tag{9.34}$$

$$\tilde{\vartheta}_{ij}^k = -\sum_{a=0}^{3}\sum_{b=1}^{2}\sum_{c=1}^{2} S_i^a \, \vartheta_{ab}^c \, \mathfrak{T}_c^k \, \mathfrak{S}_j^b. \tag{9.35}$$

The formulas (9.32), (9.33), (9.34), and (9.35) relate the θ-parameters with and without tilde. Note that, unlike those introduced in [5], the θ-parameters introduced in (9.28) and (9.30) are not symmetric with respect to i and j. The extent of their asymmetry is given by the following formulas:

$$\theta^k_{ij}-\theta^k_{j\,i}=c^k_{ij} \qquad\qquad \tilde\theta^k_{ij}-\tilde\theta^k_{j\,i}=\tilde c^k_{ij} \tag{9.36}$$

The parameters c^k_{ij} in (9.36) are taken from the following commutator relationships for the frame vectors of the orthonormal frame:

$$[\boldsymbol\Upsilon_i,\ \boldsymbol\Upsilon_j]=\sum^3_{k=1}c^k_{ij}\,\boldsymbol\Upsilon_k.$$

The parameters $\tilde c^k_{ij}$ are taken from the analogous formula for $\tilde{\boldsymbol\Upsilon}_0$, $\tilde{\boldsymbol\Upsilon}_1$, $\tilde{\boldsymbol\Upsilon}_2$, $\tilde{\boldsymbol\Upsilon}_3$.

Now we apply (9.26), (9.28), (9.29), (9.30), (9.31) to the above huge transformation formula. Then it is written so that we can assign a number to this formula:

$$\left\{\begin{aligned}&\tilde{\mathbf U}_j=\sum^3_{i=0}S^i_j\,\mathbf U_i+\\&+\sum^{J+Q}_{P=1}\sum^{2}\ldots\sum^{3}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta\\ \bar i_1,\,\ldots\,,\,\bar i_\nu\\ \bar j_1,\,\ldots\,,\,\bar j_\gamma\\ h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\Bigl(\tilde w^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\;\mathbf W^{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}[P]+\\&+\overline{\tilde w^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]}\;\bar{\mathbf W}^{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}_{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}[P]\Bigr).\end{aligned}\right. \tag{9.37}$$

The inverse transformation formula for (9.37) is written similarly:

$$\left\{\begin{aligned}&\mathbf U_j=\sum^3_{i=0}S^i_j\,\tilde{\mathbf U}_i+\\&+\sum^{J+Q}_{P=1}\sum^{2}\ldots\sum^{3}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta\\ \bar i_1,\,\ldots\,,\,\bar i_\nu\\ \bar j_1,\,\ldots\,,\,\bar j_\gamma\\ h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\Bigl(w^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\;\tilde{\mathbf W}^{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}[P]+\\&+\overline{w^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]}\;\tilde{\bar{\mathbf W}}^{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}_{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}[P]\Bigr).\end{aligned}\right. \tag{9.38}$$

Here are the formulas for $\tilde w^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]$ and $w^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]$ in the above transformation formulas (9.37) and (9.38):

$$\tilde w^{i_1\dots\,i_\alpha\,\bar i_1\dots\,\bar i_\nu\,h_1\dots\,h_m}_{j\,j_1\dots\,j_\beta\,\bar j_1\dots\,\bar j_\gamma\,k_1\dots\,k_n}[P]=\sum^2_{\substack{a_1,\,\dots,\,a_\alpha\\ b_1,\,\dots,\,b_\beta}}\!\!\dots\!\sum^2\sum^2_{\substack{\bar a_1,\,\dots,\,\bar a_\nu\\ \bar b_1,\,\dots,\,\bar b_\gamma}}\!\!\dots\!\sum^2\sum^3_{\substack{c_1,\,\dots,\,c_m\\ d_1,\,\dots,\,d_n}}\!\!\dots\!\sum^3\mathfrak{S}^{\,i_1}_{a_1}\dots\,\mathfrak{S}^{\,i_\alpha}_{a_\alpha}\,\times$$

$$\times\,\mathfrak{T}^{b_1}_{j_1}\dots\,\mathfrak{T}^{b_\beta}_{j_\beta}\,\overline{\mathfrak{S}^{\,\bar i_1}_{\bar a_1}}\dots\,\overline{\mathfrak{S}^{\,\bar i_\nu}_{\bar a_\nu}}\,\,\overline{\mathfrak{T}^{\bar b_1}_{\bar j_1}}\dots\,\overline{\mathfrak{T}^{\bar b_\gamma}_{\bar j_\gamma}}\,S^{h_1}_{c_1}\dots\,S^{h_m}_{c_m}\,T^{d_1}_{k_1}\dots\,T^{d_n}_{k_n}\,\times$$

$$\times\left(\sum^{\alpha}_{\mu=1}\sum^{2}_{v_\mu=1}\tilde\vartheta^{\,a_\mu}_{j\,v_\mu}\,\,\tilde S^{\,a_1\dots\,v_\mu\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,\dots\,\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P]-\sum^{\beta}_{\mu=1}\sum^{2}_{w_\mu=1}\tilde\vartheta^{\,w_\mu}_{j\,b_\mu}\,\times\right.$$

$$\times\,\tilde S^{\,a_1\dots\,\dots\,\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,w_\mu\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P]+\sum^{\nu}_{\mu=1}\sum^{2}_{v_\mu=1}\overline{\tilde\vartheta^{\,\bar a_\mu}_{j\,v_\mu}}\,\,\tilde S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\dots\,\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,w_\mu\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P]\,-$$

$$-\sum^{\gamma}_{\mu=1}\sum^{2}_{w_\mu=1}\overline{\tilde\vartheta^{\,w_\mu}_{j\,\bar b_\mu}}\,\,\tilde S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\dots\,\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,w_\mu\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P]+\sum^{m}_{\mu=1}\sum^{3}_{v_\mu=0}\tilde\theta^{\,c_\mu}_{j\,v_\mu}\,\times$$

$$\left.\times\,\tilde S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,v_\mu\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,\dots\,\dots\,d_n}[P]-\sum^{n}_{\mu=1}\sum^{3}_{w_\mu=0}\tilde\theta^{\,w_\mu}_{j\,d_\mu}\tilde S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,\dots\,\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,w_\mu\dots\,d_n}[P]\right),$$

$$w^{i_1\dots\,i_\alpha\,\bar i_1\dots\,\bar i_\nu\,h_1\dots\,h_m}_{j\,j_1\dots\,j_\beta\,\bar j_1\dots\,\bar j_\gamma\,k_1\dots\,k_n}[P]=\sum^2_{\substack{a_1,\,\dots,\,a_\alpha\\ b_1,\,\dots,\,b_\beta}}\!\!\dots\!\sum^2\sum^2_{\substack{\bar a_1,\,\dots,\,\bar a_\nu\\ \bar b_1,\,\dots,\,\bar b_\gamma}}\!\!\dots\!\sum^2\sum^3_{\substack{c_1,\,\dots,\,c_m\\ d_1,\,\dots,\,d_n}}\!\!\dots\!\sum^3\mathfrak{T}^{\,i_1}_{a_1}\dots\,\mathfrak{T}^{\,i_\alpha}_{a_\alpha}\,\times$$

$$\times\,\mathfrak{S}^{b_1}_{j_1}\dots\,\mathfrak{S}^{b_\beta}_{j_\beta}\,\overline{\mathfrak{T}^{\,\bar i_1}_{\bar a_1}}\dots\,\overline{\mathfrak{T}^{\,\bar i_\nu}_{\bar a_\nu}}\,\,\overline{\mathfrak{S}^{\bar b_1}_{\bar j_1}}\dots\,\overline{\mathfrak{S}^{\bar b_\gamma}_{\bar j_\gamma}}\,T^{h_1}_{c_1}\dots\,T^{h_m}_{c_m}\,S^{d_1}_{k_1}\dots\,S^{d_n}_{k_n}\,\times$$

$$\times\left(\sum^{\alpha}_{\mu=1}\sum^{2}_{v_\mu=1}\vartheta^{\,a_\mu}_{j\,v_\mu}\,\,S^{\,a_1\dots\,v_\mu\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,\dots\,\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P]-\sum^{\beta}_{\mu=1}\sum^{2}_{w_\mu=1}\vartheta^{\,w_\mu}_{j\,b_\mu}\,\times\right.$$

$$\times\,S^{\,a_1\dots\,\dots\,\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,w_\mu\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P]+\sum^{\nu}_{\mu=1}\sum^{2}_{v_\mu=1}\overline{\vartheta^{\,\bar a_\mu}_{j\,v_\mu}}\,\,S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\dots\,\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,w_\mu\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P]\,-$$

$$-\sum^{\gamma}_{\mu=1}\sum^{2}_{w_\mu=1}\overline{\vartheta^{\,w_\mu}_{j\,\bar b_\mu}}\,\,S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\dots\,\dots\,\bar a_\nu\,c_1\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,w_\mu\dots\,\bar b_\gamma\,d_1\dots\,d_n}[P]+\sum^{m}_{\mu=1}\sum^{3}_{v_\mu=0}\theta^{\,c_\mu}_{j\,v_\mu}\,\times$$

$$\left.\times\,S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,v_\mu\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,\dots\,\dots\,d_n}[P]-\sum^{n}_{\mu=1}\sum^{3}_{w_\mu=0}\theta^{\,w_\mu}_{j\,d_\mu}S^{\,a_1\dots\,a_\alpha\,\bar a_1\dots\,\bar a_\nu\,c_1\dots\,\dots\,\dots\,c_m}_{\,b_1\dots\,b_\beta\,\bar b_1\dots\,\bar b_\gamma\,d_1\dots\,w_\mu\dots\,d_n}[P]\right).$$

These formulas are again rather huge so that we do not mark them with a number.

10. Extended tensorial and spin-tensorial fields.

Definition 10.1. Let N be a composite spin-tensorial bundle over the space-time manifold M (in the sense of the formula (9.7)). An extended tensor field $\mathbf{X}$ of the type (e,f) is a tensor-valued function in N such that it takes each point $q\in N$ to some tensor $\mathbf{X}(q)\in T^e_f(p,M)$, where $p=\pi(q)$ is the projection of q.

Informally speaking, an extended tensor field $\mathbf{X}$ is a tensorial function with one point argument $p\in M$ and several spin-tensorial arguments $\mathbf{S}[1],\,\dots\,,\,\mathbf{S}[J+Q]$ as shown in (9.8). An extended spin-tensorial field is defined in a similar way.

Definition 10.2. Let N be a composite spin-tensorial bundle over the space-time manifold M (in the sense of the formula (9.7)). An extended spin-tensorial field $\mathbf{X}$ of the type $(\varepsilon,\eta|\sigma,\zeta|e,f)$ is a spin-tensor-valued function in N such that it takes each point $q\in N$ to some spin-tensor $\mathbf{X}(q)\in S^{\varepsilon}_{\eta}\bar S^{\sigma}_{\zeta}T^{e}_{f}(p,M)$, where $p=\pi(q)$ is the projection of q.

Remember that upon complexification the tensor bundle T^e_fM can be identified with the spin-tensorial bundle $S^0_0\bar S^0_0T^e_fM$ (compare (9.5) and (9.6) above):

$$\mathbb{C}T^e_fM=\mathbb{C}\otimes T^e_fM=S^0_0\bar S^0_0T^e_fM. \tag{10.1}$$

Due to the formula (10.1) extended tensorial fields introduced in the definition 10.1 can be understood as some special cases of extended spin-tensorial fields introduced in the definition 10.2. With this remark in mind, below we do not study extended tensorial fields in a separate way. They are naturally included into the general framework of the study of extended spin-tensorial fields.

In a local chart $U\subset M$ equipped with a positively polarized right orthonormal frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and with its associated spinor frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ an extended spin-tensorial field $\mathbf{X}$ is represented by its components $X^{i_1\ldots\,i_\varepsilon\bar i_1\ldots\,\bar i_\sigma h_1\ldots\,h_e}_{j_1\ldots\,j_\eta\bar j_1\ldots\,\bar j_\zeta k_1\ldots\,k_f}$ each of which is a function of the variables (9.11) and (9.12). When passing from U to an overlapping chart $\tilde U$ equipped with another positively polarized right orthonormal frame $\tilde{\boldsymbol{\Upsilon}}_0$, $\tilde{\boldsymbol{\Upsilon}}_1$, $\tilde{\boldsymbol{\Upsilon}}_2$, $\tilde{\boldsymbol{\Upsilon}}_3$ and associated spinor frame $\tilde{\boldsymbol{\Psi}}_1$, $\tilde{\boldsymbol{\Psi}}_2$ the components of $\mathbf{X}$ are transformed according to the standard transformation formulas

$$\left\{\begin{aligned}&\tilde X^{i_1\ldots\,i_\varepsilon\bar i_1\ldots\,\bar i_\sigma h_1\ldots\,h_e}_{j_1\ldots\,j_\eta\bar j_1\ldots\,\bar j_\zeta k_1\ldots\,k_f}=\sum^{2}_{\substack{a_1,\,\ldots\,,\,a_\varepsilon\\ b_1,\,\ldots\,,\,b_\eta}}\!\!\ldots\!\sum^{2}\ \sum^{2}_{\substack{\bar a_1,\,\ldots\,,\,\bar a_\sigma\\ \bar b_1,\,\ldots\,,\,\bar b_\zeta}}\!\!\ldots\!\sum^{2}\ \sum^{3}_{\substack{c_1,\,\ldots\,,\,c_e\\ d_1,\,\ldots\,,\,d_f}}\!\!\ldots\!\sum^{3}\mathfrak{T}^{i_1}_{a_1}\ldots\,\mathfrak{T}^{i_\varepsilon}_{a_\varepsilon}\,\times\\ &\qquad\times\,\mathfrak{S}^{b_1}_{j_1}\ldots\,\mathfrak{S}^{b_\eta}_{j_\eta}\,\overline{\mathfrak{T}^{\bar i_1}_{\bar a_1}}\ldots\,\overline{\mathfrak{T}^{\bar i_\sigma}_{\bar a_\sigma}}\ \overline{\mathfrak{S}^{\bar b_1}_{\bar j_1}}\ldots\,\overline{\mathfrak{S}^{\bar b_\zeta}_{\bar j_\zeta}}\ T^{h_1}_{c_1}\ldots\,T^{h_e}_{c_e}\,\times\\ &\qquad\times\,S^{d_1}_{k_1}\ldots\,S^{d_f}_{k_f}\ X^{a_1\ldots\,a_\varepsilon\,\bar a_1\ldots\,\bar a_\sigma\,c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\,\bar b_1\ldots\,\bar b_\zeta\,d_1\ldots\,d_f},\end{aligned}\right. \tag{10.2}$$

$$\left\{\begin{aligned}&X^{i_1\ldots\,i_\varepsilon\bar i_1\ldots\,\bar i_\sigma h_1\ldots\,h_e}_{j_1\ldots\,j_\eta\bar j_1\ldots\,\bar j_\zeta k_1\ldots\,k_f}=\sum^{2}_{\substack{a_1,\,\ldots\,,\,a_\varepsilon\\ b_1,\,\ldots\,,\,b_\eta}}\!\!\ldots\!\sum^{2}\ \sum^{2}_{\substack{\bar a_1,\,\ldots\,,\,\bar a_\sigma\\ \bar b_1,\,\ldots\,,\,\bar b_\zeta}}\!\!\ldots\!\sum^{2}\ \sum^{3}_{\substack{c_1,\,\ldots\,,\,c_e\\ d_1,\,\ldots\,,\,d_f}}\!\!\ldots\!\sum^{3}\mathfrak{S}^{i_1}_{a_1}\ldots\,\mathfrak{S}^{i_\varepsilon}_{a_\varepsilon}\,\times\\ &\qquad\times\,\mathfrak{T}^{b_1}_{j_1}\ldots\,\mathfrak{T}^{b_\eta}_{j_\eta}\,\overline{\mathfrak{S}^{\bar i_1}_{\bar a_1}}\ldots\,\overline{\mathfrak{S}^{\bar i_\sigma}_{\bar a_\sigma}}\ \overline{\mathfrak{T}^{\bar b_1}_{\bar j_1}}\ldots\,\overline{\mathfrak{T}^{\bar b_\zeta}_{\bar j_\zeta}}\ S^{h_1}_{c_1}\ldots\,S^{h_e}_{c_e}\,\times\\ &\qquad\times\,T^{d_1}_{k_1}\ldots\,T^{d_f}_{k_f}\ \tilde X^{a_1\ldots\,a_\varepsilon\,\bar a_1\ldots\,\bar a_\sigma\,c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\,\bar b_1\ldots\,\bar b_\zeta\,d_1\ldots\,d_f},\end{aligned}\right. \tag{10.3}$$

while their arguments are transformed according to the formulas (9.13), (9.14), and (9.15) as described above in section 9.

Definition 10.3. An extended spin-tensorial field $\mathbf{X}$ associated with a composite spin-tensorial bundle N is called *smooth* if its components are smooth function of their arguments (9.11) and (9.12) in any local chart U equipped with a positively polarized right orthonormal frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and associated frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$.

11. The algebra of extended spin-tensorial fields.

Suppose that some composite spin-tensorial bundle N given by the formula (9.7) is fixed. Let's denote by $S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)$ the set of all smooth extended spin-tensorial fields of the type $(\varepsilon,\eta|\sigma,\zeta|e,f)$. Then the direct sum

$$\mathbf{S}(M)=\bigoplus_{\varepsilon=0}^{\infty}\bigoplus_{\eta=0}^{\infty}\bigoplus_{\sigma=0}^{\infty}\bigoplus_{\zeta=0}^{\infty}\bigoplus_{e=0}^{\infty}\bigoplus_{f=0}^{\infty}S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M) \tag{11.1}$$

is called the *algebra of extended spin-tensorial fields*. It is an algebra over the ring

$$S^{0}_{0}\bar{S}^{0}_{0}T^{0}_{0}(M)=\mathfrak{F}_{\mathbb{C}}(N) \tag{11.2}$$

of smooth complex functions in N. The graded algebra (11.1) is equipped with the following four algebraic operations:

(1) $S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)+S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)\longrightarrow S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)$;
(2) $S^{0}_{0}\bar{S}^{0}_{0}T^{0}_{0}(M)\otimes S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)\longrightarrow S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)$;
(3) $S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)\otimes S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{m}_{n}(M)\longrightarrow S^{\varepsilon+\alpha}_{\eta+\beta}\bar{S}^{\sigma+\nu}_{\zeta+\gamma}T^{e+m}_{f+n}(M)$;
(4) $C\colon\ S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)\longrightarrow S^{\varepsilon-1}_{\eta-1}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)$ for $\varepsilon\geqslant 1$ and $\eta\geqslant 1$,
$C\colon\ S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)\longrightarrow S^{\varepsilon}_{\eta}\bar{S}^{\sigma-1}_{\zeta-1}T^{e}_{f}(M)$ for $\sigma\geqslant 1$ and $\zeta\geqslant 1$,
$C\colon\ S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)\longrightarrow S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e-1}_{f-1}(M)$ for $e\geqslant 1$ and $f\geqslant 1$.

These operations are called *addition*, *multiplication by scalars*, *tensor product*, and *contraction*. The last item (4) is subdivided into three parts indicating that the contraction operation is allowed only within certain groups of indices, i. e. a spinor index can be contracted only with a spinor index, a barred spinor index only with another barred spinor index, and a tensorial index with another tensorial index.

12. Differentiation of extended spin-tensorial fields.

Suppose that some composite spin-tensorial bundle N over the space-time manifold M is fixed (see (9.7)). Then the algebra $\mathbf{S}(M)$ is also fixed.

Definition 12.1. A mapping $D\colon\ \mathbf{S}(M)\to\mathbf{S}(M)$ is called a *differentiation* of the algebra of extended spin-tensorial fields if the following conditions are fulfilled:

(1) D is concordant with the grading: $D(S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M))\subset S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(M)$;
(2) D is $\mathbb{C}$-linear: $D(\mathbf{X}+\mathbf{Y})=D(\mathbf{X})+D(\mathbf{Y})$
and $D(\lambda\mathbf{X})=\lambda D(\mathbf{X})$ for $\lambda\in\mathbb{C}$;
(3) D commutates with the contractions: $D(C(\mathbf{X}))=C(D(\mathbf{X}))$;
(4) D obeys the Leibniz rule: $D(\mathbf{X}\otimes\mathbf{Y})=D(\mathbf{X})\otimes\mathbf{Y}+\mathbf{X}\otimes D(\mathbf{Y})$.

Let's consider the set of all differentiations of the extended algebra $\mathbf{S}(M)$. We denote it $\mathfrak{D}_{\mathbf{S}}(M)$. It is easy to check up that

(1) the sum of two differentiations is a differentiation of the algebra $\mathbf{S}(M)$;
(2) the product of a differentiation by a smooth complex function in N is a differentiation of the algebra $\mathbf{S}(M)$.

Now we see that $\mathfrak{D}_{\mathbf{S}}(M)$ is equipped with the structure of a module over the ring of smooth complex functions $\mathfrak{F}_{\mathbb{C}}(N)$ (see (11.2)).

In the module $\mathfrak{D}_{\mathbf{S}}(M)$ the composition of two differentiations D_1 and D_2 is not a differentiation, but their commutator

$$[D_1,\,D_2]=D_1\circ D_2-D_2\circ D_1$$

is a differentiation. Therefore, $\mathfrak{D}_{\mathbf{S}}(M)$ is a Lie algebra. Note, however, that $\mathfrak{D}_{\mathbf{S}}(M)$ is not a Lie algebra over the ring of smooth complex functions $\mathfrak{F}_{\mathbb{C}}(N)$. It is only a Lie algebra over the field of complex numbers $\mathbb{C}$.

13. Localization.

The results of this section are very similar to those in section 7 of the paper [5]. Nevertheless, for the reader's convenience we give them in full details.

Smooth extended spin-tensorial fields are global objects related to the spin-tensorial bundle N in whole. But they are functions and their values are local objects. This means that two different fields $\mathbf{A}\neq\mathbf{B}$ can take the same values at some particular points. Whenever this happens, we write $\mathbf{A}_q=\mathbf{B}_q$, where $q\in N$ is a point of the composite spin-tensorial bundle (9.7).

Differentiations of the algebra $\mathbf{S}(M)$, as they are introduced above in the definitions 12.1, are global objects without any explicit subdivision into parts related to separate points of the bundle N. Below in this section we shall show that they also can be represented as functions taking their values in some linear spaces associated with separate points of the manifold N.

Let $D\in\mathfrak{D}_{\mathbf{S}}(M)$ be a differentiation of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$. Let's denote by δ the restriction of the mapping $D\colon\mathbf{S}(M)\to\mathbf{S}(M)$ to the module $S^0_0\bar S^0_0T^0_0(M)$ of extended scalar fields in (11.1):

$$\delta\colon\ S^0_0\bar S^0_0T^0_0(M)\to S^0_0\bar S^0_0T^0_0(M). \tag{13.1}$$

Since $S^0_0\bar S^0_0T^0_0(M)=\mathfrak{F}_{\mathbb{C}}(N)$, the mapping δ in (13.1) is a differentiation of the ring of smooth complex functions in the smooth real manifold N. It is known (see § 1 in Chapter I of [11]) that any differentiation of the ring of smooth functions of an arbitrary smooth manifold is determined by some vector field $\mathbf{Z}$ in this manifold. In our case $\mathbf{Z}$ is a complexified vector field, i. e.

$$\mathbf{Z}\in\mathbb{C}T^1_0(N)=\mathbb{C}\otimes T^1_0(N).$$

Applying this fact, we get the following representation for the operator δ in terms of the differential operators (9.26), (9.19), and (9.20):

$$\begin{gathered}\delta=\mathbf{Z}=\sum^3_{i=0}Z^i\ \mathbf{U}_i\,+\\+\sum^{J+Q}_{P=1}\sum^2_{\substack{i_1,\,\ldots,\,i_\alpha\\ j_1,\,\ldots,\,j_\beta\\ \bar i_1,\,\ldots,\,\bar i_\nu\\ \bar j_1,\,\ldots,\,\bar j_\gamma\\ h_1,\,\ldots,\,h_m\\ k_1,\,\ldots,\,k_n}}\!\!\ldots\sum^3\Bigl(Z^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\ \mathbf{W}^{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}[P]\,+\\+\bar Z^{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]\ \bar{\mathbf{W}}^{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}_{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}[P]\Bigr).\end{gathered}\tag{13.2}$$

Note that Z^i, $Z^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]$, and $\bar{Z}^{\bar{i}_1\ldots\,\bar{i}_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\bar{j}_1\ldots\,\bar{j}_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]$ in (13.2) are arbitrary smooth complex functions within the the local chart U where the representation (13.2) is defined.

Lemma 13.1. *Let ψ be an extended scalar field (a smooth function) identically constant within some open subset $O \subset N$ and let $\varphi = D(\psi)$ for some differentiation D. Then $\varphi = 0$ within the open set O.*

Proof. Since $D(\psi) = \delta(\psi)$, choosing some local chart and applying the differential operators (9.26), (9.19), and (9.20) to a constant, from the formula (13.2) we derive that $\delta(\psi) = \mathbf{Z}\psi = 0$ at any point q of the open set O. □

Lemma 13.2. *Let $\mathbf{X}$ be an extended spin-tensorial field. If $\mathbf{X} \equiv 0$, then for any differentiation D the field $D(\mathbf{X})$ is also identically equal to zero.*

The proof is trivial. Since $\mathbf{X} \equiv 0$, we can write $\mathbf{X} = \lambda\,\mathbf{X}$ with $\lambda \neq 1$. Then, applying the item (2) of the definition 12.1, we get $D(\mathbf{X}) = \lambda\,D(\mathbf{X})$. Since $\lambda \neq 1$, this yields the required equality $D(\mathbf{X}) \equiv 0$.

Lemma 13.3. *Let $\mathbf{X}$ be an extended spin-tensorial field identically zero within some open set $O \subset N$. If $\mathbf{Y} = D(\mathbf{X})$ for some differentiation D, then $\mathbf{Y}_q = 0$ at any point $q \in O$.*

Proof. Let's choose some arbitrary point $q \in O$ and take some smooth scalar function η such that it is identically equal to the unity in some open neighborhood $O' \subset O$ of the point q and identically equal to zero outside the open set O. The existence of such a function η is easily proved by choosing some local chart U that covers the point q. The product $\eta\,\mathbf{X}$ is identically equal to zero:

$$\eta \otimes \mathbf{X} = \eta\,\mathbf{X} \equiv 0. \tag{13.3}$$

Applying the differentiation D to (13.3), then taking into account the lemma 13.2 and the item (4) of the definition 12.1, we obtain

$$0 = D(0) = D(\eta \otimes \mathbf{X}) = D(\eta) \otimes \mathbf{X} + \eta \otimes D(\mathbf{X}) = D(\eta)\,\mathbf{X} + \eta\,D(\mathbf{X}).$$

Note that $D(\eta) = 0$ at the point q due to the lemma 13.1. Moreover, $\mathbf{X}_q = 0$ and $\eta = 1$ at the point q. Therefore, by specifying the above equality to the point q we get $D(\mathbf{X}) = 0$ at the point q. The lemma is proved. □

Lemma 13.4. *If two extended spin-tensorial fields $\mathbf{X}$ and $\mathbf{Y}$ are equal within some open neighborhood O of a point $q \in N$, then for any differentiation D their images $D(\mathbf{X})$ and $D(\mathbf{Y})$ are equal at the point q.*

The lemma 13.4 follows immediately from the lemma 13.3. This lemma is a basic tool for our purposes of localization in this section.

Let q be some point of N and let $p = \pi(q)$ be its projection in the base manifold M. Suppose that U is a local chart of M equipped with a positively polarized right oriented orthonormal frame $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$ and with its associated spinor frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ and such that it covers the point p in M. Then we can use its preimage $\pi^{-1}(U)$ as a local chart in N covering the point q. The variables (9.11) and (9.12) form a complete set of local coordinates in the chart $\pi^{-1}(U)$. Any extended spin-tensorial field $\mathbf{X}$ of the type $(\varepsilon,\eta|\sigma,\zeta|e,f)$ is represented by the formula

$$\mathbf{X}=\sum^{2}_{\substack{a_1,\,\ldots,\,a_\varepsilon\\ b_1,\,\ldots,\,b_\eta\\ \bar a_1,\,\ldots,\,\bar a_\sigma\\ \bar b_1,\,\ldots,\,\bar b_\zeta\\ c_1,\,\ldots,\,c_e\\ d_1,\,\ldots,\,d_f}}\!\!\ldots\sum^{3} X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma h_1\ldots\,h_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta k_1\ldots\,k_f}\ \mathbf{\Psi}^{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e},\tag{13.4}$$

where $\mathbf{\Psi}^{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}$ are given by the tensor products (5.33). In the case of standard (traditional) spin-tensorial fields the coefficients $X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}$ in (13.4) depend on the coordinates x^0, x^1, x^2, x^3 of a point $p\in M$ only (see (4.3) and compare (5.34) with (13.4)). In the case of extended fields they depend on the whole set of variables (9.11) and (9.12).

Taking some differentiations, we can apply them to the left hand side of (13.4), but we cannot apply them to each summand in the right hand side of this formula. The matter is that the scalars $X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}$ and the spin-tensors $\mathbf{\Psi}^{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}$ are defined locally only within the chart $\pi^{-1}(U)$. Therefore, they do not fit the definition 10.2. In order to convert them to global fields we choose some smooth real function $\eta\in\mathfrak{F}_{\mathbb{R}}(N)\subset\mathfrak{F}_{\mathbb{C}}(N)$ such that it is identically equal to the unity within some open neighborhood of the point q and is identically zero outside the chart $\pi^{-1}(U)$. Then we define the following global extended fields:

$$\hat X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}=\begin{cases}\eta\ X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f} & \text{within }\ \pi^{-1}(U),\\ 0 & \text{outside }\ \pi^{-1}(U),\end{cases}\tag{13.5}$$

$$\hat{\mathbf{\Upsilon}}_i=\begin{cases}\eta\ \mathbf{\Upsilon}_i & \text{within }\ \pi^{-1}(U),\\ 0 & \text{outside }\ \pi^{-1}(U),\end{cases}\tag{13.6}$$

$$\hat{\boldsymbol{\eta}}^i=\begin{cases}\eta\ \boldsymbol{\eta}^i & \text{within }\ \pi^{-1}(U),\\ 0 & \text{outside }\ \pi^{-1}(U),\end{cases}\tag{13.7}$$

$$\hat{\mathbf{\Psi}}_i=\begin{cases}\eta\ \mathbf{\Psi}_i & \text{within }\ \pi^{-1}(U),\\ 0 & \text{outside }\ \pi^{-1}(U),\end{cases}\tag{13.8}$$

$$\hat{\boldsymbol{\vartheta}}^i=\begin{cases}\eta\ \boldsymbol{\vartheta}^i & \text{within }\ \pi^{-1}(U),\\ 0 & \text{outside }\ \pi^{-1}(U).\end{cases}\tag{13.9}$$

Then by analogy to (5.28) we write

$$\begin{aligned}\bar{\hat{\mathbf{\Psi}}}_i&=\tau(\hat{\mathbf{\Psi}}_i),\\ \bar{\hat{\boldsymbol{\vartheta}}}^{\,i}&=\tau(\hat{\boldsymbol{\vartheta}}^{\,i})\end{aligned}\tag{13.10}$$

and by analogy to (5.30), (5.31), (5.32), and (5.33) we introduce the following tensor products, which are global extended fields:

$$\hat{\boldsymbol{\Upsilon}}^{d_1\ldots\,d_f}_{c_1\ldots\,c_e}=\hat{\boldsymbol{\Upsilon}}_{c_1}\otimes\ldots\otimes\hat{\boldsymbol{\Upsilon}}_{c_e}\otimes\hat{\boldsymbol{\eta}}^{d_1}\otimes\ldots\otimes\hat{\boldsymbol{\eta}}^{d_f},\tag{13.11}$$

$$\hat{\boldsymbol{\Psi}}^{b_1\ldots\,b_\eta}_{a_1\ldots\,a_\varepsilon}=\hat{\boldsymbol{\Psi}}_{a_1}\otimes\ldots\otimes\hat{\boldsymbol{\Psi}}_{a_\varepsilon}\otimes\hat{\boldsymbol{\vartheta}}^{b_1}\otimes\ldots\otimes\hat{\boldsymbol{\vartheta}}^{b_\eta},\tag{13.12}$$

$$\hat{\bar{\boldsymbol{\Psi}}}^{\bar{b}_1\ldots\,\bar{b}_\zeta}_{\bar{a}_1\ldots\,\bar{a}_\sigma}=\hat{\bar{\boldsymbol{\Psi}}}_{\bar{a}_1}\otimes\ldots\otimes\hat{\bar{\boldsymbol{\Psi}}}_{\bar{a}_\sigma}\otimes\hat{\bar{\boldsymbol{\vartheta}}}^{b_1}\otimes\ldots\otimes\hat{\bar{\boldsymbol{\vartheta}}}^{b_\zeta}.\tag{13.13}$$

$$\hat{\boldsymbol{\Psi}}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}=\hat{\boldsymbol{\Psi}}^{b_1\ldots\,b_\eta}_{a_1\ldots\,a_\varepsilon}\otimes\hat{\bar{\boldsymbol{\Psi}}}^{\bar{b}_1\ldots\,\bar{b}_\zeta}_{\bar{a}_1\ldots\,\bar{a}_\sigma}\otimes\hat{\boldsymbol{\Upsilon}}^{d_1\ldots\,d_f}_{c_1\ldots\,c_e}.\tag{13.14}$$

Taking into account (13.5) and (13.14), from (13.4) we derive

$$\eta^{\,\omega}\ \mathbf{X}=\sum^{2}_{\substack{a_1,\,\ldots\,,\,a_\varepsilon\\ b_1,\,\ldots\,,\,b_\eta\\ \bar{a}_1,\,\ldots\,,\,\bar{a}_\sigma\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\zeta\\ c_1,\,\ldots\,,\,c_e\\ d_1,\,\ldots\,,\,d_f}}\!\!\ldots\!\!\sum^{3}\hat{X}^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\ \hat{\boldsymbol{\Psi}}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e},\tag{13.15}$$

where $\omega=\varepsilon+\eta+\sigma+\zeta+e+f+1$. Now we can apply a differentiation D not only to left hand side of (13.15), but to each summand in the right hand side of this equality. Using the item (4) of the definitions 12.1, from (13.15) we derive

$$\begin{gathered}D(\eta^{\omega}\ \mathbf{X})=\sum^{2}_{\substack{a_1,\,\ldots\,,\,a_\varepsilon\\ b_1,\,\ldots\,,\,b_\eta\\ \bar{a}_1,\,\ldots\,,\,\bar{a}_\sigma\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\zeta\\ c_1,\,\ldots\,,\,c_e\\ d_1,\,\ldots\,,\,d_f}}\!\!\ldots\!\!\sum^{3}D(\hat{X}^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f})\ \hat{\boldsymbol{\Psi}}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}+\\ +\sum^{2}_{\substack{a_1,\,\ldots\,,\,a_\varepsilon\\ b_1,\,\ldots\,,\,b_\eta\\ \bar{a}_1,\,\ldots\,,\,\bar{a}_\sigma\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\zeta\\ c_1,\,\ldots\,,\,c_e\\ d_1,\,\ldots\,,\,d_f}}\!\!\ldots\!\!\sum^{3}\hat{X}^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\ D(\hat{\boldsymbol{\Psi}}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}).\end{gathered}\tag{13.16}$$

Due to the lemma 13.4 we have $D(\eta^{\,\omega}\ \mathbf{X})=D(\mathbf{X})$ at the point q. Moreover, due to (13.5), (13.6), (13.7), (13.8), (13.9), (13.10), (13.11), (13.12), (13.13), and (13.14) and since $\eta(q)=1$, we have $\hat{\boldsymbol{\Psi}}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}=\boldsymbol{\Psi}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}$ at this point. As for the fields $D(\hat{X}^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f})$ and $D(\hat{\boldsymbol{\Psi}}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e})$ in (13.16), again due to the lemma 13.4 their values at the point q do not depend on a particular choice of the function η.

Since $\hat{X}^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}$ is a function, i. e. it is interpreted as an extended scalar field, we have $D(\hat{X}^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f})=\delta(\hat{X}^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f})$. Then from the formulas (13.2), (9.26), (9.18), (9.19), and (9.20) for the value of the function

$D(\hat X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f})$ at the point q we derive the following formula:

$$
\begin{aligned}
&D(\hat X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f})=\sum^3_{i=0}\sum^3_{j=0}Z^i\,\Upsilon^j_i\,\frac{\partial X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}}{\partial x^j}+\\
&+\sum^{J+Q}_{P=1}\sum^2_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta\\ \bar i_1,\,\ldots\,,\,\bar i_\nu\\ \bar j_1,\,\ldots\,,\,\bar j_\gamma\\ h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\sum^3 Z^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\,\frac{\partial X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}}{\partial S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]}+\\
&+\sum^{J+Q}_{P=1}\sum^2_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta\\ \bar i_1,\,\ldots\,,\,\bar i_\nu\\ \bar j_1,\,\ldots\,,\,\bar j_\gamma\\ h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\sum^3 \bar Z^{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]\,\frac{\partial X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}}{\overline{\partial S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]}}.
\end{aligned}
\tag{13.17}
$$

In order to evaluate $D(\hat{\boldsymbol\Psi}^{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e})$ at the point q let's apply D to (13.14). Using the item (4) of the definition 12.1, we obtain the following equality:

$$
\begin{aligned}
&D(\hat{\boldsymbol\Psi}^{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e})=D(\hat{\boldsymbol\Psi}^{b_1\ldots\,b_\eta}_{a_1\ldots\,a_\varepsilon})\otimes\hat{\bar{\boldsymbol\Psi}}^{\bar b_1\ldots\,\bar b_\zeta}_{\bar a_1\ldots\,\bar a_\sigma}\otimes\hat{\boldsymbol\Upsilon}^{d_1\ldots\,d_f}_{c_1\ldots\,c_e}+\\
&+\hat{\boldsymbol\Psi}^{b_1\ldots\,b_\eta}_{a_1\ldots\,a_\varepsilon}\otimes D(\hat{\bar{\boldsymbol\Psi}}^{\bar b_1\ldots\,\bar b_\zeta}_{\bar a_1\ldots\,\bar a_\sigma})\otimes\hat{\boldsymbol\Upsilon}^{d_1\ldots\,d_f}_{c_1\ldots\,c_e}+\hat{\boldsymbol\Psi}^{b_1\ldots\,b_\eta}_{a_1\ldots\,a_\varepsilon}\otimes\hat{\bar{\boldsymbol\Psi}}^{\bar b_1\ldots\,\bar b_\zeta}_{\bar a_1\ldots\,\bar a_\sigma}\otimes D(\hat{\boldsymbol\Upsilon}^{d_1\ldots\,d_f}_{c_1\ldots\,c_e}).
\end{aligned}
\tag{13.18}
$$

In order to evaluate $D(\hat{\boldsymbol\Psi}^{b_1\ldots\,b_\eta}_{a_1\ldots\,a_\varepsilon})$ in (13.18) we apply D to the equality (13.12):

$$
\begin{aligned}
D(\hat{\boldsymbol\Psi}^{b_1\ldots\,b_\eta}_{a_1\ldots\,a_\varepsilon})&=\sum^\varepsilon_{v=1}\boldsymbol\Psi_{a_1}\otimes\ldots\otimes D(\hat{\boldsymbol\Psi}_{a_v})\otimes\ldots\otimes\boldsymbol\Psi_{a_\varepsilon}\otimes\\
&\otimes\boldsymbol\vartheta^{\,b_1}\otimes\ldots\otimes\boldsymbol\vartheta^{\,b_\eta}+\sum^\eta_{w=1}\boldsymbol\Psi_{a_1}\otimes\ldots\otimes\boldsymbol\Psi_{a_\varepsilon}\otimes\\
&\otimes\boldsymbol\vartheta^{\,b_1}\otimes\ldots\otimes D(\hat{\boldsymbol\vartheta}^{\,b_w})\otimes\ldots\otimes\boldsymbol\vartheta^{\,b_\eta}.
\end{aligned}
\tag{13.19}
$$

Similarly, in order to evaluate $D(\hat{\bar{\boldsymbol\Psi}}^{\bar b_1\ldots\,\bar b_\zeta}_{\bar a_1\ldots\,\bar a_\sigma})$ in (13.18) we apply D to (13.13):

$$
\begin{aligned}
D(\hat{\bar{\boldsymbol\Psi}}^{\bar b_1\ldots\,\bar b_\zeta}_{\bar a_1\ldots\,\bar a_\sigma})&=\sum^\sigma_{v=1}\hat{\bar{\boldsymbol\Psi}}_{\bar a_1}\otimes\ldots\otimes D(\hat{\bar{\boldsymbol\Psi}}_{\bar a_v})\otimes\ldots\otimes\hat{\bar{\boldsymbol\Psi}}_{\bar a_\sigma}\otimes\\
&\otimes\hat{\bar{\boldsymbol\vartheta}}^{\,b_1}\otimes\ldots\otimes\hat{\bar{\boldsymbol\vartheta}}^{\,b_\zeta}+\sum^\zeta_{w=1}\hat{\bar{\boldsymbol\Psi}}_{\bar a_1}\otimes\ldots\otimes\hat{\bar{\boldsymbol\Psi}}_{\bar a_\sigma}\otimes\\
&\otimes\hat{\bar{\boldsymbol\vartheta}}^{\,b_1}\otimes\ldots\otimes D(\hat{\bar{\boldsymbol\vartheta}}^{\,b_w})\otimes\ldots\otimes\hat{\bar{\boldsymbol\vartheta}}^{\,b_\zeta}.
\end{aligned}
\tag{13.20}
$$

And finally, in order to evaluate $D(\hat{\mathbf{\Upsilon}}^{d_1\ldots\,d_f}_{c_1\ldots\,c_e})$ in (13.18) we apply D to (13.11). As a result we get the following equality:

$$\begin{aligned} D(\hat{\mathbf{\Upsilon}}^{d_1\ldots\,d_f}_{c_1\ldots\,c_e}) &= \sum_{v=1}^{e} \mathbf{\Upsilon}_{c_1} \otimes \ldots \otimes\, D(\hat{\mathbf{\Upsilon}}_{c_v}) \otimes \ldots \otimes \mathbf{\Upsilon}_{c_e} \otimes \\ &\otimes \boldsymbol{\eta}^{d_1} \otimes \ldots \otimes \boldsymbol{\eta}^{d_f} + \sum_{w=1}^{f} \mathbf{\Upsilon}_{c_1} \otimes \ldots \otimes \mathbf{\Upsilon}_{c_e} \otimes \\ &\otimes \boldsymbol{\eta}^{d_1} \otimes \ldots \otimes D(\hat{\boldsymbol{\eta}}^{d_w}) \otimes \ldots \otimes \boldsymbol{\eta}^{d_f}. \end{aligned} \tag{13.21}$$

Applying (13.19), (13.20), (13.21) to (13.18) and then substituting (13.18) and (13.17) into the equality (13.16), we derive the following lemma.

Lemma 13.5. *Any differentiation D of the algebra of extended spin-tensorial fields is uniquely fixed by its restrictions to the modules $S^0_0\bar{S}^0_0T^0_0(M)$, $S^1_0\bar{S}^0_0T^0_0(M)$, $S^0_1\bar{S}^0_0T^0_0(M)$, $S^0_0\bar{S}^1_0T^0_0(M)$, $S^0_0\bar{S}^0_1T^0_0(M)$, $S^0_0\bar{S}^0_0T^1_0(M)$, and $S^0_0\bar{S}^0_0T^0_1(M)$ in (11.1).*

In (13.21) the value of the extended vector field $D(\hat{\mathbf{\Upsilon}}_h)$ at the point q is a vector of $\mathbb{C}T_{\pi(q)}(M)$. We can write the following expansion for this vector:

$$D(\hat{\mathbf{\Upsilon}}_h) = \sum_{k=0}^{3} \Gamma^k_h\, \mathbf{\Upsilon}_k. \tag{13.22}$$

Due to the lemma 13.4 the left hand side of the equality (13.22) does not depend on a particular choice of the function η in (13.6). Therefore, the coefficients Γ^k_h in (13.22) represent the differentiation D at the point q for a given local chart U in M. The same is true for Z^i, $Z^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]$, $Z^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]$, and $\bar{Z}^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]$ in (13.17). Being dependent on q, all these coefficients are some smooth complex-valued functions of the variables (9.11) and (9.12). However, if q is fixed, they all are complex constants.

For the fields $D(\hat{\mathbf{\Psi}}_i)$ in (13.19) and for $D(\hat{\bar{\mathbf{\Psi}}}_i)$ in (13.20) we have the expansions similar to the above expansion (13.22):

$$\begin{aligned} D(\hat{\mathbf{\Psi}}_i) &= \sum_{k=1}^{2} \mathrm{A}^k_i\; \mathbf{\Psi}_k, \\ D(\hat{\bar{\mathbf{\Psi}}}_i) &= \sum_{k=1}^{2} \bar{\mathrm{A}}^k_i\; \bar{\mathbf{\Psi}}_k. \end{aligned} \tag{13.23}$$

In general case A^j_i and $\bar{\mathrm{A}}^j_i$ are arbitrary complex-valued functions of the variables (9.11) and (9.12). They are complex constants if $q \in N$ is fixed.

Let's return back to the formulas (13.19), (13.20), and (13.21). For the values of the fields $D(\hat{\boldsymbol{\vartheta}}^{\,j})$, $D(\hat{\bar{\boldsymbol{\vartheta}}}^{\,j})$, $D(\hat{\boldsymbol{\eta}}^k)$ at the point q we write

$$D(\hat{\boldsymbol{\eta}}^k) = -\sum_{h=0}^{3} \Gamma^k_h\; \boldsymbol{\eta}^{\,h}, \tag{13.24}$$

$$D(\hat{\boldsymbol{\vartheta}}^{\,j}) = -\sum_{h=0}^{3} \mathrm{A}^j_h\ \ \boldsymbol{\vartheta}^{\,h}, \tag{13.25}$$

$$D(\hat{\overline{\boldsymbol{\vartheta}}}{}^{\,j}) = -\sum_{h=0}^{3} \bar{\mathrm{A}}^j_h\ \ \overline{\boldsymbol{\vartheta}}{}^{\,h}. \tag{13.26}$$

Let $\mathbf{X}$ be an extended spin-tensorial field of the type $(\varepsilon,\eta|\sigma,\zeta|e,f)$ given by the expansion (13.4). Then $D(\mathbf{X})$ is given by the analogous expansion:

$$D(\mathbf{X}) = \sum^{2}_{\substack{a_1,\,\ldots\,,\,a_\varepsilon\\ b_1,\,\ldots\,,\,b_\eta\\ \bar{a}_1,\,\ldots\,,\,\bar{a}_\sigma\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\zeta\\ c_1,\,\ldots\,,\,c_e\\ d_1,\,\ldots\,,\,d_f}}\!\!\ldots\!\!\sum^{3} DX^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma h_1\ldots\,h_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta k_1\ldots\,k_f}\ \ \mathbf{\Psi}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}. \tag{13.27}$$

where

$$\begin{aligned}
&DX^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f} = \sum_{i=0}^{3}\sum_{j=0}^{3} Z^i\,\Upsilon^j_i\,\frac{\partial X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}}{\partial x^j}\ +\\
&+\sum_{P=1}^{J+Q}\sum^{2}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta\\ \bar{i}_1,\,\ldots\,,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots\,,\,\bar{j}_\gamma\\ h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\!\!\sum^{3} Z^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]\ \frac{\partial X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}}{\partial S^{\,i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]}\ +\\
&+\sum_{P=1}^{J+Q}\sum^{2}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta\\ \bar{i}_1,\,\ldots\,,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots\,,\,\bar{j}_\gamma\\ h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\!\!\sum^{3} \bar{Z}^{\bar{i}_1\ldots\,\bar{i}_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\bar{j}_1\ldots\,\bar{j}_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]\ \frac{\partial X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}}{\overline{\partial S^{\,i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]}}\ +\\
&+\sum_{\mu=1}^{\varepsilon}\sum_{v_\mu=1}^{2} \mathrm{A}^{a_\mu}_{v_\mu}\ \ X^{a_1\ldots\,v_\mu\,\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,\ldots\,\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\ -\\
&\qquad-\sum_{\mu=1}^{\eta}\sum_{w_\mu=1}^{2} \mathrm{A}^{w_\mu}_{b_\mu}\ \ X^{a_1\ldots\,\ldots\,\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,w_\mu\,\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\ +\\
&+\sum_{\mu=1}^{\sigma}\sum_{v_\mu=1}^{2} \bar{\mathrm{A}}^{\bar{a}_\mu}_{v_\mu}\ \ X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,v_\mu\,\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\ldots\,\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\ -\\
&\qquad-\sum_{\mu=1}^{\zeta}\sum_{w_\mu=1}^{2} \bar{\mathrm{A}}^{w_\mu}_{\bar{b}_\mu}\ \ X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\ldots\,\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,w_\mu\,\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\ +\\
&+\sum_{\mu=1}^{e}\sum_{v_\mu=0}^{3} \Gamma^{c_\mu}_{v_\mu}\ \ X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,v_\mu\,\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,\ldots\,\ldots\,d_f}\ -\\
&\qquad-\sum_{\mu=1}^{f}\sum_{w_\mu=0}^{3} \Gamma^{w_\mu}_{b_\mu}\ \ X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,\ldots\,\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,w_\mu\,\ldots\,d_f}.
\end{aligned} \tag{13.28}$$

Lemma 13.6. *Any differentiation D of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$ is uniquely fixed by its restrictions to the modules $S^0_0\bar{S}^0_0T^0_0(M)$, $S^1_0\bar{S}^0_0T^0_0(M)$, $S^0_0\bar{S}^1_0T^0_0(M)$, $S^0_0\bar{S}^0_0T^1_0(M)$ in the direct sum (11.1).*

This lemma 13.6 strengthens the previous lemma 13.5. It is proved by direct calculations when we apply (13.24), (13.25), (13.26) to (13.19), (13.20), and (13.21). Another proof can be based on (13.27) and on the general explicit formula (13.28) for the components of the field $D(\mathbf{X})$. Indeed, we see that any differentiation D is completely determined by the parameters Z^i, $Z^{i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]$, $\bar{Z}^{\bar{i}_1\dots\,\bar{i}_\nu\,i_1\dots\,i_\alpha\,h_1\dots\,h_m}_{\bar{j}_1\dots\,\bar{j}_\gamma\,j_1\dots\,j_\beta\,k_1\dots\,k_n}[P]$ characterizing the restriction of D to $S^0_0\bar{S}^0_0T^0_0(M)$ and by the parameters A^k_i, $\bar{\mathrm{A}}^k_i$, Γ^k_h describing the restrictions of D to $S^1_0\bar{S}^0_0T^0_0(M)$, $S^0_0\bar{S}^1_0T^0_0(M)$, and $S^0_0\bar{S}^0_0T^1_0(M)$ respectively.

Z-parameters, A-parameters and Γ-parameters determining a differentiation D in (13.28) obey some definite transformation rules under a change of a local chart. The transformation rule for Z^i is the most simple one:

$$Z^i=\sum_{j=0}^{3}S^i_j\,\tilde{Z}^j. \tag{13.29}$$

The transformation rule for $Z^{i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]$ is much more huge:

$$\begin{gathered}
Z^{i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]=\sum^{2}_{\substack{a_1,\,\dots,\,a_\alpha\\ b_1,\,\dots,\,b_\beta\\ \bar{a}_1,\,\dots,\,\bar{a}_\nu\\ \bar{b}_1,\,\dots,\,\bar{b}_\gamma}}\!\!\dots\sum^{2}\sum^{3}_{\substack{c_1,\,\dots,\,c_m\\ d_1,\,\dots,\,d_n}}\!\!\dots\sum^{3}\mathfrak{S}^{\,i_1}_{a_1}\dots\,\mathfrak{S}^{\,i_\alpha}_{a_\alpha}\times\\
\times\,\mathfrak{T}^{b_1}_{j_1}\dots\,\mathfrak{T}^{b_\beta}_{j_\beta}\,\overline{\mathfrak{S}^{\bar{i}_1}_{\bar{a}_1}}\dots\,\overline{\mathfrak{S}^{\bar{i}_\nu}_{\bar{a}_\nu}}\;\overline{\mathfrak{T}^{\bar{b}_1}_{\bar{j}_1}}\dots\,\overline{\mathfrak{T}^{\bar{b}_\gamma}_{\bar{j}_\gamma}}\;S^{h_1}_{c_1}\dots\,S^{h_m}_{c_m}\,T^{d_1}_{k_1}\dots\,T^{d_n}_{k_n}\times\\
\times\,\tilde{Z}^{a_1\dots\,a_\alpha\,\bar{a}_1\dots\,\bar{a}_\nu\,c_1\dots\,c_m}_{b_1\dots\,b_\beta\,\bar{b}_1\dots\,\bar{b}_\gamma\,d_1\dots\,d_n}[P]-\sum^{\alpha}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{2}_{v_\mu=1}\vartheta^{\,i_\mu}_{i\,v_\mu}\times\\
\times\;S^{\,i_1\dots\,v_\mu\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{\,j_1\dots\dots\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]\;S^i_j\;\tilde{Z}^j+\sum^{\beta}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{2}_{w_\mu=1}\vartheta^{\,w_\mu}_{i\,j_\mu}\times\\
\times\;S^{\,i_1\dots\dots\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,w_\mu\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]\;S^i_j\;\tilde{Z}^j-\sum^{\nu}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{2}_{v_\mu=1}\overline{\vartheta^{\,\bar{i}_\mu}_{i\,v_\mu}}\times\\
\times\;S^{\,i_1\dots\,i_\alpha\,\bar{i}_1\dots\,v_\mu\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar{j}_1\dots\dots\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]\;S^i_j\;\tilde{Z}^j+\sum^{\gamma}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{2}_{w_\mu=1}\overline{\vartheta^{\,w_\mu}_{i\,\bar{j}_\mu}}\times\\
\times\;S^{\,i_1\dots\,i_\alpha\,\bar{i}_1\dots\dots\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar{j}_1\dots\,w_\mu\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]\;S^i_j\;\tilde{Z}^j-\sum^{m}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{3}_{v_\mu=0}\theta^{\,i_\mu}_{i\,v_\mu}\times\\
\times\;S^{\,i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,v_\mu\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\dots\dots\,k_n}[P]\;S^i_j\;\tilde{Z}^j+\sum^{n}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{3}_{w_\mu=0}\theta^{\,w_\mu}_{i\,j_\mu}\times\\
\times\;S^{\,i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\dots\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,w_\mu\dots\,k_n}[P]\;S^i_j\;\tilde{Z}^j.
\end{gathered}\tag{13.30}$$

The transformation rule for $\bar{Z}^{\bar{i}_1\dots\,\bar{i}_\nu\,i_1\dots\,i_\alpha\,h_1\dots\,h_m}_{\bar{j}_1\dots\,\bar{j}_\gamma\,j_1\dots\,j_\beta\,k_1\dots\,k_n}[P]$ is equally huge as the previous transformation rule for $Z^{i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]$:

$$
\begin{aligned}
&\bar{Z}^{\bar{i}_1\dots\,\bar{i}_\nu\,i_1\dots\,i_\alpha\,h_1\dots\,h_m}_{\bar{j}_1\dots\,\bar{j}_\gamma\,j_1\dots\,j_\beta\,k_1\dots\,k_n}[P]=\sum^{2}_{\substack{a_1,\,\dots,\,a_\alpha\\ b_1,\,\dots,\,b_\beta\\ \bar{a}_1,\,\dots,\,\bar{a}_\nu\\ \bar{b}_1,\,\dots,\,\bar{b}_\gamma}}\!\!\dots\sum^{2}\sum^{3}_{\substack{c_1,\,\dots,\,c_m\\ d_1,\,\dots,\,d_n}}\!\!\dots\sum^{3}\mathfrak{S}^{\bar{i}_1}_{\bar{a}_1}\dots\,\mathfrak{S}^{\bar{i}_\nu}_{\bar{a}_\nu}\times\\
&\times\mathfrak{T}^{\bar{b}_1}_{\bar{j}_1}\dots\,\mathfrak{T}^{\bar{b}_\gamma}_{\bar{j}_\gamma}\,\overline{\mathfrak{S}^{i_1}_{a_1}}\dots\,\overline{\mathfrak{S}^{i_\alpha}_{a_\alpha}}\,\,\overline{\mathfrak{T}^{b_1}_{j_1}}\dots\,\overline{\mathfrak{T}^{b_\beta}_{j_\beta}}\,S^{h_1}_{c_1}\dots\,S^{h_m}_{c_m}\,T^{d_1}_{k_1}\dots\,T^{d_n}_{k_n}\times\\
&\times\tilde{\bar{Z}}^{\bar{a}_1\dots\,\bar{a}_\nu\,a_1\dots\,a_\alpha\,c_1\dots\,c_m}_{\bar{b}_1\dots\,\bar{b}_\gamma\,b_1\dots\,b_\beta\,d_1\dots\,d_n}[P]-\sum^{\alpha}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{2}_{v_\mu=1}\overline{\vartheta^{i_\mu}_{i\,v_\mu}}\times\\
&\times\overline{S^{\,i_1\dots\,v_\mu\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,\dots\,\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]}\,S^i_j\,\tilde{Z}^j+\sum^{\beta}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{2}_{w_\mu=1}\overline{\vartheta^{w_\mu}_{i\,j_\mu}}\times\\
&\times\overline{S^{\,i_1\dots\,\dots\,\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,w_\mu\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]}\,S^i_j\,\tilde{Z}^j-\sum^{\nu}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{2}_{v_\mu=1}\vartheta^{\bar{i}_\mu}_{i\,v_\mu}\times\\
&\times\overline{S^{\,i_1\dots\,i_\alpha\,\bar{i}_1\dots\,v_\mu\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar{j}_1\dots\,\dots\,\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]}\,S^i_j\,\tilde{Z}^j+\sum^{\gamma}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{2}_{w_\mu=1}\vartheta^{w_\mu}_{i\,\bar{j}_\mu}\times\\
&\times\overline{S^{\,i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\dots\,\dots\,\bar{i}_\nu\,h_1\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar{j}_1\dots\,w_\mu\dots\,\bar{j}_\gamma\,k_1\dots\,k_n}[P]}\,S^i_j\,\tilde{Z}^j-\sum^{m}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{3}_{v_\mu=0}\theta^{i_\mu}_{i\,v_\mu}\times\\
&\times\overline{S^{\,i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,v_\mu\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,\dots\,\dots\,k_n}[P]}\,S^i_j\,\tilde{Z}^j+\sum^{n}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{j=0}\sum^{3}_{w_\mu=0}\theta^{w_\mu}_{i\,j_\mu}\times\\
&\times\overline{S^{\,i_1\dots\,i_\alpha\,\bar{i}_1\dots\,\bar{i}_\nu\,h_1\dots\,\dots\,\dots\,h_m}_{\,j_1\dots\,j_\beta\,\bar{j}_1\dots\,\bar{j}_\gamma\,k_1\dots\,w_\mu\dots\,k_n}[P]}\,S^i_j\,\tilde{Z}^j.
\end{aligned}
\tag{13.31}
$$

The transformation rules (13.29), (13.30), (13.31) are completed with the transformation rules for A^k_i, $\bar{\mathrm{A}}^k_i$ and Γ^k_h. They are less huge:

$$\mathrm{A}^k_i=\sum^{2}_{b=1}\sum^{2}_{a=1}\mathfrak{S}^k_a\,\mathfrak{T}^b_i\,\tilde{\mathrm{A}}^a_b+\sum^{3}_{a=0}Z^a\,\vartheta^k_{ai},\tag{13.32}$$

$$\bar{\mathrm{A}}^k_i=\sum^{2}_{b=1}\sum^{2}_{a=1}\overline{\mathfrak{S}^k_a}\,\overline{\mathfrak{T}^b_i}\,\tilde{\bar{\mathrm{A}}}^a_b+\sum^{3}_{a=0}Z^a\,\overline{\vartheta^k_{ai}},\tag{13.33}$$

$$\Gamma^k_h=\sum^{3}_{b=0}\sum^{3}_{a=0}S^k_a\,T^b_h\,\tilde{\Gamma}^a_b+\sum^{3}_{a=0}Z^a\,\theta^k_{ah}.\tag{13.34}$$

These transformation rules (13.29), (13.30), (13.31) are derived from (13.2) using (9.21), (9.23), and (9.37). The transformation rules (13.32), (13.33), and (13.34) are derived from (13.23) and (13.22) using the formulas (9.17), (9.16), (9.29), (9.28), (9.33), and (9.35).

Note that a differentiation D acts as a first order linear differential operator upon the components of an extended spin-tensorial field $\mathbf{X}$ in the formula (13.28). The coefficients Z^i, $Z^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]$, $Z^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]$, $\bar Z^{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]$, A^k_i, $\bar{\mathrm{A}}^k_i$, Γ^k_i of the linear operator in this formula are not differentiated. Therefore, fixing some point $q \in N$ and taking the values of these coefficients at the point q, we can say that we know this linear operator at that particular point q even if we don't know the values of theese coefficients at other points.

Definition 13.1. Let N be a composite spin-tensorial bundle over the space-time manifold M (in the sense of the formula (9.7)). A *spin-tensorial first order differential operator* D_q at a point $q \in N$ is a geometric object associated with the point q and represented by the set of constants Z^i, $Z^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]$, $\bar Z^{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]$, Γ^k_i, A^k_i, $\bar{\mathrm{A}}^k_i$ in a proper local chart such that they obey the transformation rules (13.29), (13.30), (13.31), (13.32), (13.33), (13.34).

Spin-tensorial first order differential operators at a fixed point q constitute a finite-dimensional linear space over the field of complex numbers $\mathbb{C}$, we denote it with the symbol $\mathfrak{D}(q, M)$. Its dimension is given by the formula

$$\dim_{\mathbb{C}} \mathfrak{D}(q, M) = \dim_{\mathbb{R}} N + 4^2 + 2^2 + 2^2. \tag{13.35}$$

The dimension of the composite spin-tensorial bundle N in (13.35) is given by the formula (9.10). The linear spaces $\mathfrak{D}(q, M)$ with q running over N are glued into a complex vector bundle over N.

Theorem 13.1. *Any differentiation D of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$ is represented as a field of differential operators $D_q \in \mathfrak{D}(q, M)$, one per each point $q \in N$. Conversely, each smooth field of spin-tensorial first order differential operators is a differentiation of the algebra $\mathbf{S}(M)$.*

The theorem 13.1 solves the problem of localization announced in the very beginning of this section.

14. Degenerate differentiations.

Definition 14.1. A differentiation D of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$ is called a *degenerate differentiation* if its restriction (13.1) to the module $S^0_0\bar S^0_0 T^0_0(M)$ is identically zero.

For a degenerate differentiation D from the formula (13.2) we derive that all its Z-components are identically equal to zero:

$$Z^i = 0, \tag{14.1}$$

$$Z^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P] = 0, \tag{14.2}$$

$$\bar Z^{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P] = 0. \tag{14.3}$$

Unlike Z-components, A-components and Γ-components of such a differentiation D in general case are not zero. Substituting (14.1) into the formulas (13.32), (13.33), and (13.33), we find that the transformation rules for A-components and Γ-components of a degenerate differentiation D reduce to the following ones:

$$\mathrm{A}^k_i=\sum_{b=1}^{2}\sum_{a=1}^{2}\mathfrak{S}^k_a\,\mathfrak{T}^b_i\;\tilde{\mathrm{A}}^a_b,\tag{14.4}$$

$$\bar{\mathrm{A}}^k_i=\sum_{b=1}^{2}\sum_{a=1}^{2}\overline{\mathfrak{S}^k_a}\;\overline{\mathfrak{T}^b_i}\;\tilde{\bar{\mathrm{A}}}^a_b.\tag{14.5}$$

$$\Gamma^k_h=\sum_{b=0}^{3}\sum_{a=0}^{3}S^k_a\,T^b_h\;\tilde{\Gamma}^a_b.\tag{14.6}$$

Note that these transformation rules (14.4), (14.5), (14.6) are special cases of the transformation rule (10.3) for the components of an extended spin-tensorial field, while (10.2) is inverse to (10.3). This observation proves the following theorem.

Theorem 14.1. *Defining a degenerate differentiation D of the algebra $\mathbf{S}(M)$ is equivalent to defining three extended spin-tensorial fields $\boldsymbol{\mathfrak{S}}$, $\bar{\boldsymbol{\mathfrak{S}}}$, and $\mathbf{S}$ of the types* $(1,1|0,0|0,0)$, $(0,0|1,1|0,0)$, *and* $(0,0|0,0|1,1)$ *respectively.*

By tradition we use the symbols $\boldsymbol{\mathfrak{S}}$, $\bar{\boldsymbol{\mathfrak{S}}}$, and $\mathbf{S}$ for the extended spin-tensorial fields determined by the A and the Γ-components of a degenerate differentiation D. One should be careful for not to confuse the components of the field $\mathbf{S}$ and the components of the direct transition matrix S (e. g. in the formula (14.6)).

15. Covariant differentiations.

The set of differentiations of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$ possesses the structure of a module over the ring of smooth complex functions $\mathfrak{F}_{\mathbb{C}}(N)$. The set of complexified extended vector fields $\mathbb{C}T^1_0(M)=S^0_0\bar{S}^0_0T^1_0(M)$ (see (10.1)) also is a module over the same ring $\mathfrak{F}_{\mathbb{C}}(N)$. Therefore, the following definition is consistent.

Definition 15.1. Say that in the space-time manifold M a covariant differentiation of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$ is given if some homomorphism of $\mathfrak{F}_{\mathbb{C}}(N)$-modules $\nabla\colon\ \mathbb{C}T^1_0(M)\to\mathfrak{D}_{\mathbf{S}}(M)$ is given. The image of a vector field $\mathbf{Y}$ under such homomorphism is denoted by $\nabla_{\mathbf{Y}}$. The differentiation $D=\nabla_{\mathbf{Y}}$ is called the *covariant differentiation along the vector field* $\mathbf{Y}$.

Let's note that the $\mathfrak{F}_{\mathbb{C}}(N)$-module $\mathfrak{D}_{\mathbf{S}}(M)$ admits a localization in the sense of the following definition (compare with the definition 8.2 in [5]).

Definition 15.2. Let A be a module over the ring of smooth complex functions $\mathfrak{F}_{\mathbb{C}}(M)$ in some smooth real manifold M. We say that the module A admits a *localization* if it is isomorphic to a functional module so that each element $\mathbf{a}\in A$ is represented as some function $\mathbf{a}(q)=\mathbf{a}_q$ in M taking its values in some $\mathbb{C}$-linear spaces A_q associated with each point q of the manifold M.

Indeed, according to the theorem 13.1 each differentiation D is a field of differential operators. As for the differential operators themselves, they form finite-dimensional $\mathbb{C}$-linear spaces $\mathfrak{D}(q,M)$ of the dimension (13.35), one per each point $q\in N$.

The definition 15.1 is a complexified version of the definition 8.2 from [5]. One can easily formulate complexified versions for the definition 8.3 and for the theorems 8.2 and 8.3 from [5].

Definition 15.3. Let A be a module that admits a localization in the sense of the definition 15.2. We say that the localization of A is a *complete localization* if the following two conditions are fulfilled:

(1) for any point $q \in M$ and for any vector $\mathbf{v} \in A_q$ there exists an element $\mathbf{a} \in A$ such that $\mathbf{a}_q = \mathbf{v}$;
(2) if $\mathbf{a}_q = 0$ at some point $q \in M$, then there exist some finite set of elements $\mathbf{E}_0, \ldots, \mathbf{E}_n$ in A and some smooth complex functions $\alpha_0 \ \ldots, \alpha_n$ vanishing at the point q such that $\mathbf{a} = \alpha_0\,\mathbf{E}_0 + \ldots + \alpha_n\,\mathbf{E}_n$.

Theorem 15.1. *Let A and B be two $\mathfrak{F}_{\mathbb{C}}(M)$-modules that admit localizations. If the localization of A is a complete localization, then each homomorphism $f\colon A \to B$ is represented by a family of $\mathbb{C}$-linear mappings*

$$F_q\colon A_q \to B_q \tag{15.1}$$

so that if $\mathbf{a} \in A$ and $\mathbf{b} = f(\mathbf{a})$, then $\mathbf{b}_q = F_q(\mathbf{a}_q)$ for each point $q \in M$.

Theorem 15.2. *Let $\pi : VM \to M$ be a smooth n-dimensional complex vector bundle over some smooth real base manifold M and let A be the $\mathfrak{F}_{\mathbb{C}}(M)$-module of all global smooth sections*[1] *of this bundle. Then A admits a complete localization in the sense of the definition 15.3.*

The proof of the theorems 15.1 and 15.2 is analogous to the proof of the theorems 8.1 and 8.2 in [5]. We leave this proof to the reader.

Now let's return back to the spaces $\mathfrak{D}(q, M)$ with q running over N. Remember that they are glued into a complex vector bundle for which N is a base manifold. Similarly, the complexified tangent spaces $\mathbb{C}T_{\pi(q)}(M) = \mathbb{C}T^1_0(\pi(q), M)$ are also glued into a complex vector bundle over N. This bundle is called an *induced bundle*. It is induced by the projection map (9.9) from the complexified tangent bundle $\mathbb{C}TM$.

Note that the $\mathfrak{F}_{\mathbb{C}}(N)$ modules $\mathbb{C}T^1_0(M)$ and $\mathfrak{D}_{\mathbf{S}}(M)$ in the definition 15.1 are represented as the sets of all smooth global sections for the above two bundles. For this reason we can apply the theorem 15.2 to $\mathbb{C}T^1_0(M)$ and $\mathfrak{D}_{\mathbf{S}}(M)$. As a result we conclude that each covariant differentiation ∇ of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$ is composed by $\mathbb{C}$-linear maps $\mathbb{C}T_{\pi(q)}(M) \to \mathfrak{D}(q, M)$ specific to each point $q \in N$. This fact is expressed by the following formula:

$$\nabla_{\mathbf{Y}}\mathbf{X} = C(\mathbf{Y} \otimes \nabla\mathbf{X}). \tag{15.2}$$

If $\mathbf{Y}$ in (15.2) is given by the expansion

$$\mathbf{Y} = \sum_{j=0}^{3} Y^j\ \boldsymbol{\Upsilon}_j$$

[1] See the definition of *sections* and *smooth sections* in [10].

in some positively polarized right orthonormal frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and if $\mathbf{X}$ is an extended spin-tensorial field of the type $(\varepsilon,\eta|\sigma,\zeta|e,f)$ given by the expansion (13.4), then the formula (15.2) is specified to the following one:

$$\nabla_{\mathbf{Y}}\mathbf{X}=\sum^{3}_{j=0}\sum^{2}_{\substack{a_1,\,\ldots,\,a_\varepsilon\\ b_1,\,\ldots,\,b_\eta\\ \bar a_1,\,\ldots,\,\bar a_\sigma\\ \bar b_1,\,\ldots,\,\bar b_\zeta\\ c_1,\,\ldots,\,c_e\\ d_1,\,\ldots,\,d_f}}\!\!\ldots\sum^{3}Y^j\ \nabla_{\!j}X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}\ \boldsymbol{\Psi}^{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}. \tag{15.3}$$

Looking at the formula (15.2), we see that each covariant differentiation ∇ can be treated as an operator producing the extended spin-tensorial field $\nabla\mathbf{X}$ of the type $(\varepsilon,\eta|\sigma,\zeta|e,f+1)$ from any given extended spin-tensorial field $\mathbf{X}$ of the type $(\varepsilon,\eta|\sigma,\zeta|e,f)$. This operator increases by one the number of covariant tensorial indices of a spin-tensorial field $\mathbf{X}$. It is called the operator of *covariant differential* associated with the covariant differentiation ∇. By $\nabla_{\!j}X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}$ in (15.3) we denote the components of the field $\nabla\mathbf{X}$. Comparing (13.27) and (15.3), we can write the following equality for the differentiation $D=\nabla_{\mathbf{Y}}$:

$$DX^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma h_1\ldots\,h_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta k_1\ldots\,k_f}=\sum^{3}_{j=0}Y^j\ \nabla_{\!j}X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}. \tag{15.4}$$

Note that $\nabla_{\!j}$ in (15.4) is the symbol of a covariant derivative. Unlike ∇ and $\nabla_{\mathbf{Y}}$, covariant derivatives are applied not to spin-tensorial fields, but to its components in expansions like (13.4).

Let's consider the linear map $\mathbb{C}T_{\pi(q)}(M)\to\mathfrak{D}(q,M)$ produced by some covariant differentiation ∇ at some particular point $q\in N$. In a local chart this map is given by some linear functions expressing the components of the differential operator D_q, where $D=\nabla_{\mathbf{Y}}$, through the components of the vector $\mathbf{Y}_q$:

$$Z^{\,i}=\sum^{3}_{j=0}Z^{\,i}_j\ Y^j, \tag{15.5}$$

$$Z^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]=\sum^{3}_{j=0}Z^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\ Y^j, \tag{15.6}$$

$$\bar Z^{\,\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]=\sum^{3}_{j=0}\bar Z^{\,\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\,j\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]\ Y^j \tag{15.7}$$

$$\Gamma^k_i=\sum^{3}_{j=0}\Gamma^k_{j\,i}\ Y^j, \tag{15.8}$$

$$\mathrm{A}^k_i=\sum^{3}_{j=0}\mathrm{A}^k_{j\,i}\ \ Y^j, \tag{15.9}$$

$$\bar{\mathrm{A}}^k_i=\sum^{3}_{j=0}\bar{\mathrm{A}}^k_{j\,i}\ Y^j. \tag{15.10}$$

Substituting (15.5), (15.6), (15.7), (15.8), (15.9), and (15.10) into the formula (13.28), then taking into account (15.4), we derive the following formula:

$$
\begin{aligned}
&\nabla_j X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}=\sum_{i=0}^{3}\sum_{k=0}^{3}Z^{\,i}_j\,\Upsilon^k_i\,\frac{\partial X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}}{\partial x^k}\,+\\
&+\sum_{P=1}^{J+Q}\sum_{\substack{i_1,\,\ldots,\,i_\alpha\\ j_1,\,\ldots,\,j_\beta\\ \bar{i}_1,\,\ldots,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots,\,\bar{j}_\gamma\\ h_1,\,\ldots,\,h_m\\ k_1,\,\ldots,\,k_n}}^{2}\!\!\ldots\!\sum^{3} Z^{\,i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]\;\frac{\partial X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}}{\partial S^{\,i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]}\,+\\
&+\sum_{P=1}^{J+Q}\sum_{\substack{i_1,\,\ldots,\,i_\alpha\\ j_1,\,\ldots,\,j_\beta\\ \bar{i}_1,\,\ldots,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots,\,\bar{j}_\gamma\\ h_1,\,\ldots,\,h_m\\ k_1,\,\ldots,\,k_n}}^{2}\!\!\ldots\!\sum^{3} \bar{Z}^{\,\bar{i}_1\ldots\,\bar{i}_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{j\,\bar{j}_1\ldots\,\bar{j}_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]\;\frac{\partial X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}}{\overline{\partial S^{\,i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]}}\,+\\
&+\sum_{\mu=1}^{\varepsilon}\sum_{v_\mu=1}^{2}\mathrm{A}^{a_\mu}_{j\,v_\mu}\;X^{a_1\ldots\,v_\mu\,\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,\ldots\,\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\,-\\
&\qquad\qquad-\sum_{\mu=1}^{\eta}\sum_{w_\mu=1}^{2}\mathrm{A}^{w_\mu}_{j\,b_\mu}\;X^{a_1\ldots\,\ldots\,\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,w_\mu\,\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\,+\\
&+\sum_{\mu=1}^{\sigma}\sum_{v_\mu=1}^{2}\bar{\mathrm{A}}^{\bar{a}_\mu}_{j\,v_\mu}\;X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,v_\mu\,\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\ldots\,\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\,-\\
&\qquad\qquad-\sum_{\mu=1}^{\zeta}\sum_{w_\mu=1}^{2}\bar{\mathrm{A}}^{w_\mu}_{j\,\bar{b}_\mu}\;X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\ldots\,\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,w_\mu\,\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\,+\\
&+\sum_{\mu=1}^{e}\sum_{v_\mu=0}^{3}\Gamma^{c_\mu}_{j\,v_\mu}\;X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,v_\mu\,\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,\ldots\,\ldots\,d_f}\,-\\
&\qquad\qquad-\sum_{\mu=1}^{f}\sum_{w_\mu=0}^{3}\Gamma^{w_\mu}_{j\,b_\mu}\;X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,\ldots\,\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,w_\mu\,\ldots\,d_f}.
\end{aligned}
\tag{15.11}
$$

The formula (15.11) is an explicit formula for the covariant derivative ∇_j associated with the covariant differentiation ∇. It is also called a *coordinate representation* of the covariant differentiation ∇. The quantities $Z^{\,i}_j$, $Z^{\,i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]$, $\bar{Z}^{\,\bar{i}_1\ldots\,\bar{i}_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{j\,\bar{j}_1\ldots\,\bar{j}_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]$, $\mathrm{A}^k_{j\,i}$, $\bar{\mathrm{A}}k_{j\,i}$, and $\Gamma^k_{j\,i}$ are the components of the covariant differentiation ∇ in this coordinate representation. They obey some definite transformation rules under a change of a local chart. From (13.29), (13.30), (13.31), (13.32), (13.33), and (13.34) we derive a series of transformation formulas for these quantities $Z^{\,i}_j$, $Z^{\,i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}[P]$, $\bar{Z}^{\,\bar{i}_1\ldots\,\bar{i}_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{j\,\bar{j}_1\ldots\,\bar{j}_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]$, $\mathrm{A}^k_{j\,i}$, $\bar{\mathrm{A}}k_{j\,i}$, and $\Gamma^k_{j\,i}$.

Here is the transformation rule for $Z^{\,i}_j$. It is the most simple:

$$Z^{\,i}_j=\sum^{3}_{h=0}\sum^{3}_{k=0}S^i_h\;T^k_j\;\tilde Z^{\,h}_k. \tag{15.12}$$

It is followed by the transformation rule for $Z^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]$:

$$\begin{aligned}
&Z^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]=\sum^{3}_{d=0}\sum^{2}_{\substack{a_1,\,\ldots\,,\,a_\alpha\\ b_1,\,\ldots\,,\,b_\beta\\ \bar a_1,\,\ldots\,,\,\bar a_\nu\\ \bar b_1,\,\ldots\,,\,\bar b_\gamma}}\!\!\ldots\sum^{2}\sum^{3}_{\substack{c_1,\,\ldots\,,\,c_m\\ d_1,\,\ldots\,,\,d_n}}\!\!\ldots\sum^{3}\mathfrak{S}^{\,i_1}_{a_1}\ldots\,\mathfrak{S}^{\,i_\alpha}_{a_\alpha}\times\\
&\times\mathfrak{T}^{b_1}_{j_1}\ldots\,\mathfrak{T}^{b_\beta}_{j_\beta}\;\overline{\mathfrak{S}^{\bar i_1}_{\bar a_1}}\ldots\,\overline{\mathfrak{S}^{\bar i_\nu}_{\bar a_\nu}}\;\;\overline{\mathfrak{T}^{\bar b_1}_{\bar j_1}}\ldots\,\overline{\mathfrak{T}^{\bar b_\gamma}_{\bar j_\gamma}}\;S^{h_1}_{c_1}\ldots\,S^{h_m}_{c_m}\,T^{d_1}_{k_1}\ldots\,T^{d_n}_{k_n}\times\\
&\times T^d_j\;\tilde Z^{a_1\ldots\,a_\alpha\,\bar a_1\ldots\,\bar a_\nu\,c_1\ldots\,c_m}_{d\,b_1\ldots\,b_\beta\,\bar b_1\ldots\,\bar b_\gamma\,d_1\ldots\,d_n}[P]-\sum^{\alpha}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{2}_{v_\mu=1}\vartheta^{\,i_\mu}_{i\,v_\mu}\times\\
&\times S^{\,i_1\ldots\,v_\mu\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,\ldots\,\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\;S^i_c\;T^d_j\;\tilde Z^c_d+\sum^{\beta}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{2}_{w_\mu=1}\vartheta^{\,w_\mu}_{i\,j_\mu}\times\\
&\times S^{\,i_1\ldots\,\ldots\,\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,w_\mu\,\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\;S^i_c\;T^d_j\;\tilde Z^c_d-\sum^{\nu}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{2}_{v_\mu=1}\overline{\vartheta^{\,\bar i_\mu}_{i\,v_\mu}}\times\\
&\times S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,v_\mu\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\ldots\,\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\;S^i_c\;T^d_j\;\tilde Z^c_d+\sum^{\gamma}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{2}_{w_\mu=1}\overline{\vartheta^{\,w_\mu}_{i\,\bar j_\mu}}\times\\
&\times S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\ldots\,\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,w_\mu\,\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\;S^i_c\;T^d_j\;\tilde Z^c_d-\sum^{m}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{3}_{v_\mu=0}\theta^{\,i_\mu}_{i\,v_\mu}\times\\
&\times S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,v_\mu\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,\ldots\,\ldots\,k_n}[P]\;S^i_c\;T^d_j\;\tilde Z^c_d+\sum^{n}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{3}_{w_\mu=0}\theta^{\,w_\mu}_{i\,j_\mu}\times\\
&\times S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,\ldots\,\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,w_\mu\,\ldots\,k_n}[P]\;S^i_c\;T^d_j\;\tilde Z^c_d.
\end{aligned}\tag{15.13}$$

The transformation rules for the quantities $\mathrm{A}^k_{j\,i}$, $\bar{\mathrm{A}}^k_{j\,i}$, and $\Gamma^k_{j\,i}$ are presented by the following three formulas, which are not so huge as (15.13):

$$\Gamma^k_{j\,i}=\sum^{3}_{b=0}\sum^{3}_{a=0}\sum^{3}_{c=0}S^k_a\,T^b_i\,T^c_j\;\tilde\Gamma^a_{c\,b}+\sum^{3}_{a=0}Z^{\,a}_j\;\theta^k_{ai}, \tag{15.14}$$

$$\mathrm{A}^k_{j\,i}=\sum^{2}_{b=1}\sum^{2}_{a=1}\sum^{3}_{c=0}\mathfrak{S}^k_a\,\mathfrak{T}^b_i\,T^c_j\;\tilde{\mathrm{A}}^a_{c\,b}+\sum^{3}_{a=0}Z^{\,a}_j\;\vartheta^k_{ai}, \tag{15.15}$$

$$\bar{\mathrm{A}}^k_{j\,i}=\sum^{2}_{b=1}\sum^{2}_{a=1}\sum^{3}_{c=0}\overline{\mathfrak{S}^k_a}\;\overline{\mathfrak{T}^b_i}\,T^c_j\;\tilde{\bar{\mathrm{A}}}^a_{b}+\sum^{3}_{a=0}Z^{\,a}_j\;\overline{\vartheta^k_{ai}}. \tag{15.16}$$

And finally, we write the transformation rule for $\bar{Z}^{\bar{i}_1\dots\bar{i}_\nu\,i_1\dots i_\alpha\,h_1\dots h_m}_{j\,\bar{j}_1\dots\bar{j}_\gamma\,j_1\dots j_\beta\,k_1\dots k_n}[P]$:

$$
\begin{aligned}
&\bar{Z}^{\bar{i}_1\dots\bar{i}_\nu\,i_1\dots i_\alpha\,h_1\dots h_m}_{j\,\bar{j}_1\dots\bar{j}_\gamma\,j_1\dots j_\beta\,k_1\dots k_n}[P]=\sum^{3}_{d=0}\sum^{2}_{\substack{a_1,\,\dots,\,a_\alpha\\ b_1,\,\dots,\,b_\beta\\ \bar{a}_1,\,\dots,\,\bar{a}_\nu\\ \bar{b}_1,\,\dots,\,\bar{b}_\gamma}}\!\!\dots\sum^{2}\sum^{3}_{\substack{c_1,\,\dots,\,c_m\\ d_1,\,\dots,\,d_n}}\!\!\dots\sum^{3}\mathfrak{S}^{\bar{i}_1}_{\bar{a}_1}\dots\,\mathfrak{S}^{\bar{i}_\nu}_{\bar{a}_\nu}\,\times\\
&\times\mathfrak{T}^{\bar{b}_1}_{\bar{j}_1}\dots\,\mathfrak{T}^{\bar{b}_\gamma}_{\bar{j}_\gamma}\,\overline{\mathfrak{S}^{i_1}_{a_1}}\dots\,\overline{\mathfrak{S}^{i_\alpha}_{a_\alpha}}\,\,\overline{\mathfrak{T}^{b_1}_{j_1}}\dots\,\overline{\mathfrak{T}^{b_\beta}_{j_\beta}}\,S^{h_1}_{c_1}\dots\,S^{h_m}_{c_m}\,T^{d_1}_{k_1}\dots\,T^{d_n}_{k_n}\,\times\\
&\times T^d_j\,\tilde{\bar{Z}}^{\bar{a}_1\dots\bar{a}_\nu\,a_1\dots a_\alpha\,c_1\dots c_m}_{d\,\bar{b}_1\dots\bar{b}_\gamma\,b_1\dots b_\beta\,d_1\dots d_n}[P]-\sum^{\alpha}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{2}_{v_\mu=1}\overline{\vartheta^{i_\mu}_{i\,v_\mu}}\,\times\\
&\times\overline{S^{i_1\dots v_\mu\dots i_\alpha\,\bar{i}_1\dots\bar{i}_\nu\,h_1\dots h_m}_{j_1\dots\dots\dots j_\beta\,\bar{j}_1\dots\bar{j}_\gamma\,k_1\dots k_n}[P]}\,S^i_c\,T^d_j\,\tilde{Z}^c_d+\sum^{\beta}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{2}_{w_\mu=1}\overline{\vartheta^{w_\mu}_{i\,j_\mu}}\,\times\\
&\times\overline{S^{i_1\dots\dots\dots i_\alpha\,\bar{i}_1\dots\bar{i}_\nu\,h_1\dots h_m}_{j_1\dots w_\mu\dots j_\beta\,\bar{j}_1\dots\bar{j}_\gamma\,k_1\dots k_n}[P]}\,S^i_c\,T^d_j\,\tilde{Z}^c_d-\sum^{\nu}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{2}_{v_\mu=1}\vartheta^{\bar{i}_\mu}_{i\,v_\mu}\,\times\\
&\times\overline{S^{i_1\dots i_\alpha\,\bar{i}_1\dots v_\mu\dots\bar{i}_\nu\,h_1\dots h_m}_{j_1\dots j_\beta\,\bar{j}_1\dots\dots\dots\bar{j}_\gamma\,k_1\dots k_n}[P]}\,S^i_c\,T^d_j\,\tilde{Z}^c_d+\sum^{\gamma}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{2}_{w_\mu=1}\vartheta^{w_\mu}_{i\,\bar{j}_\mu}\,\times\\
&\times\overline{S^{i_1\dots i_\alpha\,\bar{i}_1\dots\dots\dots\bar{i}_\nu\,h_1\dots h_m}_{j_1\dots j_\beta\,\bar{j}_1\dots w_\mu\dots\bar{j}_\gamma\,k_1\dots k_n}[P]}\,S^i_c\,T^d_j\,\tilde{Z}^c_d-\sum^{m}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{3}_{v_\mu=0}\theta^{i_\mu}_{i\,v_\mu}\,\times\\
&\times\overline{S^{i_1\dots i_\alpha\,\bar{i}_1\dots\bar{i}_\nu\,h_1\dots v_\mu\dots h_m}_{j_1\dots j_\beta\,\bar{j}_1\dots\bar{j}_\gamma\,k_1\dots\dots\dots k_n}[P]}\,S^i_c\,T^d_j\,\tilde{Z}^c_d+\sum^{n}_{\mu=1}\sum^{3}_{i=0}\sum^{3}_{c=0}\sum^{3}_{d=0}\sum^{3}_{w_\mu=0}\theta^{w_\mu}_{i\,j_\mu}\,\times\\
&\times\overline{S^{i_1\dots i_\alpha\,\bar{i}_1\dots\bar{i}_\nu\,h_1\dots\dots\dots h_m}_{j_1\dots j_\beta\,\bar{j}_1\dots\bar{j}_\gamma\,k_1\dots w_\mu\dots k_n}[P]}\,S^i_c\,T^d_j\,\tilde{Z}^c_d.
\end{aligned}
\tag{15.17}
$$

The formulas (15.3) and (15.11) yield the explicit expressions for arbitrary covariant derivatives in general case. However, below we consider some specializations of these formulas which appear to be more valuable than the initial formulas (15.3) and (15.11) themselves.

16. Degenerate covariant differentiations.

Definition 16.1. A covariant differentiation ∇ is said to be *degenerate* if $\nabla_{\mathbf{Y}}\psi=0$ for any extended scalar field ψ and for any extended vector field $\mathbf{Y}$.

This definition is concordant with the definition 14.1. For degenerate covariant differentiations we have a theorem which is analogous to the theorems 14.1.

Theorem 16.1. *Defining a degenerate covariant differentiation* ∇ *of the algebra of extended spin-tensorial fields* $\mathbf{S}(M)$ *is equivalent to defining three extended spin-tensorial fields* $\mathfrak{S}$, $\bar{\mathfrak{S}}$, *and* $\mathbf{S}$ *of the types* $(1,1|0,0|0,1)$, $(0,0|1,1|0,1)$, *and* $(0,0|0,0|1,2)$ *respectively.*

Proof. Let ∇ be an arbitrary degenerate covariant differentiation. For this differentiation from (15.5), (15.6), and (15.7) we derive the following equalities:

$$Z^{\,i}_{j} = 0, \tag{16.1}$$

$$Z^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P] = 0, \tag{16.2}$$

$$\bar Z^{\,\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{j\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P] = 0. \tag{16.3}$$

The equalities (16.1), (16.2), and (16.3) are analogs of the equalities (14.1), (14.2), and (14.3) respectively. Applying (16.1) to (15.14), (15.15), and (15.16), we obtain the following transformation rules for the A-components and the Γ-components of the degenerate covariant differentiation ∇:

$$\mathrm{A}^k_{j\,i} = \sum^2_{b=1}\sum^2_{a=1}\sum^3_{c=0}\mathfrak{S}^k_a\,\mathfrak{T}^b_i\,T^c_j\;\tilde{\mathrm{A}}^a_{c\,b}, \tag{16.4}$$

$$\bar{\mathrm{A}}^k_{j\,i} = \sum^2_{b=1}\sum^2_{a=1}\sum^3_{c=0}\overline{\mathfrak{S}^k_a}\;\overline{\mathfrak{T}^b_i}\,T^c_j\;\tilde{\bar{\mathrm{A}}}^a_{b}, \tag{16.5}$$

$$\Gamma^k_{j\,i} = \sum^3_{b=0}\sum^3_{a=0}\sum^3_{c=0}S^k_a\,T^b_i\,T^c_j\;\tilde\Gamma^a_{c\,b}. \tag{16.6}$$

From (16.4), (16.5), and (16.6), we see that A-components of the degenerate covariant differentiation ∇ define two extended spin-tensorial fields of the types $(1,1|0,0|0,1)$ and $(0,0|1,1|0,1)$, while its Γ-components define an extended spin-tensorial field of the type $(0,0|0,0|1,2)$. The theorem is proved. □

17. Horizontal and vertical covariant differentiations.

Suppose again that N is a composite spin-tensorial bundle over the space-time manifold M (in the sense of (9.7)). Let ∇ be a covariant differentiation of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$. Then $D = \nabla_{\mathbf{Y}}$ is a differentiation of $\mathbf{S}(M)$, its restriction to the set of scalar fields is given by some vector field $\mathbf{Z} = \mathbf{Z}(\mathbf{Y})$ in N (see formula (13.2) above). In other words, we have a homomorphism

$$\mathbb{C}T^1_0(M) \to \mathbb{C}T^1_0(N) \tag{17.1}$$

that maps an extended vector field $\mathbf{Y}$ of M to some regular vector field of N. Applying the localization theorem 15.1 to the homomorphism (17.1), we come to the following definition and to the theorem after it.

Definition 17.1. Suppose that for each point q of the composite spin-tensorial bundle N over the space-time M some $\mathbb{C}$-linear map of the vector spaces

$$f_q\colon\ \mathbb{C}T_{\pi(q)}(M) \to \mathbb{C}T_q(N) \tag{17.2}$$

is given. Then we say that a *lift* of vectors from M to the bundle N is defined.

Theorem 17.1. *Any homomorphism of $\mathfrak{F}_{\mathbb{C}}(N)$-modules* (17.1) *is uniquely associated with some smooth lift of vectors from M to N. It is represented by this lift as a collection of $\mathbb{C}$-linear maps* (17.2) *specific to each point $q \in N$.*

The $\mathbb{C}$-linear maps (17.2) here are special cases of general $\mathbb{C}$-linear maps (15.1) declared in the localization theorem 15.1.

Now let's consider the canonical projection $\pi: N \to M$. The differential of this map (upon complexification) acts in the direction opposite to the lift of vectors (17.2) introduced in the definition 17.1. Indeed, we have $\pi_*: \mathbb{C}T_q(N) \to \mathbb{C}T_{\pi(q)}(M)$ at each point $q \in N$. Therefore, the composition $f \circ \pi_*$ acts from $\mathbb{C}T_{\pi(q)}(M)$ to $\mathbb{C}T_{\pi(q)}(M)$. This composite map determines an extended operator field (a spin-tensorial field of the type $(0,0|0,0|1,1)$).

Definition 17.2. A lift of vectors f from M to N is called *vertical* if $\pi_* \circ f = 0$.

Definition 17.3. A lift of vectors f from M to N is called *horizontal* if $\pi_* \circ f = \mathbf{id}$, i. e. if the composition $\pi_* \circ f$ coincides with the field of identical operators.

Like any other bundle, the composite spin-tensorial bundle N naturally subdivides into fibers over the points of the base manifold M. The set of vectors tangent to the fiber at a point q (upon complexification) is a $\mathbb{C}$-linear subspace within the complexified tangent space $\mathbb{C}T_q(N)$. This subspace coincides with the kernel of the complexified mapping π_*. We denote this subspace

$$V_q(N) = \operatorname{Ker} \pi_* \tag{17.3}$$

and call it the *vertical subspace*. Any vertical lift of vectors determines a set linear mappings from $\mathbb{C}T_{\pi(q)}$ to the vertical subspace (17.3) at each point $q \in N$.

Lemma 17.1. *The difference of two horizontal lifts is a vertical lift of vectors from the base manifold M to the bundle N.*

Indeed, if one takes two horizontal lifts of vectors f_1 and f_2, then $\pi_* \circ (f_1 - f_2) = \pi_* \circ f_1 - \pi_* \circ f_2 = \mathbf{id} - \mathbf{id} = 0$. This means that the difference $f_1 - f_2$ is a vertical lift according to the definition 17.2.

Each covariant differentiation ∇ is associated with some lift of vectors (see the definition 17.1, and the theorem 17.1 above).

Definition 17.4. A covariant differentiation ∇ is called a *horizontal covariant differentiation* (or a *vertical covariant differentiation*) if the corresponding lift of vectors is *horizontal* (or *vertical*).

Lemma 17.2. *The difference of two horizontal covariant differentiations is a vertical covariant differentiation.*

The lemma 17.2 is an immediate consequence of the lemma 17.1.

18. Native extended spin-tensorial fields and vertical multivariate differentiations.

Let N be a composite spin-tensorial bundle over the space-time manifold M. Then each its point q is represented by a list $q = (p, \mathbf{S}[1], \ldots, \mathbf{S}[J+Q])$, where $p \in M$ and $\mathbf{S}[1], \ldots, \mathbf{S}[J+Q]$ are some spin-tensors at the point p (see formula (9.8) above). Let's consider the map that takes q to the P-th spin-tensor $\mathbf{S}[P]$ in this list. According to the definition 10.2, this map is an extended spin-tensorial field of the type $(\alpha_P, \beta_P|\nu_P, \gamma_P|m_P, n_P)$. It is canonically associated with the bundle N. Therefore, it is called a *native extended spin-tensorial field*. Totally, we have $J+Q$ native extended spin-tensorial fields associated with the composite spin-tensorial bundle N, we denote them $\mathbf{S}[1],\,\ldots\,,\,\mathbf{S}[J+Q]$.

Definition 18.1. A *multivariate differentiation* of the type $(\beta, \alpha|\gamma, \nu|n, m)$ in the algebra $\mathbf{S}(M)$ is a homomorphism of $\mathfrak{F}_{\mathbb{C}}(N)$-modules

$$\nabla\colon\ S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{m}_{n}(M) \to \mathfrak{D}_{\mathbf{S}}(M). \tag{18.1}$$

If $\mathbf{Y}$ is an extended spin-tensorial field of the type $(\alpha, \beta|\nu, \gamma|m, n)$, then we can apply the homomorphism (18.1) to it. As a result we get the differentiation $D = \nabla_{\mathbf{Y}}$ of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$. It is called the *multivariate differentiation along the spin-tensorial field* $\mathbf{Y}$.

Note that the type of a multivariate differentiation $(\beta, \alpha|\gamma, \nu|n, m)$ in the above definition 18.1 is dual to the type of the module $S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{m}_{n}(M)$ in the formula (18.1). If $\alpha = 0$, $\beta = 0$, $\nu = 0$, $\gamma = 0$, $m = 1$, and $n = 0$, then the definition 18.1 reduces to the definition 15.1. This means that a covariant differentiation is a special multivariate differentiation whose type is $(0, 0|0, 0|0, 1)$. Similarly, a multivariate differentiation of the type $(0, 0|0, 0|1, 0)$ is called a *contravariant differentiation*.

Let ∇ be some multivariate differentiation of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$. Then, applying the localization theorem 15.1 to the homomorphism $\nabla:\ S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{m}_{n}(M) \to \mathfrak{D}_{\mathbf{S}}(M)$, we find that this homomorphism is composed by $\mathbb{C}$-linear maps $S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{m}_{n}(\pi(q), M) \to \mathfrak{D}(q, M)$ specific to each point $q \in N$. This fact is expressed by the formula coinciding with (15.2):

$$\nabla_{\mathbf{Y}}\mathbf{X} = C(\mathbf{Y} \otimes \nabla\mathbf{X}). \tag{18.2}$$

However, instead of (15.3) in this case we have

$$\begin{gathered}
\nabla_{\mathbf{Y}}\mathbf{X} = \sum^{2}_{\substack{a_1,\,\ldots\,,\,a_\varepsilon\\ b_1,\,\ldots\,,\,b_\eta\\ \bar{a}_1,\,\ldots\,,\,\bar{a}_\sigma\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\zeta\\ c_1,\,\ldots\,,\,c_e\\ d_1,\,\ldots\,,\,d_f}}\!\!\ldots\!\sum^{3}\ \sum^{2}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta\\ \bar{i}_1,\,\ldots\,,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots\,,\,\bar{j}_\gamma\\ h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\!\sum^{3} Y^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}\,\times\\
\times\,\nabla^{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m} X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}\ \boldsymbol{\Psi}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}.
\end{gathered}\tag{18.3}$$

Looking at (18.3), we see that each multivariate differentiation ∇ of the type $(\beta, \alpha|\gamma, \nu|n, m)$ can be treated as an operator producing the extended spin-tensorial field $\nabla\mathbf{X}$ of the type $(\varepsilon+\beta, \eta+\alpha|\sigma+\gamma, \zeta+\nu|r+n, s+m)$ from any given extended spin-tensorial field $\mathbf{X}$ of the type $(\varepsilon, \eta|\sigma, \zeta|r, s)$. This operator is called the operator of *multivariate differential* of the type $(\beta, \alpha|\gamma, \nu|n, m)$.

Let P be an integer number such that $1 \leqslant P \leqslant J+Q$ and let $\mathbf{Y}$ be an extended spin-tensorial field of the type $(\alpha_P, \beta_P|\nu_P, \gamma_P|m_P, n_P)$. Remember that each point q of the composite spin-tensorial bundle N is a list of the form (9.8):

$$q = (p,\,\mathbf{S}[1],\,\ldots\,,\,\mathbf{S}[J+Q]), \tag{18.4}$$

Note that the P-th spin-tensor $\mathbf{S}[P]$ in the list (18.4) has the same type as the spin-tensor $\mathbf{Y} = \mathbf{Y}_q$ (the value of the extended spin-tensorial field $\mathbf{Y}$ at the point q). They both belong to the same spin-tensorial space $S^{\alpha_P}_{\beta_P}\bar{S}^{\nu_P}_{\gamma_P}T^{m_P}_{n_P}(p,M)$, therefore we can add them. This means that we can treat the list

$$q(t) = (p,\, \mathbf{S}[1],\, \ldots\,,\, \mathbf{S}[P] + t\,\mathbf{Y}_q,\, \ldots\,,\, \mathbf{S}[J+Q]) \tag{18.5}$$

as a one-parametric set of points in N, the scalar variable t being its parameter. Thus in (18.5) we have a line (a straight line) passing through the initial point $q \in N$ and lying completely within the fiber over the point $p = \pi(q) \in M$. Suppose that $\mathbf{X}$ is some extended spin-tensorial field of the type $(\varepsilon, \eta|\sigma, \zeta|e, f)$. Denote by $\mathbf{X}(t)$ the values of this field at the points of the above parametric line (18.5):

$$\mathbf{X}(t) = \mathbf{X}_{q(t)}. \tag{18.6}$$

Since $\pi(q(t)) = p = \mathrm{const}$ for any t, the values of the spin-tensor-valued function (18.6) all belong to the same space $S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(p,M)$. Hence, we can add and subtract them, and, since $\mathbf{X}$ is smooth, we can take the following limit of the ratio:

$$\dot{X}(t) = \lim_{\tau\to 0}\frac{\mathbf{X}(t+\tau) - \mathbf{X}(t)}{\tau}. \tag{18.7}$$

Let's denote by $\mathbf{Z}_q$ the value of the derivative (18.7) for $t = 0$:

$$\mathbf{Z}_q = \dot{X}(0) = \frac{d\mathbf{X}_{q(t)}}{dt}\,\biggr|_{t=0}. \tag{18.8}$$

It is easy to understand that, when q is fixed, $\mathbf{Z}_q$ is a spin-tensor from the space $S^{\varepsilon}_{\eta}\bar{S}^{\sigma}_{\zeta}T^{e}_{f}(p,M)$ at the point $p = \pi(q)$. By varying $q \in N$, we find that the spin-tensors $\mathbf{Z}_q$ constitute a smooth extended spin-tensorial field $\mathbf{Z}$. As a result of the above considerations we have constructed a map

$$D\colon\ \mathbf{S}(M) \to \mathbf{S}(M). \tag{18.9}$$

It is easy to check up that the map (18.9) defined by means of the formulas (18.5), (18.6), (18.7), and (18.8) is a differentiation of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$, i. e. $D \in \mathfrak{D}_{\mathbf{S}}(M)$ (see the definition 12.1 above). Moreover, due to the formula (18.5) this differentiation D depends on the extended spin-tensorial field $\mathbf{Y}$. The easiest way to study this dependence $D = D(\mathbf{Y})$ is to write the equality (18.8) in a local chart, i. e. in some local coordinates (9.11) and (9.12):

$$\begin{aligned}\mathbf{Z} = \sum^{2}_{\substack{a_1,\ldots,a_\varepsilon\\ b_1,\ldots,b_\eta\\ \bar{a}_1,\ldots,\bar{a}_\sigma\\ \bar{b}_1,\ldots,\bar{b}_\zeta\\ c_1,\ldots,c_e\\ d_1,\ldots,d_f}}\!\!\ldots\sum^{3}\ \sum^{2}_{\substack{i_1,\ldots,i_\alpha\\ j_1,\ldots,j_\beta\\ \bar{i}_1,\ldots,\bar{i}_\nu\\ \bar{j}_1,\ldots,\bar{j}_\gamma\\ h_1,\ldots,h_m\\ k_1,\ldots,k_n}}\!\!\ldots\sum^{3}\Biggl(&Y^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}\ \frac{\partial X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}}{\partial S^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}[P]}\,+\\ &+\overline{Y^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}}\ \frac{\partial X^{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}}{\overline{\partial S^{i_1\ldots\,i_\alpha\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}[P]}}\Biggr)\,\mathbf{\Psi}^{b_1\ldots\,b_\eta\bar{b}_1\ldots\,\bar{b}_\zeta d_1\ldots\,d_f}_{a_1\ldots\,a_\varepsilon\bar{a}_1\ldots\,\bar{a}_\sigma c_1\ldots\,c_e}.\end{aligned} \tag{18.10}$$

Here $\alpha = \alpha_P$, $\beta = \beta_P$, $\nu = \nu_P$, $\gamma = \gamma_P$, $m = m_P$, and $n = n_P$. Two partial derivatives in (18.10) behave like the components of two extended spin-tensorial fields under a change of a local chart. We denote these fields by $\nabla\mathbf{X}$ and $\bar{\nabla}\mathbf{X}$ respectively. Here ∇ is a special sign, the «double bar nabla». It was used in [5] for the first time. Now we can write (18.10) as follows:

$$\mathbf{Z} = C(\mathbf{Y} \otimes \nabla\mathbf{X}) + C(\tau(\mathbf{Y}) \otimes \bar{\nabla}\mathbf{X}). \tag{18.11}$$

Comparing (18.11) with (18.2), we see that the formula (18.11) introduces two multivariate differentiations of the types $(\beta, \alpha|\gamma, \nu|n, m)$ and $(\gamma, \nu|\beta, \alpha|n, m)$:

$$\begin{aligned} &\nabla[P]\colon\ S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{m}_{n}(M) \to \mathfrak{D}_{\mathbf{S}}(M),\\ &\bar{\nabla}[P]\colon\ S^{\nu}_{\gamma}\bar{S}^{\alpha}_{\beta}T^{m}_{n}(M) \to \mathfrak{D}_{\mathbf{S}}(M). \end{aligned} \tag{18.12}$$

In terms of these multivariate differentiations the formula (18.11) is rewritten as

$$\mathbf{Z} = \nabla_{\mathbf{Y}}\mathbf{X} + \bar{\nabla}_{\tau(\mathbf{Y})}\mathbf{X}. \tag{18.13}$$

In a local chart the multivariate differentiations (18.12) are represented by the partial derivatives taken from the formula (18.10):

$$\nabla^{j_1\ldots\, j_\beta \bar{j}_1\ldots\, \bar{j}_\gamma k_1\ldots\, k_n}_{i_1\ldots\, i_\alpha \bar{i}_1\ldots\, \bar{i}_\nu h_1\ldots\, h_m}[P] = \frac{\partial}{\partial S^{i_1\ldots\, i_\alpha \bar{i}_1\ldots\, \bar{i}_\nu h_1\ldots\, h_m}_{j_1\ldots\, j_\beta \bar{j}_1\ldots\, \bar{j}_\gamma k_1\ldots\, k_n}[P]}, \tag{18.14}$$

$$\bar{\nabla}^{j_1\ldots\, j_\gamma\, \bar{j}_1\ldots\, \bar{j}_\beta\, k_1\ldots\, k_n}_{i_1\ldots\, i_\nu\, \bar{i}_1\ldots\, \bar{i}_\alpha\, h_1\ldots\, h_m}[P] = \frac{\partial}{\overline{\partial S^{\bar{i}_1\ldots\, \bar{i}_\alpha\, i_1\ldots\, i_\nu\, h_1\ldots\, h_m}_{\bar{j}_1\ldots\, \bar{j}_\beta\, j_1\ldots\, j_\gamma\, k_1\ldots\, k_n}[P]}}, \tag{18.15}$$

Here $\alpha = \alpha_P$, $\beta = \beta_P$, $\nu = \nu_P$, $\gamma = \gamma_P$, $m = m_P$, and $n = n_P$. Following the tradition, we shall use the term *multivariate derivatives* for the differential operators representing the differentiation $\nabla[P]$ and $\bar{\nabla}[P]$ in (18.14) and (18.15).

Definition 18.2. The multivariate differentiations $\nabla[P]$ and $\bar{\nabla}[P]$ defined through the formulas (18.5), (18.6), (18.7), (18.8), (18.13) and represented by the formulas (18.14) and (18.15) in local coordinates are called the *P-th canonical vertical multivariate differentiation* and the *barred P-th canonical*[1] *vertical multivariate differentiation* respectively.

Let $\mathbf{S}[R]$ be R-th native extended spin-tensorial field associated with the composite spin-tensorial bundle N. Then by means of direct calculations in local coordinates one can derive the following formulas:

$$\nabla_{\mathbf{Y}}[P]\mathbf{S}[R] = \begin{cases} \mathbf{Y} & \text{for}\quad P = R,\\ 0 & \text{for}\quad P \neq R, \end{cases} \tag{18.16}$$

$$\bar{\nabla}_{\mathbf{Y}}[P]\tau(\mathbf{S}[R]) = \begin{cases} \mathbf{Y} & \text{for}\quad P = R,\\ 0 & \text{for}\quad P \neq R, \end{cases} \tag{18.17}$$

[1] Note that $\nabla[P]$ and $\bar{\nabla}[P]$ are canonically associated with the bundle N, their definition does not require any auxiliary structures like metrics and connections.

Apart from (18.16) and (18.17) we can write the following formulas:

$$\nabla_{\mathbf{Y}}[P]\tau(\mathbf{S}[R]) = 0 \quad \text{for all } P \text{ and } R, \tag{18.18}$$

$$\bar{\nabla}_{\mathbf{Y}}[P]\mathbf{S}[R] = 0 \quad \text{for all } P \text{ and } R. \tag{18.19}$$

These formulas (18.18) and (18.19) are also easily derived by means of direct calculations in local coordinates.

Like covariant differentiations (see theorem 17.1 and definition 17.1), multivariate differentiations are associated with some lifts. However, unlike covariant differentiations, they lift not vectors, but spin-tensors, though converting them into tangent vectors of the bundle N. In the case of the canonical multivariate differentiation $\nabla[P]$ for each point $q \in N$ we have some $\mathbb{C}$-linear map

$$f[P]\colon\ S^{\alpha_P}_{\beta_P}\bar{S}^{\nu_P}_{\gamma_P}T^{m_P}_{n_P}(\pi(q),M) \to \mathbb{C}T_q(N), \tag{18.20}$$

The map (18.20) takes a spin-tensor $\mathbf{Y} \in S^{\alpha_P}_{\beta_P}\bar{S}^{\nu_P}_{\gamma_P}T^{m_P}_{n_P}(\pi(q),M)$ to the following vector in the tangent space $\mathbb{C}T_q(N)$ of the manifold N at the point q:

$$f[P](\mathbf{Y}) = \sum^{2}_{\substack{i_1,\ldots,i_\alpha\\ j_1,\ldots,j_\beta\\ \bar{i}_1,\ldots,\bar{i}_\nu\\ \bar{j}_1,\ldots,\bar{j}_\gamma\\ h_1,\ldots,h_m\\ k_1,\ldots,k_n}}\!\!\ldots\sum^{3} Y^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}\ \mathbf{W}^{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}[P]. \tag{18.21}$$

Similarly, in the case of the barred canonical multivariate differentiation $\bar{\nabla}[P]$ for each point $q \in N$ we have some $\mathbb{C}$-linear map

$$\bar{f}[P]\colon\ S^{\nu_P}_{\gamma_P}\bar{S}^{\alpha_P}_{\beta_P}T^{m_P}_{n_P}(\pi(q),M) \to \mathbb{C}T_q(N) \tag{18.22}$$

This map (18.22) is determined by the formula

$$\bar{f}[P](\mathbf{Y}) = \sum^{2}_{\substack{i_1,\ldots,i_\alpha\\ j_1,\ldots,j_\beta\\ \bar{i}_1,\ldots,\bar{i}_\nu\\ \bar{j}_1,\ldots,\bar{j}_\gamma\\ h_1,\ldots,h_m\\ k_1,\ldots,k_n}}\!\!\ldots\sum^{3} Y^{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}_{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}\ \bar{\mathbf{W}}^{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}_{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}[P]. \tag{18.23}$$

Here again $\alpha = \alpha_P$, $\beta = \beta_P$, $\nu = \nu_P$, $\gamma = \gamma_P$, $m = m_P$, $n = n_P$, while the vectors $\mathbf{W}^{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}[P]$ and $\bar{\mathbf{W}}^{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}_{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}[P]$ in (18.21) and (18.23) are given by the formulas (9.19) and (9.20). In a coordinate-free form the formulas (18.21) and (18.23) can be interpreted as follows: the vector $f[P](\mathbf{Y}) + \bar{f}[P](\tau(\mathbf{Y}))$ is the tangent vector of the parametric curve (18.5) at its initial point $q = q(0)$.

Let's consider the image of the $\mathbb{C}$-linear map (18.20). We denote it $V_q[P](N)$. Then from (18.21) one easily derives that $V_q[P](N)$ is a subspace within the vertical subspace $V_q(N)$ of the tangent space $\mathbb{C}T_q(N)$. Similarly, we denote by $\bar{V}_q[P](N)$

the image of the $\mathbb{C}$-linear map (18.22). This image also is a subspace within the vertical subspace $V_q(N)$. Moreover, we have

$$\begin{aligned} V_q(N) = V_q[1](N) \oplus \ldots \oplus V_q[J+Q](N) +\\ + \bar{V}_q[1](N) \oplus \ldots \oplus \bar{V}_q[J+Q](N). \end{aligned} \tag{18.24}$$

The formula (18.24) is a well-known fact, it follows from (9.7). Due to the inclusions

$$\begin{aligned} &\operatorname{Im} f[P] = V_q[P](N) \subset V_q(N),\\ &\operatorname{Im} \bar{f}[P] = \bar{V}_q[P](N) \subset V_q(N) \end{aligned} \tag{18.25}$$

the multivariate differentiations (18.12) both are vertical differentiations.

19. Horizontal covariant differentiations and extended connections.

Let ∇ be some horizontal covariant differentiation of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$ and let f be the horizontal lift of vectors associated with it (see the definition 17.3). The horizontality of f means that the image of the linear map (17.2) is some 4-dimensional subspace $H_q(N)$ within the tangent space $\mathbb{C}T_q(N)$. It is called a *horizontal subspace*. Due to $\pi_* \circ f = \mathbf{id}$ the mappings

$$f\colon \mathbb{C}T_{\pi(q)}(M) \to H_q(N), \qquad\qquad \pi_*\colon H_q(N) \to \mathbb{C}T_{\pi(q)}(M) \tag{19.1}$$

are inverse to each other. Due to the same equality $\pi_* \circ f = \mathbf{id}$ the sum of the vertical and horizontal subspaces is a direct sum:

$$H_q(N) \oplus V_q(N) = \mathbb{C}T_q(N). \tag{19.2}$$

Theorem 19.1. *Defining a horizontal lift of vectors from M to N is equivalent to fixing some direct complement $H_q(N)$ of the vertical subspace $V_q(N)$ within the tangent space $\mathbb{C}T_q(N)$ at each point $q \in N$.*

Proof. Suppose that some horizontal lift of vectors f is given. Then the subspace $H_q(N)$ at the point q is determined as the image of the mapping (17.2), while the relationship (19.2) is derived from $\pi_* \circ f = \mathbf{id}$ and from (17.3).

Conversely, assume that at each point $q \in N$ we have a subspace $H_q(N)$ complementary to $V_q(N)$. Then at each point $q \in N$ the relationship (19.2) is fulfilled. The kernel of the mapping $\pi_*\colon \mathbb{C}T_q(N) \to \mathbb{C}T_{\pi(q)}(M)$ coincides with $V_q(N)$, therefore, the restriction of π_* to the horizontal subspace $H_q(N)$ is a bijection. The lift of vectors f from M to N then can be defined as the inverse mapping for $\pi_*: H_q(N) \to \mathbb{C}T_{\pi(q)}(M)$. If f is defined in this way, then the mappings (19.1) appear to be inverse to each other and we get the equality $\pi_* \circ f = \mathbf{id}$. According to the definition 17.3, it means that f is a horizontal lift of vectors. The theorem is completely proved. □

Let's study a horizontal lift of vectors f in a coordinate form. Upon choosing some local chart U in M equipped with a positively polarized right orthonormal frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and its associated spinor frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ we can apply the lift f to the frame vector fields $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$. As a result we get

$$\begin{gathered} f(\boldsymbol{\Upsilon}_j)=\mathbf{U}_j-\\ -\sum^{J+Q}_{P=1}\sum^{2}_{\substack{i_1,\,\ldots,\,i_\alpha\\ j_1,\,\ldots,\,j_\beta\\ \bar i_1,\,\ldots,\,\bar i_\nu\\ \bar j_1,\,\ldots,\,\bar j_\gamma\\ h_1,\,\ldots,\,h_m\\ k_1,\,\ldots,\,k_n}}\!\!\ldots\sum^{3}\Gamma^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\;\mathbf{W}^{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}[P]\,-\\ -\sum^{J+Q}_{P=1}\sum^{2}_{\substack{i_1,\,\ldots,\,i_\alpha\\ j_1,\,\ldots,\,j_\beta\\ \bar i_1,\,\ldots,\,\bar i_\nu\\ \bar j_1,\,\ldots,\,\bar j_\gamma\\ h_1,\,\ldots,\,h_m\\ k_1,\,\ldots,\,k_n}}\!\!\ldots\sum^{3}\bar\Gamma^{\,\bar i_1\ldots\,\bar i_\nu\;i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\,j\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]\;\bar{\mathbf{W}}^{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}_{\bar i_1\ldots\,\bar i_\nu\;i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}[P]. \end{gathered}\tag{19.3}$$

Here $\alpha=\alpha_P$, $\beta=\beta_P$, $\nu=\nu_P$, $\gamma=\gamma_P$, $m=m_P$, $n=n_P$, while the vectors $\mathbf{U}_j$, $\mathbf{W}^{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}[P]$ and $\bar{\mathbf{W}}^{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}_{\bar i_1\ldots\,\bar i_\nu\;i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}[P]$ are determined by the formulas (9.26), (9.18), (9.19), and (9.20). The formula (19.3) for $f(\mathbf{E}_j)$ follows from $\pi_*\circ f=\mathbf{id}$ due to the equalities

$$\begin{aligned} &\pi_*(\mathbf{U}_j)=\mathbf{E}_j,\\ &\pi_*\Bigl(\mathbf{W}^{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}[P]\Bigr)=0,\\ &\pi_*\Bigl(\bar{\mathbf{W}}^{\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}_{\bar i_1\ldots\,\bar i_\nu\;i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}[P]\Bigr)=0. \end{aligned}$$

The quantities $\Gamma^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]$ and $\bar\Gamma^{\,\bar i_1\ldots\,\bar i_\nu\;i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\,j\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]$ in (19.3) are called the *components* of a horizontal lift of vectors in a local chart U. If the lift f is induced by some horizontal covariant differentiation ∇, then for its components in the above formula (19.3) we have

$$\begin{aligned} &\Gamma^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]=-Z^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P],\\ &\bar\Gamma^{\,\bar i_1\ldots\,\bar i_\nu\;i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\,j\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]=-\bar Z^{\,\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\,j\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]. \end{aligned}\tag{19.4}$$

The quantities $Z^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]$ and $\bar Z^{\,\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{\,j\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]$ in (19.4) are the same as in (15.6), (15.7), (15.11), (15.13), and (15.17). As for the quantities Z^i_j in (15.5) and (15.12), in the case of a horizontal covariant differentiation they are given by the Kronecker's delta-symbol: $Z^i_j=\delta^i_j$. Substituting $Z^i_j=\tilde Z^i_j=\delta^i_j$ into (15.13) and (15.17) and taking into account (19.4), we derive the transformation rules for the components of a horizontal lift of spin-tensors in the formula (19.3):

$$\Gamma^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]=\sum^{3}_{d=0}\ \sum^{2}\ldots\sum^{3}_{\substack{a_1,\,\ldots\,,\,a_\alpha\\ b_1,\,\ldots\,,\,b_\beta\\ \bar a_1,\,\ldots\,,\,\bar a_\nu\\ \bar b_1,\,\ldots\,,\,\bar b_\gamma\\ c_1,\,\ldots\,,\,c_m\\ d_1,\,\ldots\,,\,d_n}}\mathfrak{S}^{i_1}_{a_1}\ldots\,\mathfrak{S}^{i_\alpha}_{a_\alpha}\ \mathfrak{T}^{b_1}_{j_1}\ldots\,\mathfrak{T}^{b_\beta}_{j_\beta}\ \overline{\mathfrak{S}^{\bar i_1}_{\bar a_1}}\ldots\,\overline{\mathfrak{S}^{\bar i_\nu}_{\bar a_\nu}}\times$$

$$\times\,\overline{\mathfrak{T}^{\bar b_1}_{\bar j_1}}\ldots\,\overline{\mathfrak{T}^{\bar b_\gamma}_{\bar j_\gamma}}\ S^{h_1}_{c_1}\ldots\,S^{h_m}_{c_m}\ T^{d_1}_{k_1}\ldots\,T^{d_n}_{k_n}\ T^{d}_{j}\ \tilde\Gamma^{a_1\ldots\,a_\alpha\,\bar a_1\ldots\,\bar a_\nu\,c_1\ldots\,c_m}_{d\,b_1\ldots\,b_\beta\,\bar b_1\ldots\,\bar b_\gamma\,d_1\ldots\,d_n}[P]\,+$$

$$+\sum^{\alpha}_{\mu=1}\sum^{2}_{v_\mu=1}\vartheta^{i_\mu}_{j\,v_\mu}\ S^{i_1\ldots\,v_\mu\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,\ldots\,\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]-\sum^{\beta}_{\mu=1}\sum^{2}_{w_\mu=1}\vartheta^{w_\mu}_{j\,j_\mu}\times$$

$$\times\,S^{i_1\ldots\,\ldots\,\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,w_\mu\,\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]+\sum^{\nu}_{\mu=1}\sum^{2}_{v_\mu=1}\overline{\vartheta^{\bar i_\mu}_{j\,v_\mu}}\ S^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,v_\mu\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\ldots\,\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]\,-$$

$$-\sum^{\gamma}_{\mu=1}\sum^{2}_{w_\mu=1}\overline{\vartheta^{w_\mu}_{j\,\bar j_\mu}}\ S^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\ldots\,\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,w_\mu\,\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]+\sum^{m}_{\mu=1}\sum^{3}_{v_\mu=0}\theta^{i_\mu}_{j\,v_\mu}\times$$

$$\times\,S^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,v_\mu\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,\ldots\,\ldots\,k_n}[P]-\sum^{n}_{\mu=1}\sum^{3}_{w_\mu=0}\theta^{w_\mu}_{j\,j_\mu}\ S^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,\ldots\,\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,w_\mu\,\ldots\,k_n}[P],$$

$$\bar\Gamma^{\bar i_1\ldots\,\bar i_\nu\,i_1\ldots\,i_\alpha\,h_1\ldots\,h_m}_{j\,\bar j_1\ldots\,\bar j_\gamma\,j_1\ldots\,j_\beta\,k_1\ldots\,k_n}[P]=\sum^{3}_{d=0}\ \sum^{2}\ldots\sum^{3}_{\substack{a_1,\,\ldots\,,\,a_\alpha\\ b_1,\,\ldots\,,\,b_\beta\\ \bar a_1,\,\ldots\,,\,\bar a_\nu\\ \bar b_1,\,\ldots\,,\,\bar b_\gamma\\ c_1,\,\ldots\,,\,c_m\\ d_1,\,\ldots\,,\,d_n}}\mathfrak{S}^{\bar i_1}_{\bar a_1}\ldots\,\mathfrak{S}^{\bar i_\nu}_{\bar a_\nu}\ \mathfrak{T}^{\bar b_1}_{\bar j_1}\ldots\,\mathfrak{T}^{\bar b_\gamma}_{\bar j_\gamma}\ \overline{\mathfrak{S}^{i_1}_{a_1}}\ldots\,\overline{\mathfrak{S}^{i_\alpha}_{a_\alpha}}\times$$

$$\times\,\overline{\mathfrak{T}^{b_1}_{j_1}}\ldots\,\overline{\mathfrak{T}^{b_\beta}_{j_\beta}}\ S^{h_1}_{c_1}\ldots\,S^{h_m}_{c_m}\ T^{d_1}_{k_1}\ldots\,T^{d_n}_{k_n}\ T^{d}_{j}\ \tilde{\bar\Gamma}^{\bar a_1\ldots\,\bar a_\nu\,a_1\ldots\,a_\alpha\,c_1\ldots\,c_m}_{d\,\bar b_1\ldots\,\bar b_\gamma\,b_1\ldots\,b_\beta\,d_1\ldots\,d_n}[P]\,+$$

$$+\sum^{\alpha}_{\mu=1}\sum^{2}_{v_\mu=1}\overline{\vartheta^{i_\mu}_{j\,v_\mu}}\ \overline{S^{i_1\ldots\,v_\mu\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,\ldots\,\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]}-\sum^{\beta}_{\mu=1}\sum^{2}_{w_\mu=1}\overline{\vartheta^{w_\mu}_{j\,j_\mu}}\times$$

$$\times\,\overline{S^{i_1\ldots\,\ldots\,\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,w_\mu\,\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]}+\sum^{\nu}_{\mu=1}\sum^{2}_{v_\mu=1}\vartheta^{\bar i_\mu}_{j\,v_\mu}\ \overline{S^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,v_\mu\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\ldots\,\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]}\,-$$

$$-\sum^{\gamma}_{\mu=1}\sum^{2}_{w_\mu=1}\vartheta^{w_\mu}_{j\,\bar j_\mu}\ \overline{S^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\ldots\,\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,w_\mu\,\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]}+\sum^{m}_{\mu=1}\sum^{3}_{v_\mu=0}\theta^{i_\mu}_{j\,v_\mu}\times$$

$$\times\,\overline{S^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,v_\mu\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,\ldots\,\ldots\,k_n}[P]}-\sum^{n}_{\mu=1}\sum^{3}_{w_\mu=0}\theta^{w_\mu}_{j\,j_\mu}\ \overline{S^{i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,\ldots\,\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,w_\mu\,\ldots\,k_n}[P]}.$$

Other geometric structures associated with a horizontal covariant differentiation ∇ reveal when we apply $D=\nabla_{\mathbf{Y}}$ to the modules $S^1_0\bar S^0_0T^0_0(M)$, $S^0_0\bar S^1_0T^0_0(M)$, and $S^0_0\bar S^0_0T^1_0(M)$. The action of $D=\nabla_{\mathbf{Y}}$ within these modules is determined by the formulas (13.23) and (13.22), where $\hat{\boldsymbol{\Upsilon}}_i$, $\hat{\boldsymbol{\Psi}}_i$, and $\hat{\bar{\boldsymbol{\Psi}}}_i$ are defined by the formulas

(13.6), (13.8), and (13.10) respectively. In the present case we can take $\mathbf{Y} = \hat{\boldsymbol{\Upsilon}}_j$ and write these formulas as follows:

$$\nabla_{\hat{\boldsymbol{\Upsilon}}_j}\hat{\boldsymbol{\Psi}}_i = \sum_{k=1}^{2} \mathrm{A}^k_{j\,i}\ \boldsymbol{\Psi}_k, \tag{19.5}$$

$$\nabla_{\hat{\boldsymbol{\Upsilon}}_j}\hat{\overline{\boldsymbol{\Psi}}}_i = \sum_{k=1}^{2} \bar{\mathrm{A}}^k_{j\,i}\,\overline{\boldsymbol{\Psi}}_k. \tag{19.6}$$

$$\nabla_{\hat{\boldsymbol{\Upsilon}}_j}\hat{\boldsymbol{\Upsilon}}_i = \sum_{k=0}^{3} \Gamma^k_{j\,i}\,\mathbf{E}_k. \tag{19.7}$$

The coefficients $\mathrm{A}^k_{j\,i}$, $\bar{\mathrm{A}}^k_{j\,i}$, and $\Gamma^k_{j\,i}$ in (19.5), (19.6), and (19.7) are the same as in the formulas (15.9), (15.10), and (15.8). Since $Z^i_j = \delta^i_j$ for a horizontal covariant differentiation, the transformation formulas (15.15), (15.16), and (15.14) for $\mathrm{A}^k_{j\,i}$, $\bar{\mathrm{A}}^k_{j\,i}$, and $\Gamma^k_{j\,i}$ now reduce to the following ones:

$$\mathrm{A}^k_{j\,i} = \sum_{b=1}^{2}\sum_{a=1}^{2}\sum_{c=0}^{3} \mathfrak{S}^k_a\,\mathfrak{T}^b_i\,T^c_j\ \tilde{\mathrm{A}}^a_{c\,b} + \vartheta^k_{j\,i}, \tag{19.8}$$

$$\bar{\mathrm{A}}^k_{j\,i} = \sum_{b=1}^{2}\sum_{a=1}^{2}\sum_{c=0}^{3} \overline{\mathfrak{S}^k_a}\ \overline{\mathfrak{T}^b_i}\,T^c_j\ \tilde{\bar{\mathrm{A}}}^a_b + \overline{\vartheta^k_{j\,i}}. \tag{19.9}$$

$$\Gamma^k_{j\,i} = \sum_{b=0}^{3}\sum_{a=0}^{3}\sum_{c=0}^{3} S^k_a\,T^b_i\,T^c_j\ \tilde{\Gamma}^a_{c\,b} + \theta^k_{j\,i}, \tag{19.10}$$

Definition 19.1. Let N be a composite spin-tensorial bundle over the space-time manifold M. An extended affine connection is a geometric object such that in each local chart U of M equipped with a positively polarized right orthonormal frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and associated spinor frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ it is represented by its components $\mathrm{A}^k_{j\,i}$, $\bar{\mathrm{A}}^k_{j\,i}$, $\Gamma^k_{j\,i}$ and such that its components are smooth functions of the variables (9.11) and (9.12) transforming according to the formulas (19.8), (19.9), and (19.10) under a change of a local chart.

Theorem 19.2. *Any smooth paracompact space-time manifold M admitting the spinor structure and equipped with a composite spin-tensorial bundle N possesses at least one extended affine connection.*

Compare the theorem 19.2 with the theorem 13.2 in [5]. See also Chapter III of the thesis [12] for ideas how to prove this theorem.

Definition 19.2. A horizontal covariant differentiation ∇ of the algebra of extended spin-tensorial fields $\mathbf{S}(M)$ is called a *spatial covariant differentiation* or a *spatial gradient* if the operator ∇ annuls all native extended spin-tensorial fields and all complex conjugate fields for them:

$$\nabla\mathbf{S}[P] = 0 \quad \text{and} \quad \nabla\tau(\mathbf{S}[P]) = 0 \quad \text{for all} \quad P = 1,\,\dots\,,\,J+Q. \tag{19.11}$$

Let's study the first equality (19.11) in local coordinates. For this purpose we use the formula (15.11) substituting $Z^i_j = \delta^i_j$ and taking into account (19.4). As a result from $\nabla_{\!j} S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P] = 0$ we derive the equality

$$\Gamma^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P] = \sum^{\alpha}_{\mu=1}\sum^{2}_{v_\mu=1} \mathrm{A}^{i_\mu}_{j\,v_\mu}\ \ S^{\,i_1\ldots\,v_\mu\,\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j_1\ldots\,\ldots\,\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P] -$$
$$-\sum^{\beta}_{\mu=1}\sum^{2}_{w_\mu=1} \mathrm{A}^{w_\mu}_{j\,j_\mu}\ \ S^{\,i_1\ldots\,\ldots\,\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j_1\ldots\,w_\mu\,\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P] + \sum^{\nu}_{\mu=1}\sum^{2}_{v_\mu=1} \bar{\mathrm{A}}^{\bar i_\mu}_{j\,v_\mu}\times$$
$$\times S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,v_\mu\,\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\ldots\,\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P] - \sum^{\gamma}_{\mu=1}\sum^{2}_{w_\mu=1} \bar{\mathrm{A}}^{w_\mu}_{j\,\bar j_\mu}\ \ S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\ldots\,\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,w_\mu\,\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P] +$$
$$+\sum^{m}_{\mu=1}\sum^{3}_{v_\mu=0} \Gamma^{h_\mu}_{j\,v_\mu}\ \ S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,v_\mu\,\ldots\,h_m}_{\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,\ldots\,\ldots\,k_n}[P] - \sum^{n}_{\mu=1}\sum^{3}_{w_\mu=0} \Gamma^{w_\mu}_{j\,k_\mu}\times$$
$$\times S^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,\ldots\,\ldots\,h_m}_{\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,w_\mu\,\ldots\,k_n}[P]$$

expressing $\Gamma^{\,i_1\ldots\,i_\alpha\,\bar i_1\ldots\,\bar i_\nu\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\beta\,\bar j_1\ldots\,\bar j_\gamma\,k_1\ldots\,k_n}[P]$ through the parameters $\mathrm{A}^k_{j\,i}$, $\bar{\mathrm{A}}^k_{j\,i}$, and $\Gamma^k_{j\,i}$. Similarly, from the second equality (19.11), using the formula (4.6), we derive the equality $\nabla_{\!j} \overline{S^{\,\bar i_1\ldots\,\bar i_\alpha\,i_1\ldots\,i_\nu\,h_1\ldots\,h_m}_{\,\bar j_1\ldots\,\bar j_\beta\,j_1\ldots\,j_\gamma\,k_1\ldots\,k_n}[P]} = 0$. Applying (15.11) to it, we get

$$\bar\Gamma^{\,i_1\ldots\,i_\nu\,\bar i_1\ldots\,\bar i_\alpha\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\gamma\,\bar j_1\ldots\,\bar j_\beta\,k_1\ldots\,k_n}[P] = \sum^{\nu}_{\mu=1}\sum^{2}_{v_\mu=1} \mathrm{A}^{i_\mu}_{j\,v_\mu}\ \ \overline{S^{\,\bar i_1\ldots\,\bar i_\alpha\,i_1\ldots\,v_\mu\,\ldots\,i_\nu h_1\ldots\,h_m}_{\,\bar j_1\ldots\,\bar j_\beta\,j_1\ldots\,\ldots\,\ldots\,j_\gamma k_1\ldots\,k_n}[P]} -$$
$$-\sum^{\gamma}_{\mu=1}\sum^{2}_{w_\mu=1} \mathrm{A}^{w_\mu}_{j\,j_\mu}\ \ \overline{S^{\,\bar i_1\ldots\,\bar i_\alpha\,i_1\ldots\,\ldots\,\ldots\,i_\nu\,h_1\ldots\,h_m}_{\,\bar j_1\ldots\,\bar j_\beta\,j_1\ldots\,w_\mu\,\ldots\,j_\gamma\,k_1\ldots\,k_n}[P]} + \sum^{\alpha}_{\mu=1}\sum^{2}_{v_\mu=1} \bar{\mathrm{A}}^{\bar i_\mu}_{j\,v_\mu}\times$$
$$\times \overline{S^{\,\bar i_1\ldots\,v_\mu\,\ldots\,\bar i_\alpha\,i_1\ldots\,i_\nu\,h_1\ldots\,h_m}_{\,\bar j_1\ldots\,\ldots\,\ldots\,\bar j_\beta\,j_1\ldots\,j_\gamma\,k_1\ldots\,k_n}[P]} - \sum^{\beta}_{\mu=1}\sum^{2}_{w_\mu=1} \bar{\mathrm{A}}^{w_\mu}_{j\,\bar j_\mu}\ \ \overline{S^{\,\bar i_1\ldots\,\ldots\,\ldots\,\bar i_\alpha\,i_1\ldots\,i_\nu\,h_1\ldots\,h_m}_{\,\bar j_1\ldots\,w_\mu\,\ldots\,\bar j_\beta\,j_1\ldots\,j_\gamma\,k_1\ldots\,k_n}[P]} +$$
$$+\sum^{m}_{\mu=1}\sum^{3}_{v_\mu=0} \Gamma^{h_\mu}_{j\,v_\mu}\ \ \overline{S^{\,\bar i_1\ldots\,\bar i_\alpha\,i_1\ldots\,i_\nu\,h_1\ldots\,v_\mu\,\ldots\,h_m}_{\,\bar j_1\ldots\,\bar j_\beta\,j_1\ldots\,j_\gamma\,k_1\ldots\,\ldots\,\ldots\,k_n}[P]} - \sum^{n}_{\mu=1}\sum^{3}_{w_\mu=0} \Gamma^{w_\mu}_{j\,k_\mu}\times$$
$$\times \overline{S^{\,\bar i_1\ldots\,\bar i_\alpha\,i_1\ldots\,i_\nu\,h_1\ldots\,\ldots\,\ldots\,h_m}_{\,\bar j_1\ldots\,\bar j_\beta j_1\ldots\,j_\gamma\,k_1\ldots\,w_\mu\,\ldots\,k_n}[P]}.$$

This equality expresses $\bar\Gamma^{\,i_1\ldots\,i_\nu\,\bar i_1\ldots\,\bar i_\alpha\,h_1\ldots\,h_m}_{\,j\,j_1\ldots\,j_\gamma\,\bar j_1\ldots\,\bar j_\beta\,k_1\ldots\,k_n}[P]$ through the same parameters $\mathrm{A}^k_{j\,i}$, $\bar{\mathrm{A}}^k_{j\,i}$, and $\Gamma^k_{j\,i}$. Thus, due to the additional condition (19.11) a spacial covariant differentiation depends on less number of parameters than an arbitrary horizontal covariant differentiation. Substituting the above two expressions back into the formula (5.11) and taking into account (19.4) and $Z^i_j = \delta^i_j$, we derive the general formula for a spacial covariant derivative in local coordinates (9.11) and (9.12).

Though being more huge, this formula is a specialization of the formula (15.11):

$$
\begin{aligned}
&\nabla_j X^{a_1\dots a_\varepsilon \bar a_1\dots \bar a_\sigma c_1\dots c_e}_{b_1\dots b_\eta \bar b_1\dots \bar b_\varsigma d_1\dots d_f} = \sum_{k=0}^{3}\Upsilon^k_j\,\frac{\partial X^{a_1\dots a_\varepsilon \bar a_1\dots \bar a_\sigma c_1\dots c_e}_{b_1\dots b_\eta \bar b_1\dots \bar b_\varsigma d_1\dots d_f}}{\partial x^k}\,-\\
&-\sum_{P=1}^{J+Q}\ \sum^{2}_{\substack{i_1,\,\dots,\,i_\alpha\\ j_1,\,\dots,\,j_\beta\\ \bar i_1,\,\dots,\,\bar i_\nu\\ \bar j_1,\,\dots,\,\bar j_\gamma\\ h_1,\,\dots,\,h_m\\ k_1,\,\dots,\,k_n}}\!\!\dots\sum^{3}\Biggl(\sum_{\mu=1}^{\alpha}\sum_{v_\mu=1}^{2}\mathrm{A}^{i_\mu}_{j\,v_\mu}\ S^{i_1\dots v_\mu\dots i_\alpha \bar i_1\dots \bar i_\nu h_1\dots h_m}_{j_1\dots\dots\dots j_\beta \bar j_1\dots \bar j_\gamma k_1\dots k_n}[P]\,-\\
&-\sum_{\mu=1}^{\beta}\sum_{w_\mu=1}^{2}\mathrm{A}^{w_\mu}_{j\,j_\mu}\ S^{i_1\dots\dots\dots i_\alpha \bar i_1\dots \bar i_\nu h_1\dots h_m}_{j_1\dots w_\mu\dots j_\beta \bar j_1\dots \bar j_\gamma k_1\dots k_n}[P]+\sum_{\mu=1}^{\nu}\sum_{v_\mu=1}^{2}\bar{\mathrm{A}}^{\bar i_\mu}_{j\,v_\mu}\,\times\\
&\times S^{i_1\dots i_\alpha \bar i_1\dots v_\mu\dots \bar i_\nu h_1\dots h_m}_{j_1\dots j_\beta \bar j_1\dots\dots\dots \bar j_\gamma k_1\dots k_n}[P]-\sum_{\mu=1}^{\gamma}\sum_{w_\mu=1}^{2}\bar{\mathrm{A}}^{w_\mu}_{j\,\bar j_\mu}\,\times\\
&\times S^{i_1\dots i_\alpha \bar i_1\dots\dots\dots \bar i_\nu h_1\dots h_m}_{j_1\dots j_\beta \bar j_1\dots w_\mu\dots \bar j_\gamma k_1\dots k_n}[P]+\sum_{\mu=1}^{m}\sum_{v_\mu=0}^{3}\Gamma^{h_\mu}_{j\,v_\mu}\,\times\\
&\times S^{i_1\dots i_\alpha \bar i_1\dots \bar i_\nu h_1\dots v_\mu\dots h_m}_{j_1\dots j_\beta \bar j_1\dots \bar j_\gamma k_1\dots\dots\dots k_n}[P]-\sum_{\mu=1}^{n}\sum_{w_\mu=0}^{3}\Gamma^{w_\mu}_{j\,k_\mu}\,\times\\
&\times\ S^{i_1\dots i_\alpha \bar i_1\dots \bar i_\nu h_1\dots\dots\dots h_m}_{j_1\dots j_\beta \bar j_1\dots \bar j_\gamma k_1\dots w_\mu\dots k_n}[P]\Biggr)\,\frac{\partial X^{a_1\dots a_\varepsilon \bar a_1\dots \bar a_\sigma c_1\dots c_e}_{b_1\dots b_\eta \bar b_1\dots \bar b_\varsigma d_1\dots d_f}}{\partial S^{i_1\dots i_\alpha \bar i_1\dots \bar i_\nu h_1\dots h_m}_{j_1\dots j_\beta \bar j_1\dots \bar j_\gamma k_1\dots k_n}[P]}\,-\\
&-\sum_{P=1}^{J+Q}\ \sum^{2}_{\substack{i_1,\,\dots,\,i_\alpha\\ j_1,\,\dots,\,j_\beta\\ \bar i_1,\,\dots,\,\bar i_\nu\\ \bar j_1,\,\dots,\,\bar j_\gamma\\ h_1,\,\dots,\,h_m\\ k_1,\,\dots,\,k_n}}\!\!\dots\sum^{3}\Biggl(\sum_{\mu=1}^{\nu}\sum_{v_\mu=1}^{2}\mathrm{A}^{i_\mu}_{j\,v_\mu}\ \overline{S^{\bar i_1\dots \bar i_\alpha\, i_1\dots v_\mu\dots i_\nu h_1\dots h_m}_{\bar j_1\dots \bar j_\beta\, j_1\dots\dots\dots j_\gamma k_1\dots k_n}[P]}\,-\\
&-\sum_{\mu=1}^{\gamma}\sum_{w_\mu=1}^{2}\mathrm{A}^{w_\mu}_{j\,j_\mu}\ \overline{S^{\bar i_1\dots \bar i_\alpha\, i_1\dots\dots\dots i_\nu h_1\dots h_m}_{\bar j_1\dots \bar j_\beta\, j_1\dots w_\mu\dots j_\gamma k_1\dots k_n}[P]}+\sum_{\mu=1}^{\alpha}\sum_{v_\mu=1}^{2}\bar{\mathrm{A}}^{\bar i_\mu}_{j\,v_\mu}\,\times\\
&\times\overline{S^{\bar i_1\dots v_\mu\dots \bar i_\alpha\, i_1\dots i_\nu h_1\dots h_m}_{\bar j_1\dots\dots\dots \bar j_\beta\, j_1\dots j_\gamma k_1\dots k_n}[P]}-\sum_{\mu=1}^{\beta}\sum_{w_\mu=1}^{2}\bar{\mathrm{A}}^{w_\mu}_{j\,\bar j_\mu}\,\times\\
&\times\overline{S^{\bar i_1\dots\dots\dots \bar i_\alpha\, i_1\dots i_\nu h_1\dots h_m}_{\bar j_1\dots w_\mu\dots \bar j_\beta\, j_1\dots j_\gamma k_1\dots k_n}[P]}+\sum_{\mu=1}^{m}\sum_{v_\mu=0}^{3}\Gamma^{h_\mu}_{j\,v_\mu}\,\times\\
&\times\overline{S^{\bar i_1\dots \bar i_\alpha\, i_1\dots i_\nu h_1\dots v_\mu\dots h_m}_{\bar j_1\dots \bar j_\beta\, j_1\dots j_\gamma k_1\dots\dots\dots k_n}[P]}-\sum_{\mu=1}^{n}\sum_{w_\mu=0}^{3}\Gamma^{w_\mu}_{j\,k_\mu}\,\times\\
&\times\ \overline{S^{\bar i_1\dots \bar i_\alpha\, i_1\dots i_\nu h_1\dots\dots\dots h_m}_{\bar j_1\dots \bar j_\beta\, j_1\dots j_\gamma k_1\dots w_\mu\dots k_n}[P]}\Biggr)\,\frac{\partial X^{a_1\dots a_\varepsilon \bar a_1\dots \bar a_\sigma c_1\dots c_e}_{b_1\dots b_\eta \bar b_1\dots \bar b_\varsigma d_1\dots d_f}}{\partial\overline{S^{i_1\dots i_\alpha \bar i_1\dots \bar i_\nu h_1\dots h_m}_{j_1\dots j_\beta \bar j_1\dots \bar j_\gamma k_1\dots k_n}[P]}}\,+
\end{aligned}
\tag{19.12}
$$

$$+\sum_{\mu=1}^{\varepsilon}\sum_{v_\mu=1}^{2}\mathrm{A}^{a_\mu}_{j\,v_\mu}\;X^{a_1\ldots\,v_\mu\,\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,\ldots\,\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}-$$

$$-\sum_{\mu=1}^{\eta}\sum_{w_\mu=1}^{2}\mathrm{A}^{w_\mu}_{j\,b_\mu}\;X^{a_1\ldots\,\ldots\,\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,w_\mu\,\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,d_f}+$$

$$+\sum_{\mu=1}^{\sigma}\sum_{v_\mu=1}^{2}\bar{\mathrm{A}}^{\bar a_\mu}_{j\,v_\mu}\;X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,v_\mu\,\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\ldots\,\ldots\,\bar b_\zeta d_1\ldots\,d_f}-$$

$$-\sum_{\mu=1}^{\zeta}\sum_{w_\mu=1}^{2}\bar{\mathrm{A}}^{w_\mu}_{j\,\bar b_\mu}\;X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\ldots\,\ldots\,\bar a_\sigma c_1\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,w_\mu\,\ldots\,\bar b_\zeta d_1\ldots\,d_f}+$$

$$+\sum_{\mu=1}^{e}\sum_{v_\mu=0}^{3}\Gamma^{c_\mu}_{j\,v_\mu}\;X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,v_\mu\,\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,\ldots\,\ldots\,d_f}-$$

$$-\sum_{\mu=1}^{f}\sum_{w_\mu=0}^{3}\Gamma^{w_\mu}_{j\,b_\mu}\;X^{a_1\ldots\,a_\varepsilon\bar a_1\ldots\,\bar a_\sigma c_1\ldots\,\ldots\,\ldots\,c_e}_{b_1\ldots\,b_\eta\bar b_1\ldots\,\bar b_\zeta d_1\ldots\,w_\mu\,\ldots\,d_f}.$$

Any horizontal covariant differentiation differentiation is defined by two independent geometric structures:

(1) a horizontal lift of vectors from M to N;
(2) an extended connection.

In the case of a spacial differentiation these two structures are related to each other. This result is formulated as the following theorem.

Theorem 19.3. *Defining a spacial covariant differentiation in the algebra of extended spin-tensorial fields* $\mathbf{S}(M)$ *is equivalent to defining an extended connection.*

20. The structural theorem for differentiations.

Theorem 20.1. *Let N be a composite tensor bundle over the space-time manifold M in the sense of the formula (9.7). If M is equipped with some extended affine connection, then each differentiation D of the algebra of extended spin-tensorial fields* $\mathbf{S}(M)$ *is uniquely expanded into a sum*

$$D=\nabla_{\mathbf{X}}+\sum_{P=1}^{J+Q}\nabla_{\mathbf{Y}_P}[P]+\sum_{P=1}^{J+Q}\bar{\nabla}_{\bar{\mathbf{Y}}_P}[P]+S,\tag{20.1}$$

where $\nabla_{\mathbf{X}}$ is the spacial covariant differentiation along some extended vector field $\mathbf{X}$, $\nabla_{\mathbf{Y}_P}[P]$ is the P-th canonical vertical multivariate differentiation along some extended spin-tensorial field $\mathbf{Y}_P$, $\bar{\nabla}_{\bar{\mathbf{Y}}_P}[P]$ is the P-th barred canonical vertical multivariate differentiation along some other extended spin-tensorial field $\bar{\mathbf{Y}}_P$, and S is some degenerate differentiation of the algebra $\mathbf{S}(M)$.

Proof. Let $D\in\mathfrak{D}_{\mathbb{C}}(M)$. Then its restriction to the module $S^0_0\bar S^0_0T^0_0(M)$ is given by some vector field $\mathbf{Z}$ in N. The extended affine connection in M determines some horizontal lift of vectors f from M to N. According to the theorem 19.1, this lift of vectors determines the expansion of the tangent space $\mathbb{C}T_q(N)$ into a direct sum

(19.2) at each point $q \in N$. The vertical subspace $V_q(N)$ in (19.2) has its own expansion (18.24) into a direct sum. Combining (19.2) and (18.24), we obtain

$$\begin{aligned} T_q(N) = H_q(N) \oplus V_q[1](N) \oplus \ldots \oplus V_q[J+Q](N) +\\ + \bar{V}_q[1](N) \oplus \ldots \oplus \bar{V}_q[J+Q](N). \end{aligned} \tag{20.2}$$

Then the vector field $\mathbf{Z}$ is expanded into a sum of vector fields

$$\mathbf{Z} = \mathbf{H} + \mathbf{V}_1 + \ldots + \mathbf{V}_{J+Q} + \bar{\mathbf{V}}_1 + \ldots + \bar{\mathbf{V}}_{J+Q}. \tag{20.3}$$

uniquely determined by the expansion (20.2). Due to the maps (18.20) and (18.22) and due to the property (18.25) of these maps each vector field $\mathbf{V}_P$ in (20.3) is uniquely associated with some extended tensor field $\mathbf{Y}_P$ and each vector field $\bar{\mathbf{V}}_P$ is uniquely associated with some other extended tensor field $\bar{\mathbf{Y}}_P$. Similarly, the vector field $\mathbf{H}$ is uniquely associated with some extended vector field $\mathbf{X}$ by means of the maps (19.1). Then we can consider the sum

$$\tilde{D} = \nabla_{\mathbf{X}} + \sum_{P=1}^{J+Q} \nabla_{\mathbf{Y}_P}[P] + \sum_{P=1}^{J+Q} \bar{\nabla}_{\bar{\mathbf{Y}}_P}[P]. \tag{20.4}$$

The sum (20.4) is a differentiation of $\mathbf{S}(M)$ such that its restriction to $S^0_0\bar{S}^0_0T^0_0(M)$ is given by the vector field (20.3). Hence, $D - \tilde{D}$ is a differentiation of $\mathbf{S}(M)$ with identically zero restriction to $S^0_0\bar{S}^0_0T^0_0(M)$. This means that $D - \tilde{D}$ is a degenerate differentiation (see the definition 14.1). Denoting this degenerate differentiation by S, from (20.4) we derive the required expansion (20.1). The theorem is proved. □

The theorem 20.1 is the structural theorem for differentiations in the algebra of extended spin-tensorial fields $\mathbf{S}(M)$. It approves our previous efforts in studying the three basic types of differentiations which are used in the formula (20.1). Their application to the description of real physical fields is the subject for separate publications. The formula (19.12) is the most important explicit formula for such applications.

21. Dedicatory.

This paper is dedicated to my aunt Abdurahmanova Nailya Muhamedovna who came through the anxious times of the 20th century and had left us in the beginning of this new possibly not less anxious 21st century.

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